Higher Coinvariant Algebras, q-Stirling Numbers, and Coxeter-like Complexes

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Based partly on joint work with
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arXiv: 2001.06076, 2109.03407

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arXiv: 2205.14078

See also
arXiv: 2109.05080

Slides: https://www.jpswanson.org/talks/2022_Boston_Coinvariants.pdf

Presented at Harvard/MIT/MSR Combinatorics Seminar
October 26th, 2022

Outline

I. Higher coinvariant algebras

II. Super coinvariant algebras

III. $G(m,1,n)$-Stirling numbers

IV. Coxeter-like complexes

I. HIGHER COINVARIANT ALGEBRAS

Coinvariant Algebras

Thm (Newton) $\mathbb{Q}[x_1, \ldots, x_n]^{S_n} = \mathbb{Q}[e_1, \ldots, e_n]$ where $e_i = \underbrace{\sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k}}_{\text{elementary symmetric polynomial}}$

and $\sigma(x_i) = x_{\sigma(i)}$

Thm (Hilbert) $\langle \mathbb{Q}[x_1, \ldots, x_n]_+^{S_n} \rangle = \langle e_1, \ldots, e_n \rangle$

Def The coinvariant algebra of S_n is

$$R_n = \frac{\mathbb{Q}[x_1, \ldots, x_n]}{\langle e_1, \ldots, e_n \rangle}$$

Coinvariant Algebras

Thm (Borel) $R_n \cong \overset{\text{singular cohomology}}{\overbrace{H^*}}(\underbrace{Fl_n}_{\text{complete flag manifold}})$

Thm (Chevalley) $R_n \cong \mathbb{Q}S_n$

$\Rightarrow \dim R_n = n!$

Thm (Artin) $\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} : 0 \leq \alpha_i < i\}$ descends to a basis for R_n

Cor $\mathrm{Hilb}(R_n; q) = \sum_{d \geq 0} \dim(R_n)_d \cdot q^d = \underbrace{1 \cdot (1+q) \cdot (1+q+q^2) \cdots \underbrace{(1+q+\cdots+q^{n-1})}_{[n]_q}}_{[n]_q!}$

Diagonal Coinvariant Algebras

$\boxed{\text{Def}}$ (Garsia-Haiman '90's)

The <u>diagonal</u> coinvariant algebra of S_n is

$$DR_n = \frac{\mathbb{Q}[\underline{x}_n, \underline{y}_n]}{\langle \mathbb{Q}[\underline{x}_n, \underline{y}_n]^{S_n}_+ \rangle}$$

where S_n acts diagonally:

$$\sigma(x_i) = x_{\sigma(i)}, \quad \sigma(y_i) = y_{\sigma(i)}$$

Diagonal Coinvariant Algebras

- Haiman related DR_n to isospectral Hilbert schemes and Macdonald polynomials. Proved $n!$ conjecture and Macdonald positivity conjecture.

Diagonal Coinvariant Algebras

Thm (Haiman '02)

1) $\underline{\dim DR_n} = (n+1)^{n-1}$

2) $\underline{\mathrm{Hilb}(DR_n; q,t)}\Big|_{t=q^{-1}} = \sum_{a,b} (\dim DR_n^{a,b}) q^a t^b \Big|_{t=q^{-1}} = q^{-\binom{n}{2}} [n+1]_q^{n-1}$

3) $\underline{\mathrm{GrFrob}(DR_n; q,t)} = \sum_{a,b,\lambda} \text{mult. of } S^\lambda \text{ in } DR_n^{a,b} \cdot \underbrace{s_\lambda}_{\text{Schur function}} q^a t^b = \nabla e_n$

Diagonal Coinvariant Algebras

Conj (Haglund–Haiman–Loehr–Remmel–Ulyanov '05)

Shuffle conjecture: $\nabla e_n = \displaystyle\sum_{P \in \mathcal{LD}_n} \underbrace{q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)} x^P}_{\substack{\text{monomial expansion} \\ \text{of } \mathrm{Gr}\,\mathrm{Frob}(DR_n)}}$

Thm (Carlsson–Mellit '17)

The shuffle conjecture is true!

