

An $SL(4)$ web basis from hourglass plabic graphs

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Based on joint work with *Christian Gaetz, Oliver Pechenik, Stephan Pfannerer,*
and Jessica Striker (in preparation)

Slides: https://www.jpswanson.org/talks/2023_Davis_Webs.pdf

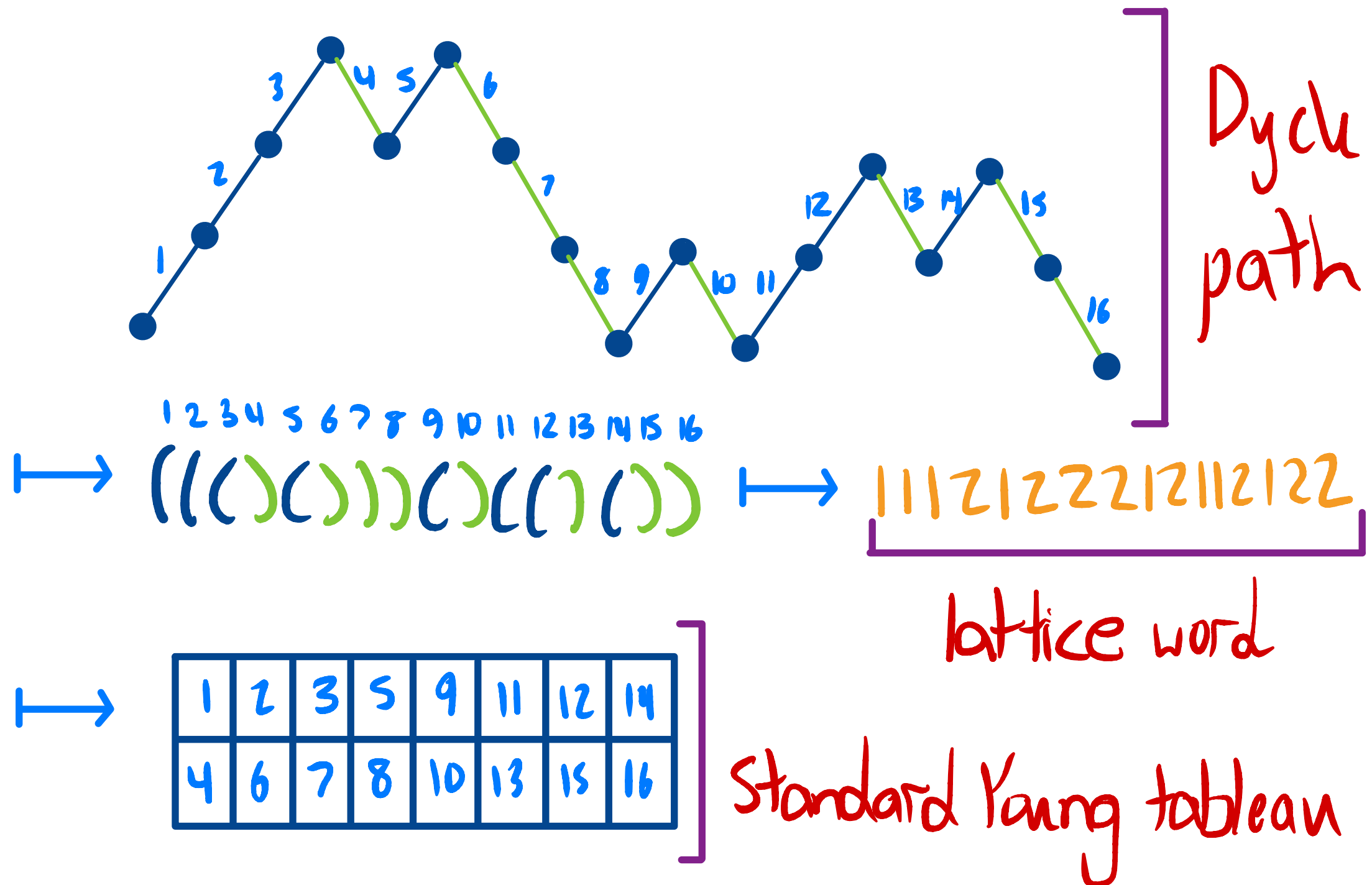
Presented at UC Davis
Combinatorics Seminar
May 1st, 2023

Outline

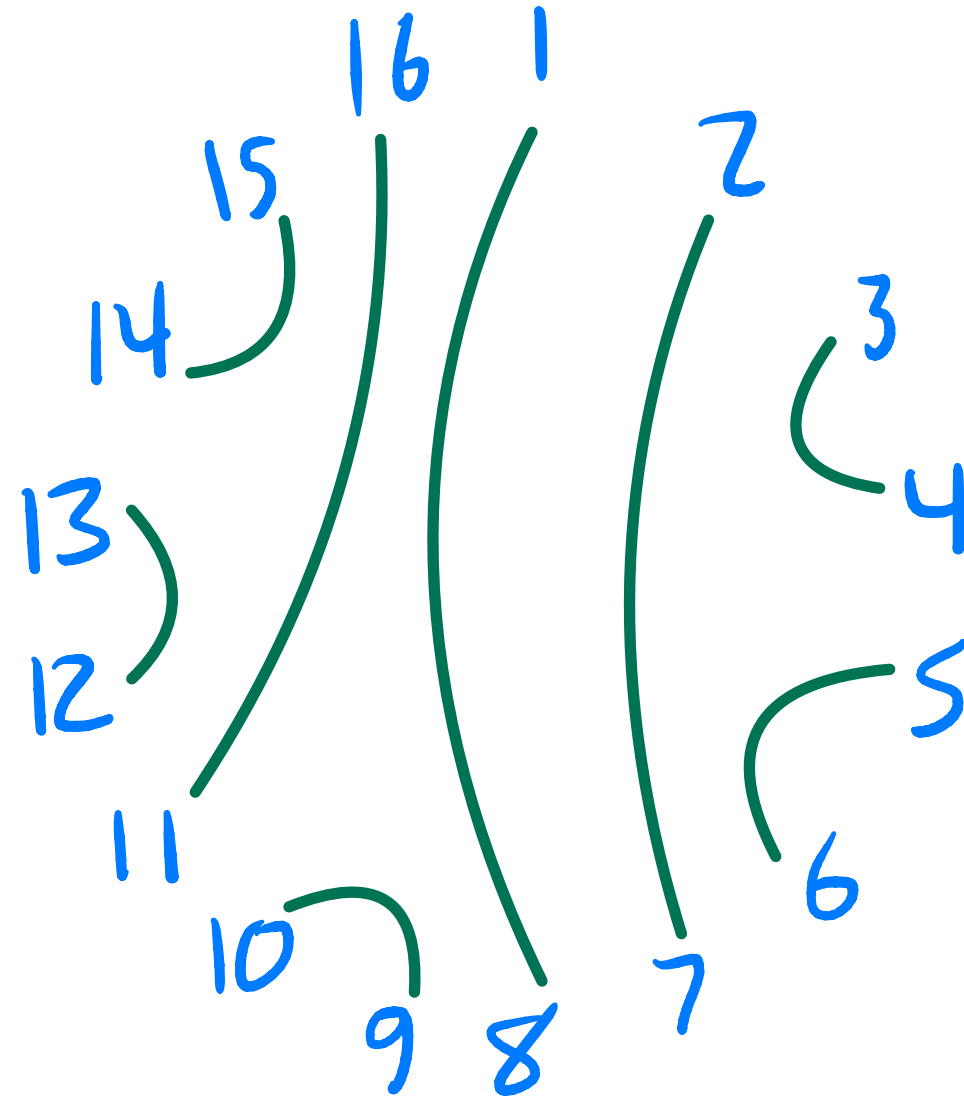
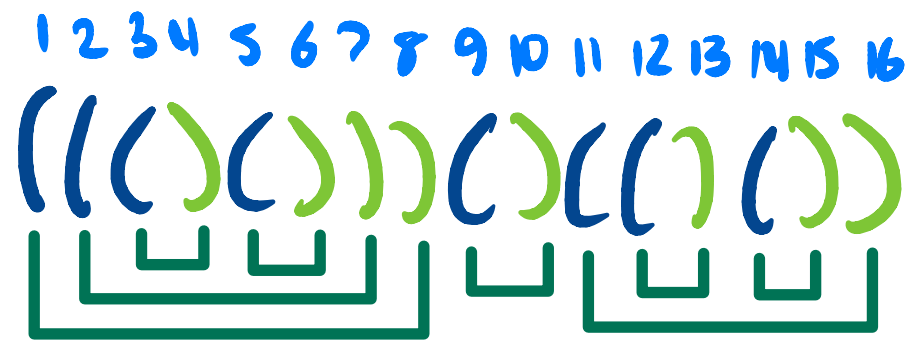
- Catalan combinatorics, 2-row webs
- 3-row web bases, plabic graphs, growth rules
- (New!) 4-row web bases, hourglass plabic graphs

Catalan objects

Some Catalan bijections:



Catalan objects



Non-crossing
matching
("spiny
picture"!!)