Delta Conjecture

Conj (Haglund–Remmel–Wilson '18)

Delta conjecture: for $0 \le k \le n-1$,

$$\Delta'_{e_k} e_n = \sum_{P \in \mathcal{LD}_n} q^{\operatorname{dinv}(P)} t^{\operatorname{area}(P)} \prod_{i : a_i(P) > a_{i-1}(P)} \left(1 + z/t^{a_i(P)}\right) x^P \Big|_{z^{n-k-1}}$$

("rise version")

Thm (D'Adderio–Mellit '22;
Blasiak–Haiman–Morse–Pun–Seelinger '22+)

The Delta conjecture is true!

Delta Conjecture

Q| What is the <u>geometry</u> behind the Delta conjecture?
Is there an underlying "higher coinvariant algebra"?

Generalized coinvariant algebras

$\boxed{\text{Def}}$ (Haglund-Rhoades-Shimozono '18)

The generalized coinvariant algebra is

$$R_{n,k} = \mathbb{Q}[\underline{x}_n]/\langle x_1^k, \ldots, x_n^k, e_n(\underline{x}_n), e_{n-1}(\underline{x}_n), \ldots, e_{n-k+1}(\underline{x}_n)\rangle.$$

fairly ad-hoc?

Generalized coinvariant algebras

$\underline{\text{Thm}}$ (Haglund–Rhoades–Shimozono '18)

1) $\mathrm{Hilb}(R_{n,k};q) = \mathrm{rev}_q([k]!\,\mathrm{Stir}[n,k])$

2) $\mathrm{grFrob}(R_{n,k};q) = \mathrm{rev}_q \circ \omega\left(\Delta'_{e_{k+1}} e_n \big|_{t=0}\right)$

$\underline{\text{Thm}}$ (Rhoades–Pawlowski '19)

There are varieties $X_{n,k}$ such that

$$R_{n,k} \cong H^*(X_{n,k})$$

Great! Though $t=0$...

II. SUPER COINVARIANT ALGEBRAS

Super Coinvariant Algebras

Def (Zabrocki '19)

The super diagonal coinvariants are

natural!

$$SDR_n = \mathbb{Q}[\underline{x}_n, \underline{y}_n, \underline{\theta}_n] / \langle \mathbb{Q}[\underline{x}_n, \underline{y}_n, \underline{\theta}_n]^{S_n}_{+} \rangle$$

where $x_i y_j = x_j y_i$, $x_i \theta_j = \theta_j x_i$, $y_i \theta_j = \theta_j y_i$, and $\underbrace{\theta_i \theta_j = -\theta_j \theta_i}_{\text{anti-commute}}$.

Super Coinvariant Algebras

<u>Conj</u> (Zabrocki '19)

$$\mathrm{GrFrob}(SDR_n; q, t, z) = \sum_{k=0}^{n-1} z^k \Delta'_{e_{n-k-1}}(e_n)$$

- That is, SDR_n is the higher coinvariant algebra associated to the Delta conjecture.

- Full case is clearly very hard! We focus on <u>$t=0$</u> from now on.

Super Coinvariant Algebras

- Superspace is $\underbrace{\mathbb{Q}[x_1,\ldots,x_n,\theta_1,\ldots,\theta_n]}_{Sym(x_1,\ldots,x_n)\otimes\Lambda(\theta_1,\ldots,\theta_n)}$ where $\theta_i\theta_j=-\theta_j\theta_i$ anti-commute (and $x_i\theta_j=\theta_j x_i$, $x_i x_j=x_j x_i$)

- S_n acts diagonally: $\sigma(x_i)=x_{\sigma(i)}$, $\sigma(\theta_i)=\theta_{\sigma(i)}$

- Think of θ variables as differential forms $\underline{\theta_i=dx_i}$,

$$\theta_i\theta_j=dx_i\wedge dx_j$$

Super Coinvariant Algebras

- The exterior derivative is

$$d = \sum_{i=1}^{n} \partial_{x_i} dx_i \in \mathrm{End}(\mathbb{Q}[\underline{x}_n, d\underline{x}_n])$$