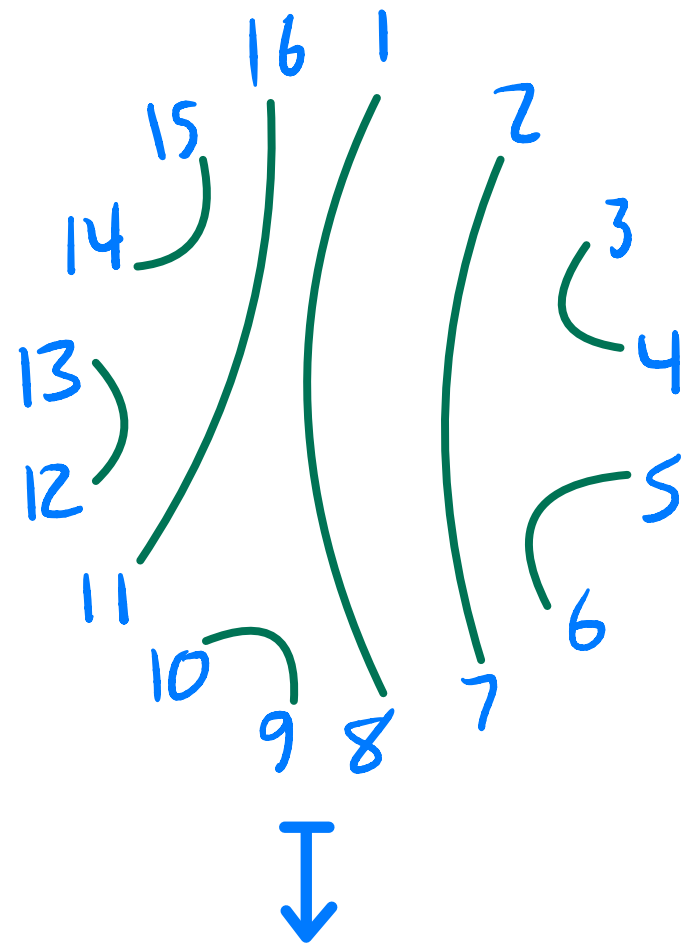
Catalan objects

Thm The bijection $NLM(2n) \xrightarrow{\sim} SYT(2 \times n)$
sends rotation to promotion
reflection to evacuation.

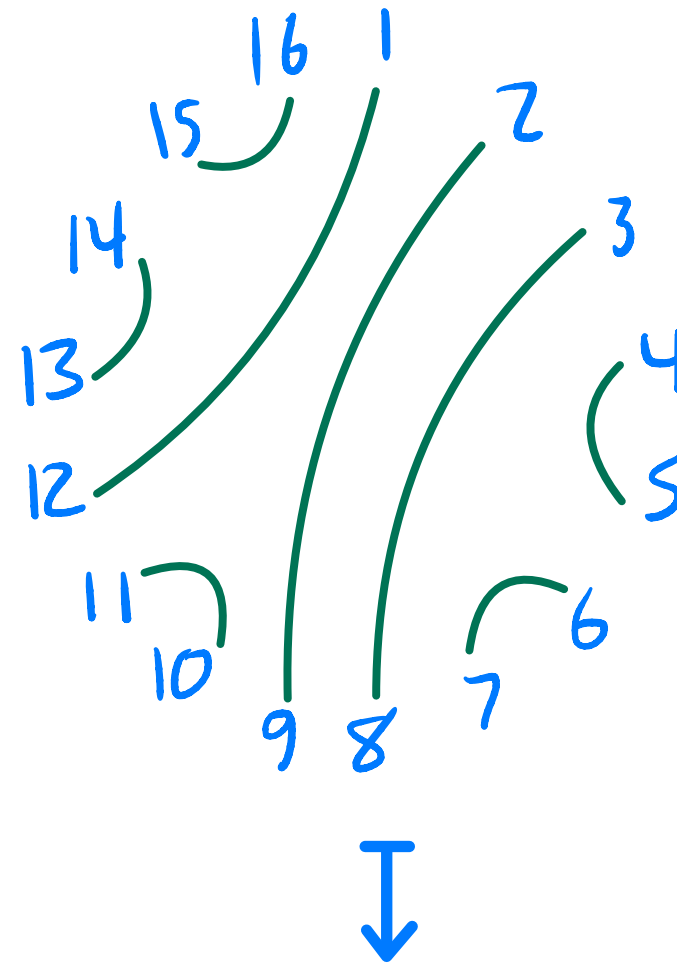
- "Hidden" dihedral action on $SYT(2 \times n)$!

Catalan objects

Ex



Rotate
→



1	2	3	5	9	11	12	14
4	6	7	8	10	13	15	16

Promote
→

1	2	3	4	6	10	13	15
5	7	8	9	11	12	14	16

SL_2 -Invariants

Let $V = \mathbb{C}^2$, $V_i \in \{V, V^*\}$.

Q What are the SL_2 -invariants of $V_1 \otimes \dots \otimes V_n$?

That is, identify $\text{Hom}_{SL_2}(V_1 \otimes \dots \otimes V_n, \mathbb{C}) \subset \mathbb{C}[x_{ij}, y_{kl}]$.

Ex $V \otimes V$: $\det \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = x_{11}x_{22} - x_{12}x_{21}$ is unique invariant

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \det(g \cdot X) = \det(g) \det(X) = \det(X)$$

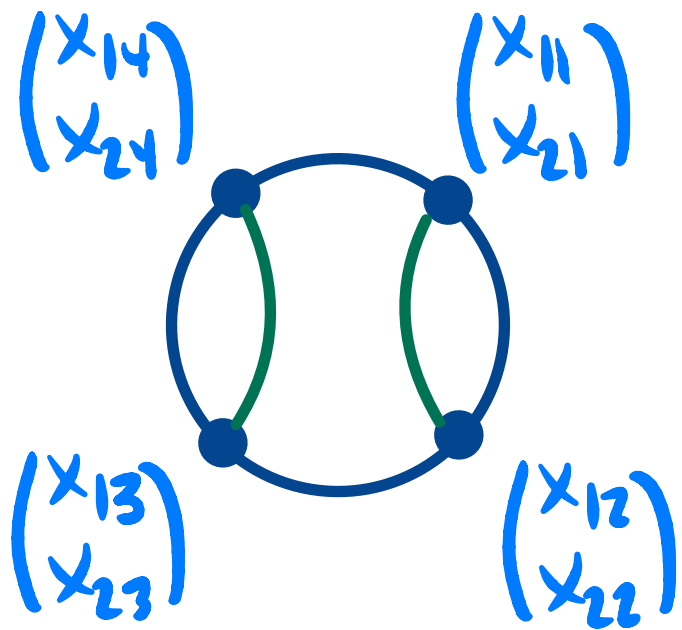
Ex $V^* \otimes V$: pairing $\langle y, x \rangle = x_1 y_1 + x_2 y_2$ is invariant

$$\begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \langle y \bar{g}^t, g x \rangle = \langle y, x \rangle$$

SL_2 -Tensor diagrams

- Encode morphisms/invariants in diagrams: $V \leftrightarrow \bullet$
 $V^* \leftrightarrow \circ$