<u>Thm</u> (Solomon)

$$\langle \mathbb{Q}[\underline{x}_n, d\underline{x}_n]^{S_n}_+ \rangle = \langle e_1, \dots, e_n, de_1, \dots, de_n \rangle$$

Super Coinvariant Algebras

Def The <u>super coinvariant algebra</u> of S_n is

$$\underline{SR_n} = \mathbb{Q}[\underline{x}_n, \underline{\theta}_n] / \langle \mathbb{Q}[\underline{x}_n, \underline{\theta}_n]^{S_n}_+ \rangle .$$

Conj (Zabrocki '19)

$$\mathrm{Hilb}(SR_n; q, z) = \sum_{k=1}^{n} [k]! \, S[n,k] \, z^{n-k}$$

Super Coinvariant Algebras

Q| What has been <u>proven</u> about SR_n?

Super Coinvariant Algebras

Thm (Wallach–S. '21)

The exterior derivative complex

$$0 \to \mathbb{Q} \to \underbrace{SR_n^0}_{=R_n} \xrightarrow{d} \underbrace{SR_n^1}_{\theta\text{-degree } 1} \xrightarrow{d} \cdots \xrightarrow{d} SR_n^{n-1} \to 0$$

is exact.

Cor $\mathrm{Hilb}(SR_n; q, -q) = \sum_{k=1}^{n} (-q)^{n-k} \mathrm{Hilb}(SR_n^k; q) = 1$

- In fact, holds for any complex reflection group G!

Super Coinvariant Algebras

Thm (Wallach–S. '22+)

$$\underbrace{SR_n^{i,k}}_{\substack{x\text{-degree } i \\ \theta\text{-degree } k}} \neq 0 \iff i + \binom{k+1}{2} \leq \binom{n}{2}$$

- Agrees with Zabrocki's conjecture!
- Generalization for $G = G(m,1,n)$
- Total degree version for $G = G(m,p,n)$ if $p \neq m$
- Super operator conjecture and special cases

Super Coinvariant Algebras

- The generalized exterior derivatives are the operators

$$d_i = \sum_{j=1}^{n} \partial_{x_j}^{i} \, dx_j \qquad (d_1 = d)$$

Thm (Wallach–S. '21)

Vandermonde

$$SR_n^{sgn} = \operatorname{Span}\{d_{i_1} \cdots d_{i_k} \Delta_n : 1 \le i_1 < \cdots < i_k \le n-1\}$$

Also have general G version

Cor $\operatorname{Hilb}(SR_n^{sgn}; q) = \prod_{i=1}^{n-1} (q^i + z)$

- Agrees with Zabrocki's conjecture!

Super Coinvariant Algebras

- Consider the generalized exterior derivative complex

$$0 \to \mathbb{Q} \to SR_n^0 \xrightarrow{d_i} SR_n^1 \xrightarrow{d_i} \cdots \xrightarrow{d_i} SR_n^{n-1} \to 0.$$

Typically not exact.

- The graded Euler characteristic is

$$\chi(H(SR_{G}^{*}, \underline{d_i}); q) = \mathrm{Hilb}(SR_{G}; q, q^{-e_i^*}) - 1.$$

Problem Is there a topological or geometric interpretation of $\chi(H(SR_G^*, d_i); q)$?

Super Coinvariant Algebras

Iraci–Rhoades–Romero '22+ solved the "Fermionic" diagonal coinvariants version

- Multiple groups have tried to show $\dim SR_n^{n-k} = k!\,S(n,k)$, but without much success so far.
  - Kelvin Chan has a proposed harmonic basis
  - Toronto group and, independently, Sagan-S. have the same proposed monomial basis
  - I have work towards an upper bound based on a "succint" modified Hall-Littlewood expression

Super Coinvariant Algebras

Thm (S. '22+) Let $I = \{i_1 < \cdots < i_k\} \subset [n-1]$. Then

$$\sum (-1)^d \partial_{e_{n-k-d}(\underline{n-1})} d_{j_1 \cdots j_k} \Delta_n = 0$$

"generic" Tanisaki witness relations

where the sum is over all subsets $J = \{j_1 < \cdots < j_k\} \subset [n-1]$ where

$$1 \leq i_1 \leq j_1 < i_2 \leq j_2 < \cdots < i_k \leq j_k < n$$

and

$$d = (j_1 - i_1) + \cdots + (j_k - i_k).$$

Super Coinvariant Algebras

- Also have "most extreme" relations with coefficients

$$(-1)^d \Delta_s(j_{u-s+1}, \ldots, j_k) \binom{d+u}{u}$$

— Appear to be positive up to $(-1)^d$!

Q: What are all the Tanisaki witness relations?
What is a combinatorial description for their coefficients?
Is there a geometric/algebraic/topological interpretation?

III. $G(m,l,n)$-STIRLING NUMBERS

Overview

Variations on unsigned/signed Stirling numbers of the first kind and unordered/ordered Stirling numbers of the second kind:

	Type A	Type B	Other Groups
Classical	Stirling 1730	Zaslavski '81?	Zaslavski '81?
q-analogue	Carlitz '33 Gould '61	S.-Wallach '21 Sagan-S. '22+ Bagno-Garber '22+	Sagan-S. '22++

Today!

Classical Stirling Numbers

Recall: The (type A) Stirling numbers of the second kind count set partitions of $[n]$ with k blocks.

Ex: $\{\{1,3\},\{5,6\},\{2\}\} \leftrightarrow \underbrace{13/2/56}$

write in increasing order of minima, say

Lem: $S(n,k) = h_{n-k}(1,2,\ldots,k)$

Classical Stirling Numbers

Def The type A_{n-1} hyperplane arrangement is

$$\{x_i = x_j : 1 \leq i < j \leq n\} \subset \mathbb{R}^n.$$

- The corresponding intersection lattice is encoded by set partitions under (reverse) refinement:

$$\{x_1 = x_3\} \cap \{x_1 = x_4\} \longleftrightarrow \{\{1,3,4\},\{2\}\} \longleftrightarrow 134/2$$

- Note that $\operatorname{codim}(X) = n - k$

Classical Type B Stirling Numbers

Def The type B_n hyperplane arrangement is

$$\{x_i = \pm x_j : 1 \leq i < j \leq n\} \cup \{x_i = 0 : 1 \leq i \leq n\} \subset \mathbb{R}^n.$$

- Intersection lattice encoded by type B_n set partitions:

$\{x_1 = -x_3\} \cap \{x_1 = x_4\} \cap \{x_5 = 0\} \cap \{x_6 = 0\}$

$\longleftrightarrow \{\{1, \bar{3}, 4\}, \{\bar{1}, 3, \bar{4}\}, \{5, \bar{5}, 6, \bar{6}\}, \{2\}, \{\bar{2}\}\}$

$\longleftrightarrow \bar{6}\bar{5}056 \left| \begin{matrix} \bar{1}\,3\,\bar{4} \\ \underline{1}\,\bar{3}\,4 \end{matrix} \right| \begin{matrix} \bar{2} \\ \underline{2} \end{matrix}$

increasing minima, say, after negative pair

Set partition of $\langle n \rangle = \{\bar{n}, \ldots, \bar{1}, 0, 1, \ldots, n\}$ s.t.

$B \in P \Rightarrow -B \in P$ and

$B = -B \Leftrightarrow \underline{0 \in B}$

zero block

Full monomial groups

$\boxed{\text{Def}}$ The full monomial group is

$$G(m,1,n) = \left\{ \begin{array}{l} n \times n \text{ pseudopermutation matrices} \\ \text{with non-zero entries } \zeta \in \mathbb{C} \text{ s.t. } \zeta^m = 1 \end{array} \right\}$$

- $G(2,1,n) = B_n =$ signed permutations
- $|G(m,1,n)| = m^n n!$
- Acts on superspace: $\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \cdot x_1 x_2^2 = b^{-1} x_2 \, a^{-2} x_1$
- Natural corresponding super coinvariants $SR_{m,n} = \dfrac{\mathbb{C}[\underline{x}_n, \underline{\theta}_n]}{\langle \mathbb{C}[\underline{x}_n, \underline{\theta}_n]_+^{G(m,1,n)} \rangle}$

$G(m,1,n)$-Stirling Numbers

Def The $G(m,1,n)$ hyperplane arrangement is

$$\{x_i = \zeta x_j : 1 \le i < j \le n,\ \zeta^m = 1\} \cup \{x_i = 0 : 1 \le i \le n\} \subset \mathbb{R}^n$$

- Intersection lattice encoded by $G(m,1,n)$-set partitions: set partition of $[n^m] = \{0\} \cup \{i^c \mid i \in [n],\ 0 \le c < m\}$ s.t. $B \in P \Rightarrow B^{+1} \in P$ and $B = B^{+1} \Rightarrow 0 \in B$.