Ex

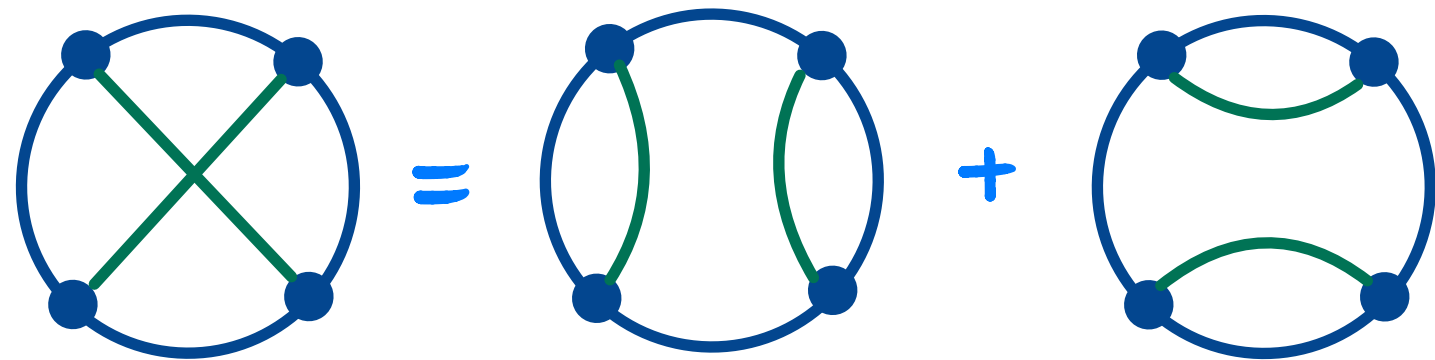


$$= \det \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \cdot \det \begin{pmatrix} x_{13} & x_{14} \\ x_{23} & x_{24} \end{pmatrix} \in \text{Inv}(V^{\otimes 4})$$

SL_2 -Tensor diagrams

- Encode morphisms/invariants in diagrams: $V \leftrightarrow \bullet$
 $V^* \leftrightarrow \circ$

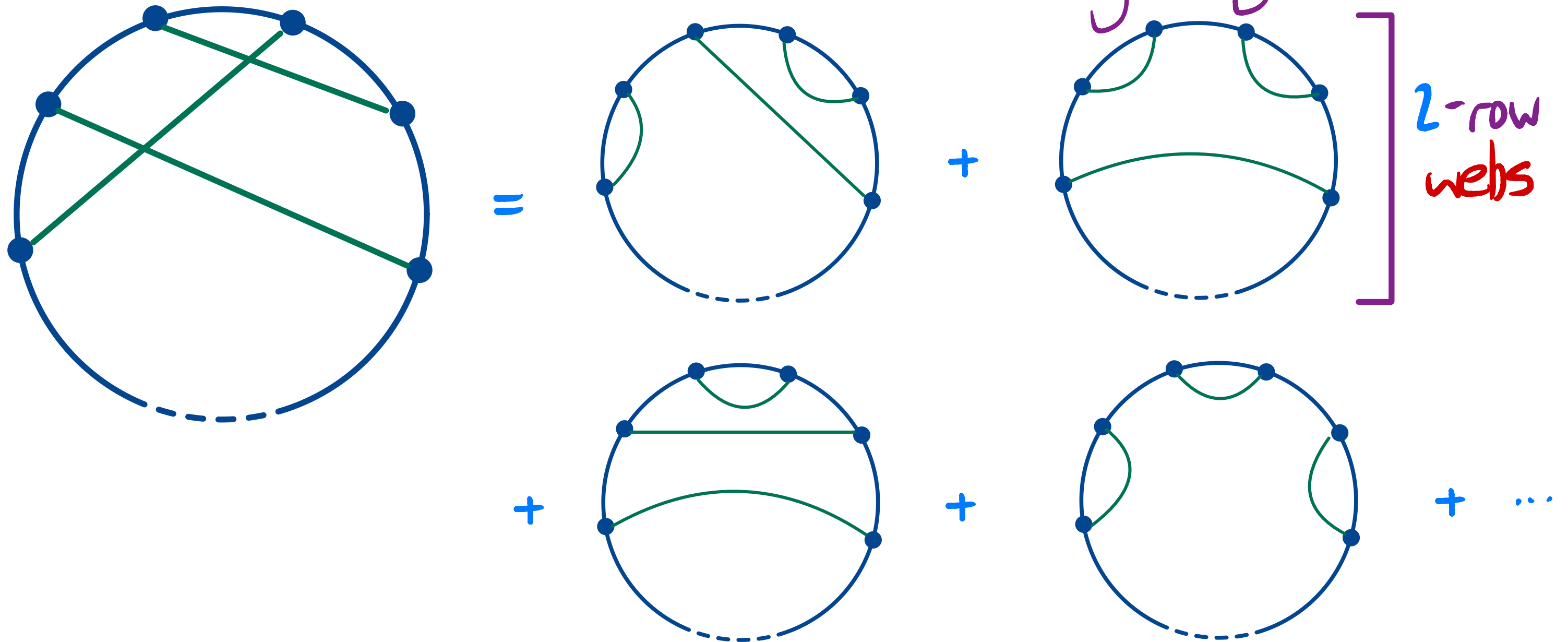
Ex] Plücker relations:



$$\begin{aligned}
 & (x_{11}x_{23} - x_{21}x_{13})(x_{12}x_{24} - x_{12}x_{14}) \\
 & = \\
 & (x_{11}x_{22} - x_{21}x_{12})(x_{13}x_{24} - x_{23}x_{14}) \\
 & + \\
 & (x_{11}x_{24} - x_{21}x_{14})(x_{12}x_{23} - x_{22}x_{13})
 \end{aligned}$$

Temperley-Lieb basis

- Using $\text{cross} = \text{cup} + \text{cap}$, can reduce any matching diagram to a linear combination of matching diagrams:



Temperley-Lieb basis

Thm The noncrossing 2-row webs are a basis for

$$\text{Inv}_{\mathfrak{sl}_2}(V_1 \otimes \dots \otimes V_n)$$

$$(V_i \in \{V, V^*\})$$

PF • Spanning: diagrams span by classical invariant theory,
noncrossing by uncrossing rule.

• Independence: by Pieri rule,

$$\dim \text{Inv}_{\mathfrak{sl}_2}(V^n) = \#\text{SYT}(2 \times \frac{n}{2})$$

Quantum Link Invariants

The quantum group $U_q(\mathfrak{sl}_2)$ is an algebra deforming the universal enveloping algebra of \mathfrak{sl}_2 .

- Sending $q \rightarrow 1$ recovers the usual case.
- $\text{Inv}(V_1 \otimes \dots \otimes V_n)$ still has a Temperley-Lieb web basis:

The diagram shows an equation between three web configurations. On the left is a crossing of two strands. On the right is the sum of two configurations: the first has two separate arcs, and the second has two parallel arcs. The coefficients are $q^{-1/4}$ and $q^{1/4}$ respectively.

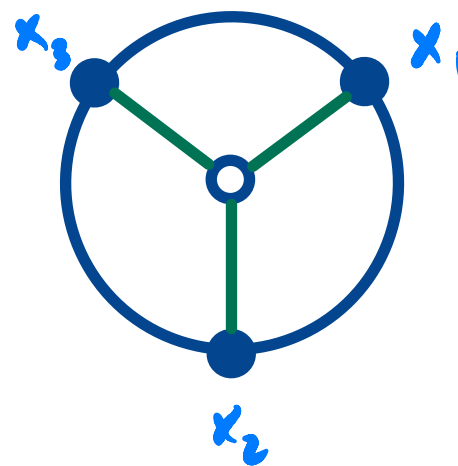
$$\text{Crossing} = q^{-1/4} \text{Two Arcs} + q^{1/4} \text{Parallel Arcs}$$

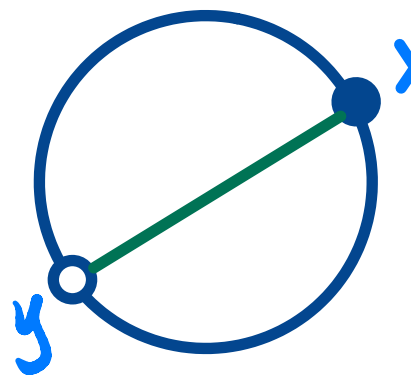
- Projections of knots/links/tangles become polynomials in q ; refined version of Jones polynomial

SL_3 -Tensor diagrams

• Let $V = \mathbb{C}^3$, $V_i \in \{V, V^*\}$.

Q Is there a web basis for $\text{Inv}_{SL_3}(V_1 \otimes \dots \otimes V_n)$?

Ex  = $\det \begin{pmatrix} | & | & | \\ x_1 & x_2 & x_3 \\ | & | & | \end{pmatrix}$

 = $\langle y, x \rangle$

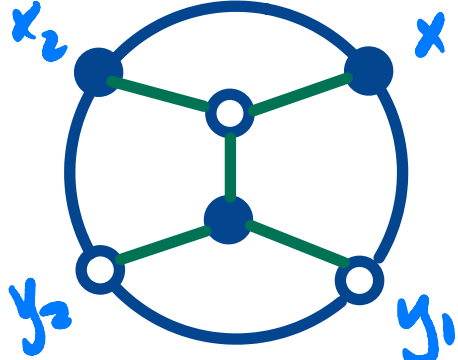
There's more! \longrightarrow

SL_3 -Tensor diagrams

• Let $V = \mathbb{C}^3$, $V_i \in \{V, V^*\}$.

Q Is there a web basis for $\text{Inv}_{SL_3}(V_1 \otimes \dots \otimes V_n)$?

Note $\Lambda^2 V \cong V^*$ since $\Lambda^3 V = \det V = 1$ over SL_3 .
 $\Lambda^2 V^* \cong V$

Ex  = $\det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$

SL_3 -Tensor diagrams

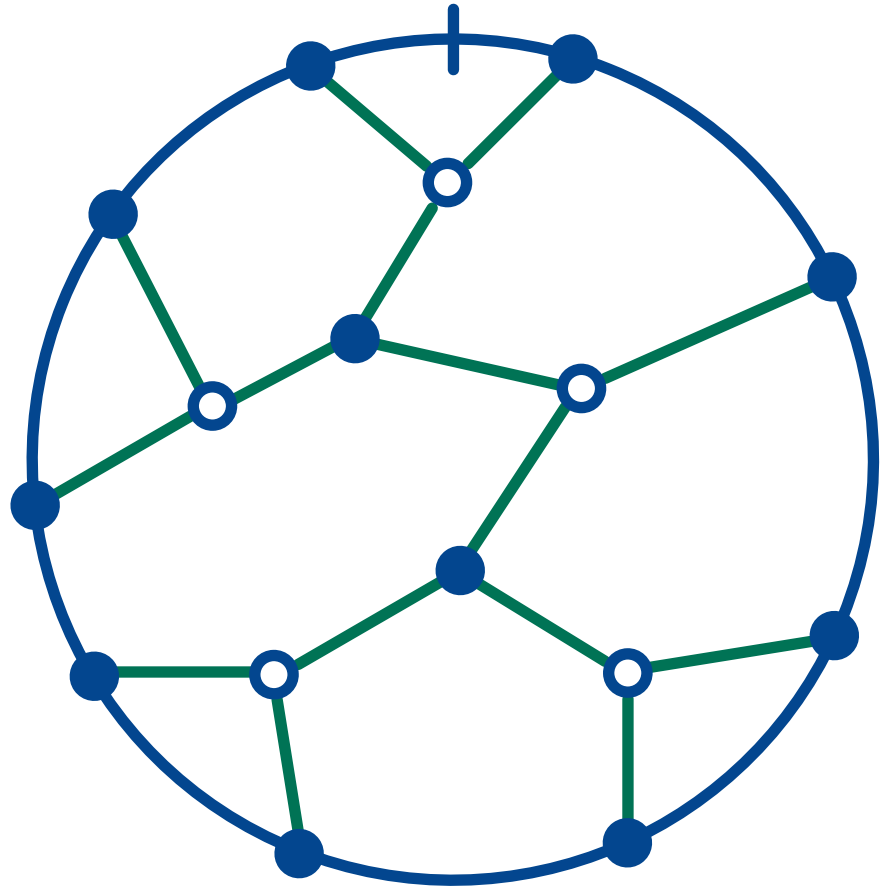
Def 3-row webs are

- planar graphs embedded in a disk
- trivalent interior vertices,
univalent boundary vertices
- bipartite.
- marked "initial" boundary vertex.

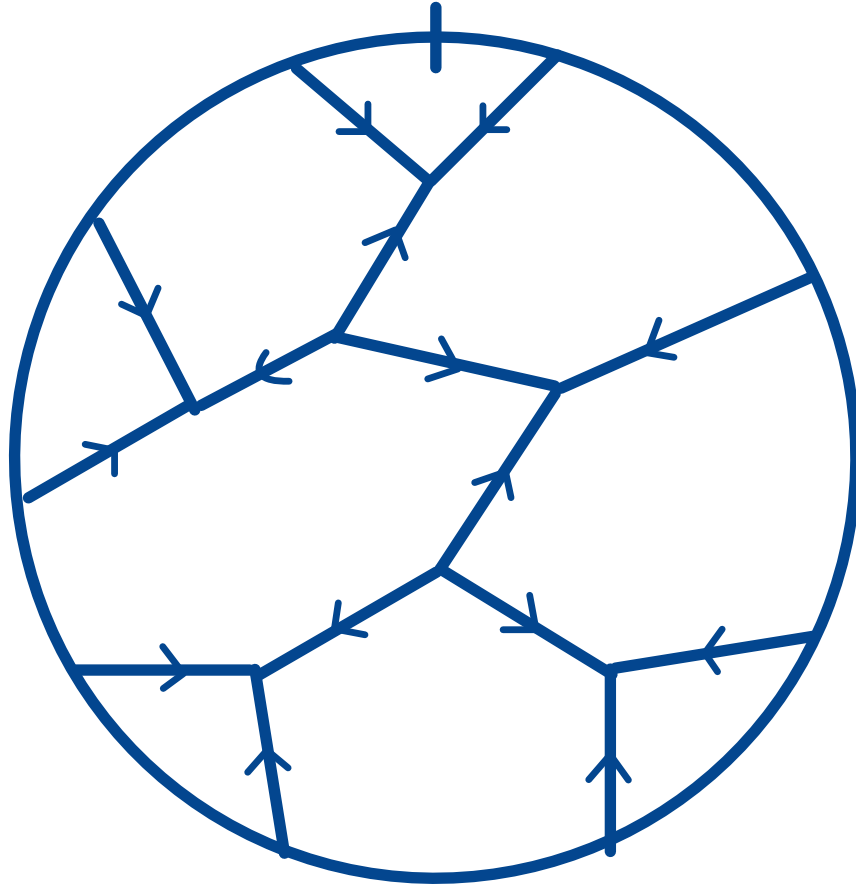
They encode $\text{Inv}_{SL_3}(V_1 \otimes \dots \otimes V_n)$.

SL_3 -Tensor diagrams

Ex



=



Spinning picture!