Ex

$$0\,4^0 4^1 4^2 \left| \begin{array}{c} 1^0 3^2 \\ 1^1 3^0 \\ 1^2 3^1 \end{array} \right| \begin{array}{c} 2^0 \\ 2^1 \\ 2^2 \end{array} = 04 \,|\, 1^0 3^2 \,|\, 2^0$$

IV. COXETER-LIKE COMPLEXES

Type A_{n-1} Coxeter complex

- Partition $\mathbb{R}^n$ into faces:

$$\bigcap_{1\le i<j\le n} \{x_i = x_j\} \text{ or } \{x_i > x_j\} \text{ or } \{x_i < x_j\}.$$

- Intersect with S^{n-1} to get a simplicial complex $\Sigma(A_{n-1})$, the Coxeter complex of type A_{n-1}.
- Faces correspond to ordered set partitions:

$$\{x_1=x_2\}\cap\{x_1<x_3\}\cap\{x_1>x_4\} \longleftrightarrow x_4<x_1=x_2<x_3$$
$$\cap\{x_2<x_3\}\cap\{x_2>x_4\}\cap\{x_3>x_4\} \longleftrightarrow \underline{(4/12/3)}$$

Hence ordered Stirlings:
$S^o(n,k) = k!\,S(n,k)$

Higher coinvariants and ordered Stirlings

Def (Carlitz '33; Gould '61)

The (type A) ordered q-Stirling numbers of the second kind

$$S^o[n,k] = [k]!\, h_{n-k}([1],[2],\ldots,[k]).$$

Thm (Haglund–Rhoades–Shimozono '18)

$$\mathrm{Hilb}(R_{n,k};q) = \mathrm{rev}_q(S^o[n,k]).$$

Higher coinvariants and ordered Stirlings

Def (S.–Wallach '21; see Sagan–S. '22)
The type B_n ordered q-Stirling numbers of the second kind

$$S_B^o[n,k] = [2][4]\cdots[2k]\, h_{n-k}([1],[3],\ldots,[2k+1]).$$

Def (Chan–Rhoades '20)
The $G(m,1,n)$-generalized coinvariant algebra is

$$R_{n,k}^{(m)} = \mathbb{C}[\underline{x}_n] / \langle x_1^{km+1},\ldots,x_n^{km+1}, e_n(\underline{x}_n^m), e_{n-1}(\underline{x}_n^m),\ldots, e_{n-k+1}(\underline{x}_n^m)\rangle.$$

Higher coinvariants and ordered Stirlings

Thm (Chan–Rhoades '20 + Sagan–S. '22+)

$$\mathrm{Hilb}(R^{(2)}_{n,k}; q) = \mathrm{rev}_q(S^{\circ}_{B}[n,k])$$

Higher coinvariants and ordered Stirlings

Thm (S.–Wallach '21) For any complex reflection group G,

$$\sum_k (-q)^{n-k}\,\mathrm{Hilb}(SR_G^k; q) = 1.$$

Lem Since $\Sigma(A_{n-1})$ and $\Sigma(B_n)$ are spheres,

$$\sum_k (-1)^{n-k} S^o(n,k) = 1 \quad \text{and} \quad \sum_k (-1)^{n-k} S_B^o(n,k) = 1.$$

- Consistent with conjectures $\mathrm{Hilb}(SR_n^{n-k}; q) = S^o[n,k]$

  $\mathrm{Hilb}(SR_{B_n}^{n-k}; q) = S_B^o[n,k]$!

(See Sagan–S. '22.)