SL_3 -Web basis

Thm (Kuperberg) The generating SL_3 -web relations are

$$\bigcirc = 3$$

$$\text{---} \circ \text{---} = 2 \cdot \text{---}$$

$$\begin{array}{c} \diagup \quad \diagdown \\ \circ \quad \bullet \\ \text{---} \quad \text{---} \\ \bullet \quad \circ \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \text{---} \\ \cup \\ \text{---} \\ + \\ \text{---} \\ \cap \\ \text{---} \end{array}$$

SL_3 -Web basis

Thm (Kuperberg; Kuperberg-Khovanov)

Call an SL_3 -web non-elliptic if it has no internal 2-faces or 4-faces. The nonelliptic webs form a basis of

$$\text{Im}_{SL_3}(V_1 \otimes \dots \otimes V_n).$$

$$(V_i \in \{V, V^*\})$$

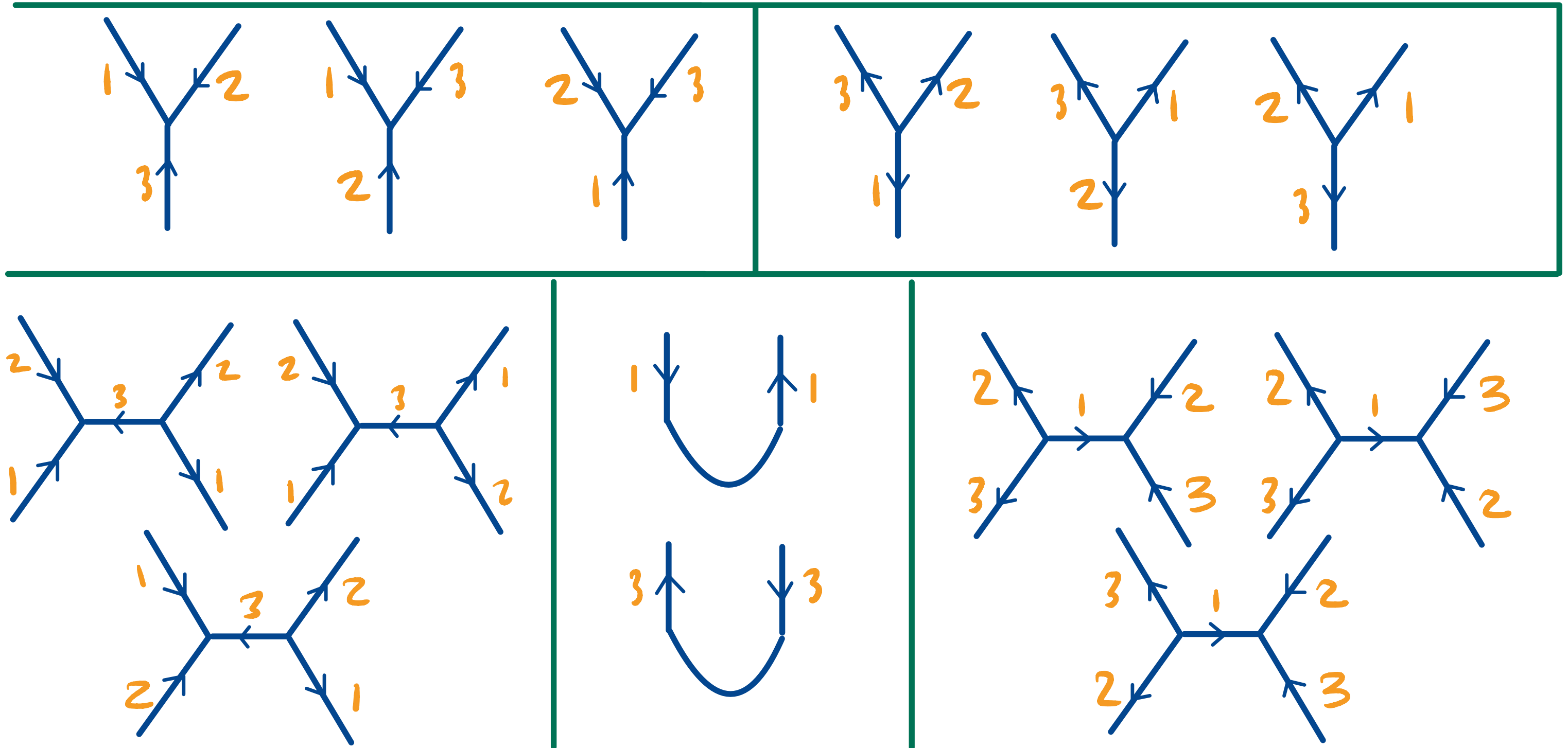
SL_3 -Web basis

PF | • Spanning: similar to SL_2 case.

• Independence: bijection to $SYT(3 \times \frac{n}{3})$ using growth rules. (Other descriptions of this bijection have since been found.)

SL_3 -Web basis

Kuperberg-Khorrami growth rules:



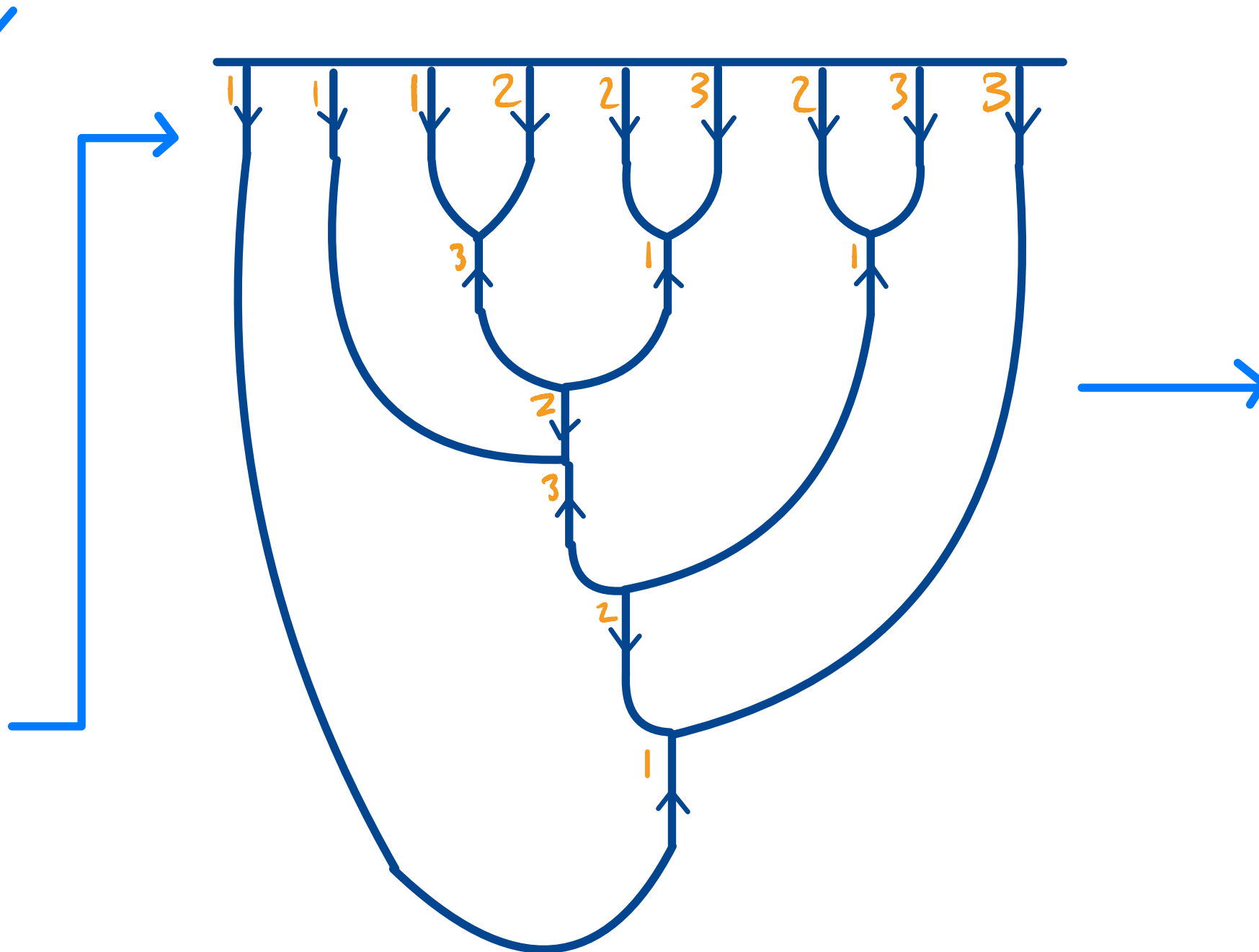
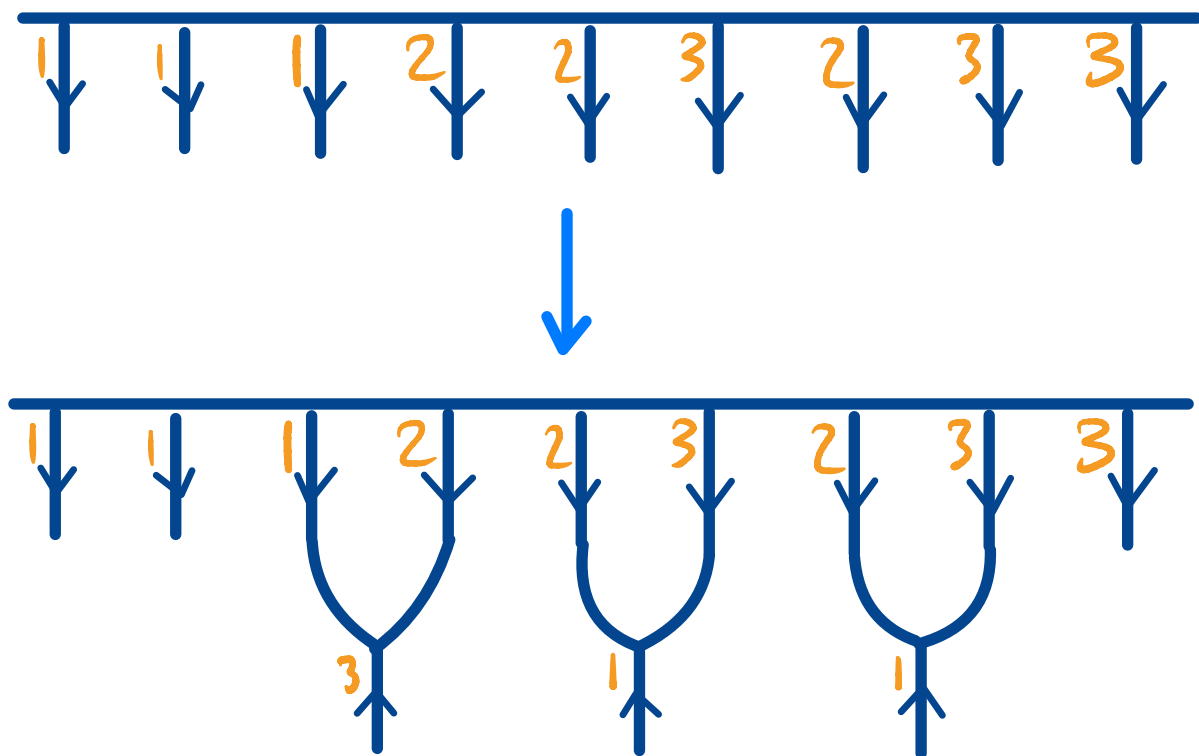
SL_3 -Web basis

Ex

$T =$

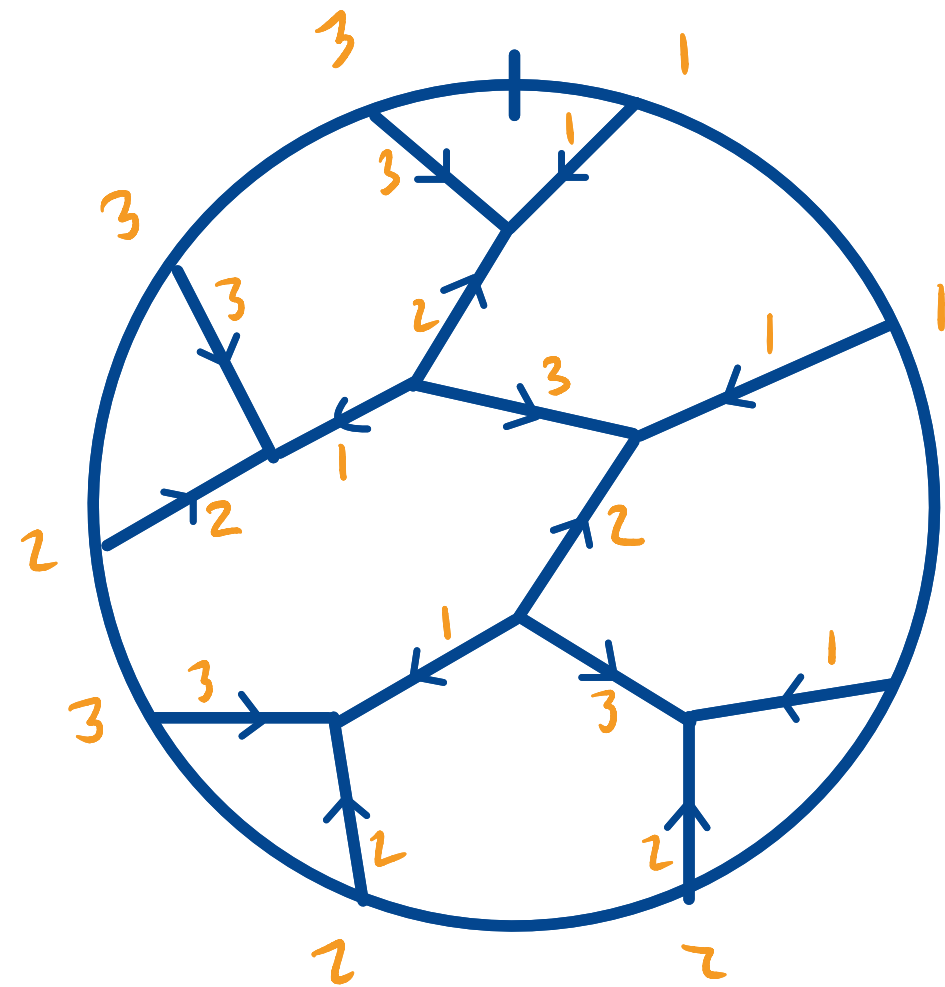
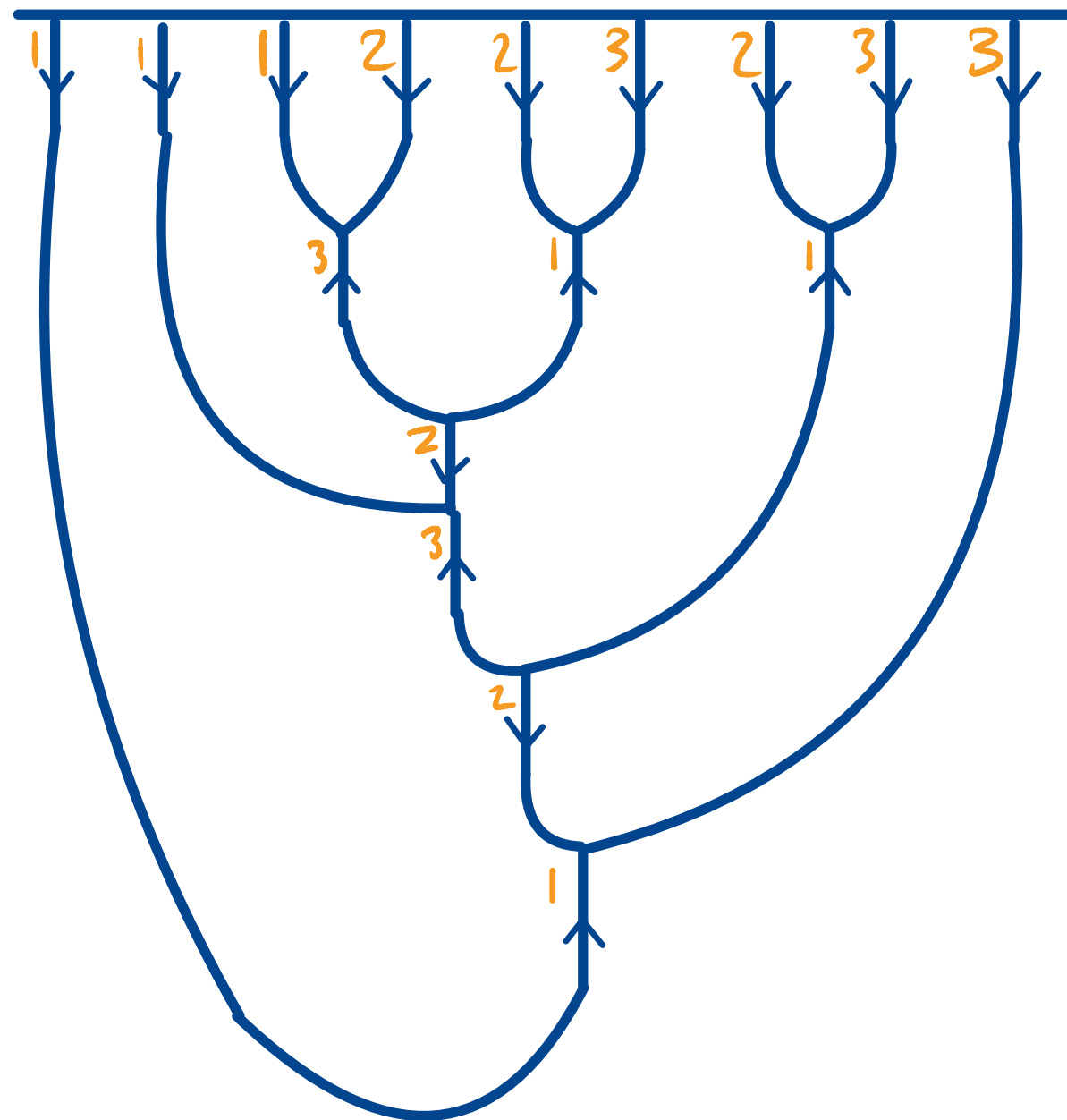
1	2	3
4	5	7
6	8	9