Higher coinvariants and ordered Stirlings

Def A Chan-Rhoades $G(m,1,n)$-ordered set partition is an ordered set partition $(B_0, B_1, \ldots)$ of $\{0\} \cup [n]$ where $0 \in B_0$ and elements in B_i for $i \neq 0$ have colors $0 \le c < m$.

Ex $(0\,4 \mid 1^0 3^2 \mid 2^1)$

Def A Chan-Rhoades $G(m,1,n)$-ordered q-Stirling number is

$$S^0_{CR}[m,n,k] = [k][2k]\cdots[mk]\,h_{n-k}([1],[m+1],\ldots,[km+1])$$

Higher coinvariants and ordered Stirlings

Thm (Chan-Rhoades '20 + Sagan-S. '22++)

$$\mathrm{Hilb}(R^{(m)}_{n,k};q) = \mathrm{rev}_q(S^{\circ}_{CR}[m,n,k]).$$

Thm (Sagan-S. '22++) For $m>1$,

$$\sum_k (-q^{m-1})^{n-k} S^{\circ}_{CR}[m,n,k] = [m-1]^n_q.$$

Note At $q=1$, RHS $\neq 1$ except for $m=2$.

Hence generalized and super coinvariants diverge for $m>2$!

Milnor fiber complex

$\boxed{\text{Thm/Def}}$ (Orlik '90) Let G be a Shephard group. Let F be a homogeneous degree r polynomial G-invariant for $r \geq 1$ minimal. There is a simplicial complex $\Sigma(G)$ in the Milnor fiber $F^{-1}(1)$, called the Milnor fiber complex.

- Generalizes Coxeter complex for G real.

$\boxed{\text{Lem}}$ (Milnor) $\Sigma(G)$ is a wedge of spheres.

Milnor fiber complex

- Gives a topological proof of

$$\sum_k \left(-q^{(m-1)}\right)^{n-k} \mathcal{S}^{0}_{CR}[m,n,k] = [m-1]^n \quad \underline{\text{at } q=1}$$

by taking the reduced Euler characteristic!

$\boxed{\text{Problem}}$ Relate $R^{(m)}_{n,k}$ to $\mathcal{E}(G(m,1,n))$ through some explicit topology/geometry/algebra.

Super coinvariants and ordered Stirlings

Def A super $G(m,1,n)$-ordered set partition is an ordered set partition $(B_0, B_1, \ldots)$ of $\{0\} \cup [n]$ where $0 \in B_0$, elements in B_i for $i \neq 0$ have colors $0 \leq c < m$, and non-zero elements in B_0 have colors $\underline{1} \leq c < m$.

Ex $(0\,\underline{4}^1 \mid 1^0 3^2 \mid 2^1)$

Def A super $G(m,1,n)$-ordered q-Stirling number is

$$\overline{S}^{\circ}[m,n,k] = [k][2k]\cdots[mk]\,h_{n-k}([m-1],[2m-1],\ldots,[(k+1)m-1])$$

Super coinvariants and ordered Stirlings

Conj (Sagan–S. '22++)

$$\mathrm{Hilb}(SR^{n-k}_{G(m,1,n)};q) = \bar{S}^{o}[m,n,k]$$

Thm (Wallach–S. '21)

$$\mathrm{Hilb}(SR_G;q,q^{-1}) = \sum_k (-q)^{n-k}\,\mathrm{Hilb}(SR^{n-k}_G;q) = 1$$

Thm (Sagan–S. '22++) For $m>1$,

$$\sum_k (-q)^{n-k}\bar{S}^{o}[m,n,k] = 1$$

- Consistent with conjecture!

Super coinvariants complex

- In ongoing work with Sagan, we use [Björner-Ziegler '92] to construct simplicial spheres $\bar{\Sigma}(G(m,1,n))$ which give a topological proof of

$$\sum_k (-q)^{n-k} \bar{\zeta}^{0} [m,n,k] = 1 \quad \underline{\text{at } q=1}.$$

Problem | Relate $SR_{G(m,1,n)}$ to $\bar{\Sigma}(G(m,1,n))$ through some explicit topology/geometry/algebra.

THANKS!