$\rightarrow 111223233$



SL_3 -Web basis

$$(\tau \rightarrow 111223233)$$



Now just erase labels!

SL_3 -Web basis

Thm | (Petersen-Pilyavskyy-Rhoades)

The bijection from non-elliptic webs to $SYT(3 \times \frac{n}{3})$
sends rotation to promotion
reflection to evacuation.

- "Hidden" dihedral action on $SYT(3 \times m)$!

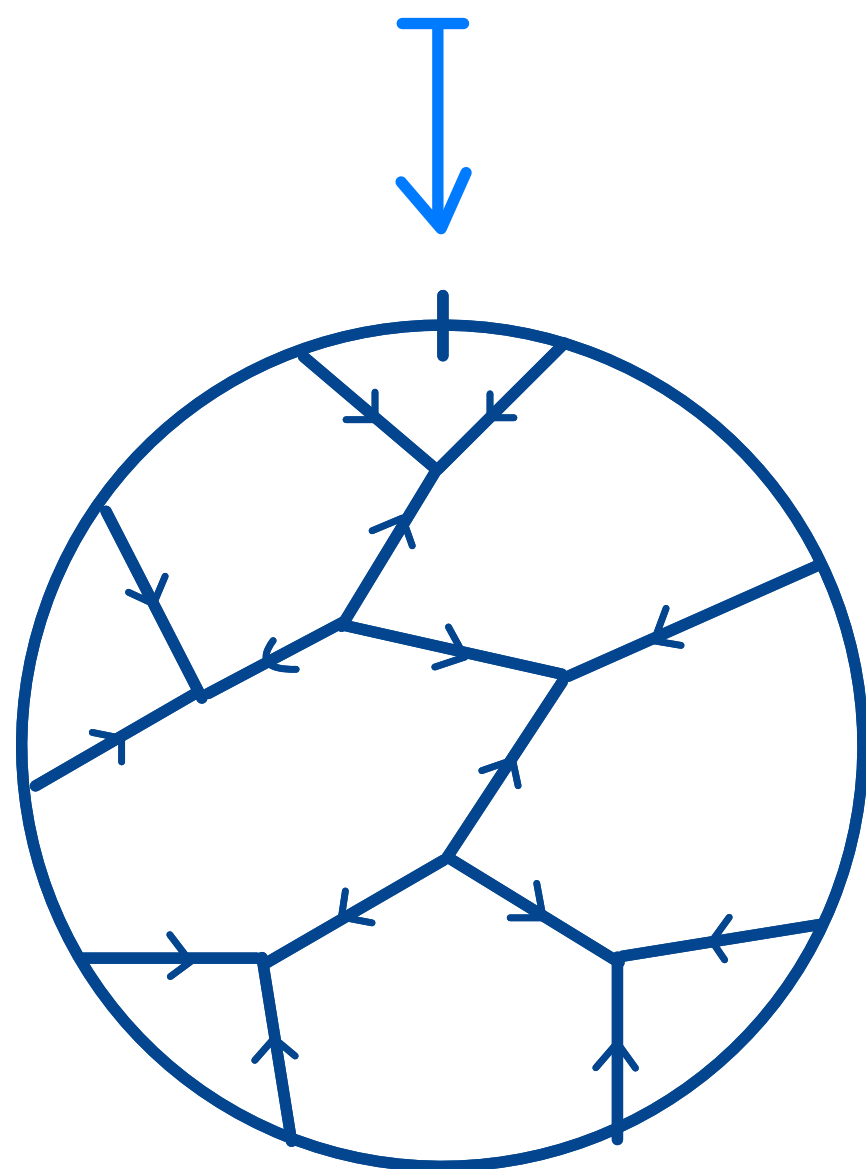
SL_3 -Web basis

Ex

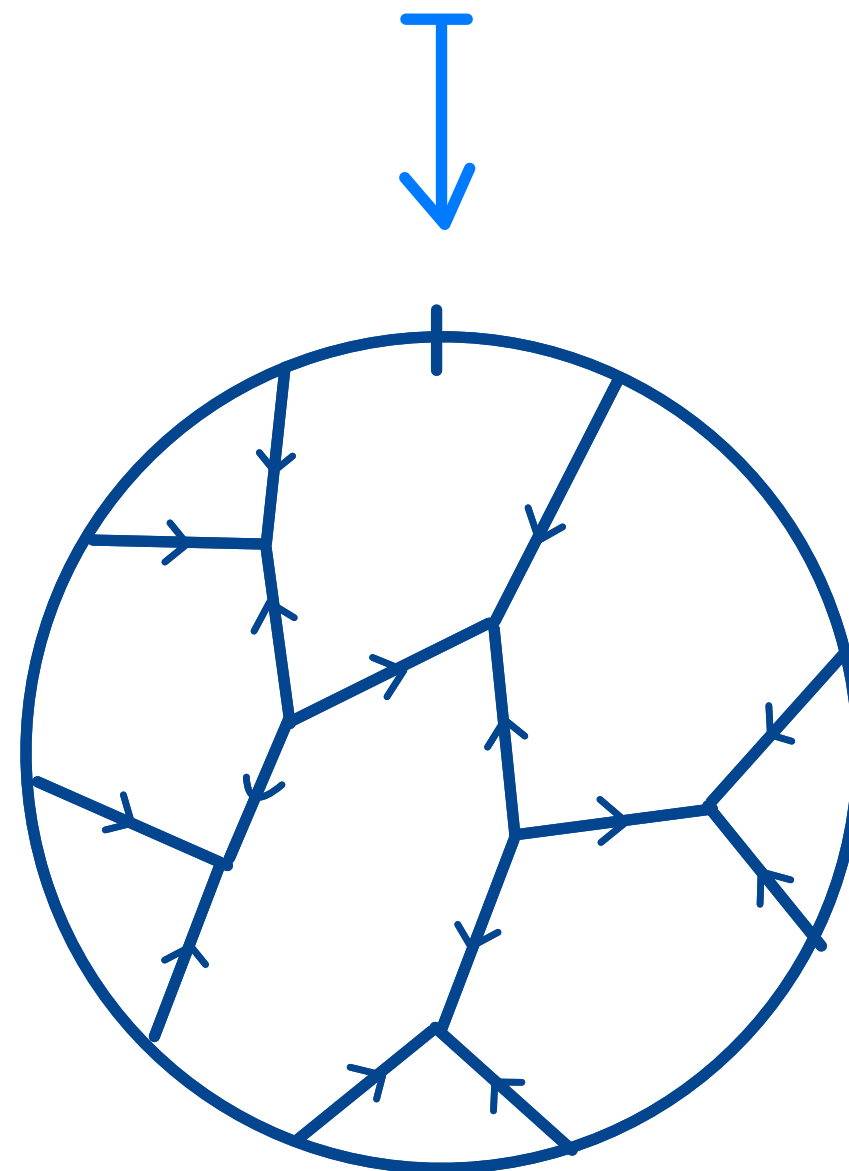
1	2	3
4	5	7
6	8	9

Prom
→

1	2	6
3	4	8
5	7	9



Rotation
→



SL_3 -Web basis

Applications

- Quantum link invariants
 - SL_3 link polynomials, foams, ...
- Cluster algebras
 - Cluster structures on $\mathbb{C}[x_{ij}, y_{kl}]^{SL_3}$
- Enumerative combinatorics
 - promotion, evacuation, cyclic sieving
- Dimer models
- Representation theory

The web basis problem

Problem (Khovanov-Kuperberg '96)

Give a web basis* for $\text{In}_{\text{SL}_r}(V_1 \otimes \cdots \otimes V_n)$ for $r \geq 4$.

*with desirable properties for use in applications:

— testability

— reduction rules

— rotation invariance

The web basis problem

Next: our solution for $r=4$!

- Unifies $r=2,3,4$
- Many pieces work for general r (TBD!)
- Introduces new planar graphs with multiple trips
- Combinatorially beautiful,
e.g. connects ASM's and PP's in a certain sense

Plabic graphs

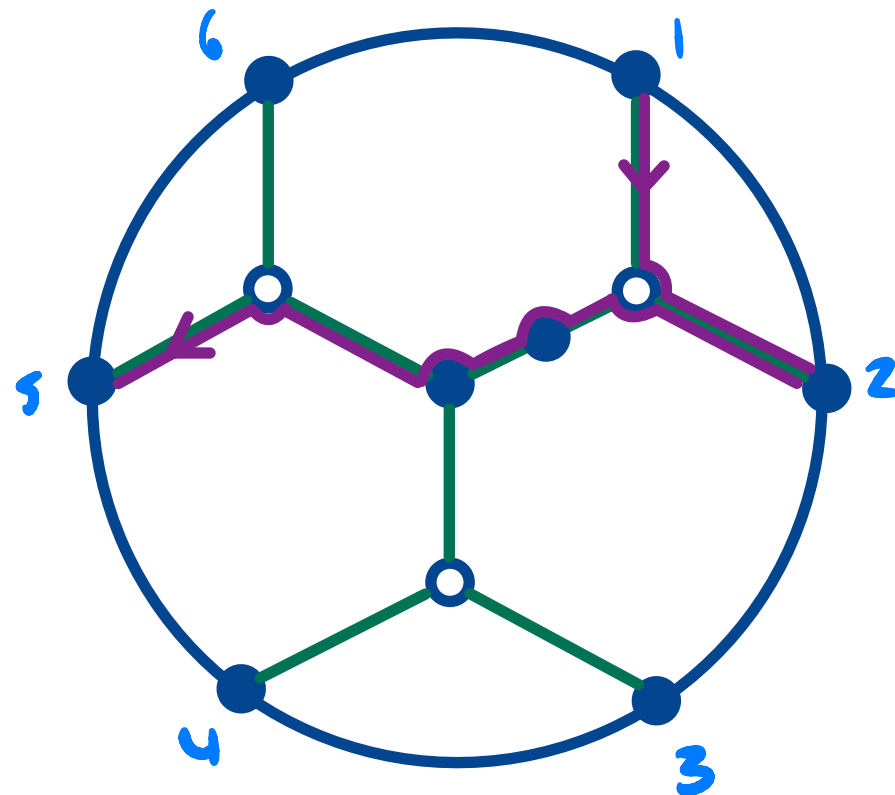
- Postnikov introduced plabic graphs to parametrize cells in $\mathcal{G}_{2,0}(k,n)$. "planar bicolored"
- They have trip permutations:

Rules of the road

Left at ○

Right at ●

Ex



$\{1, \dots, 6\}$

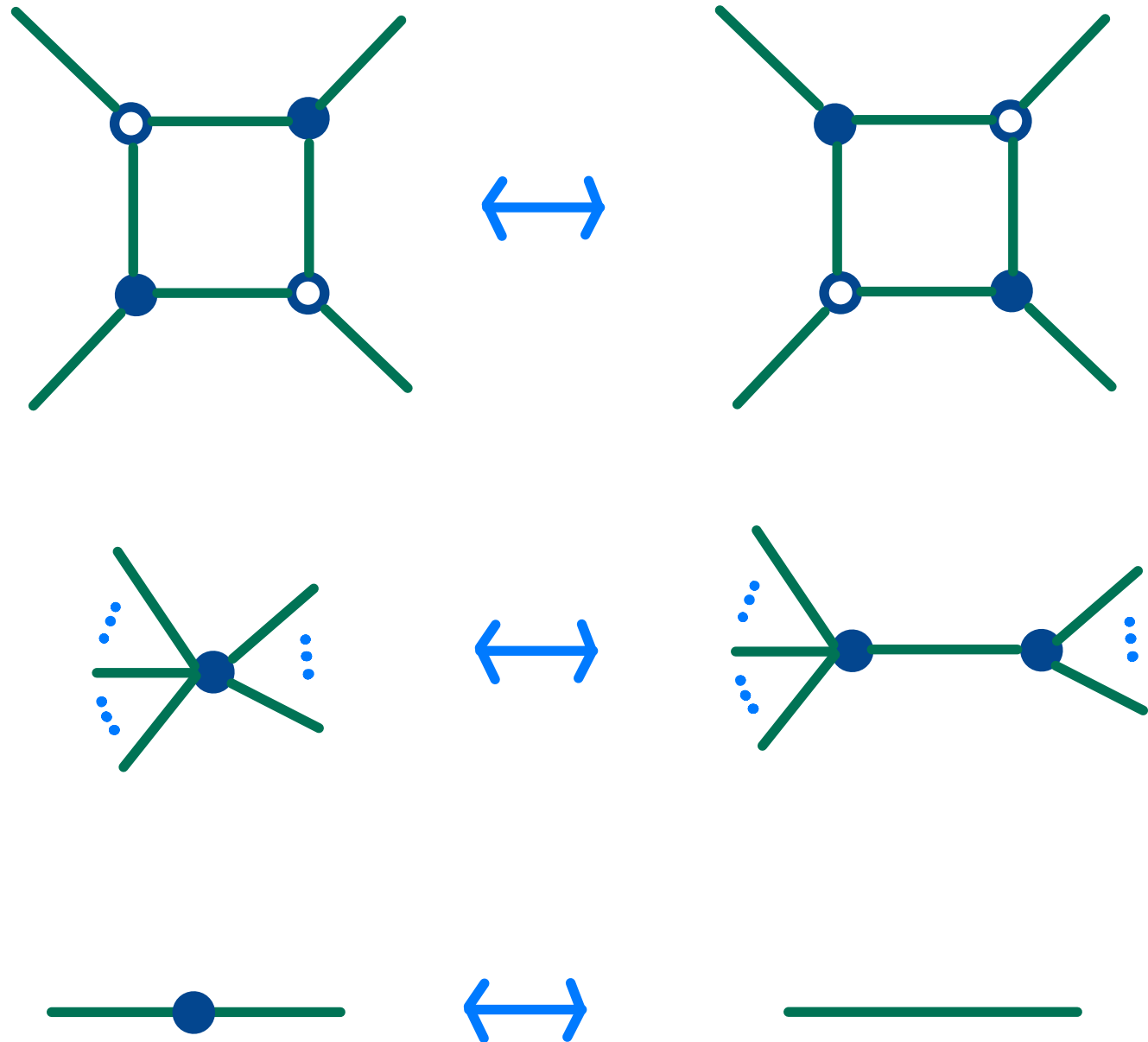
$1 \mapsto 2$

$2 \mapsto 5$

Trip = (1 2 5 6 3 4)

Plabic graphs

- Moves preserve Trip:



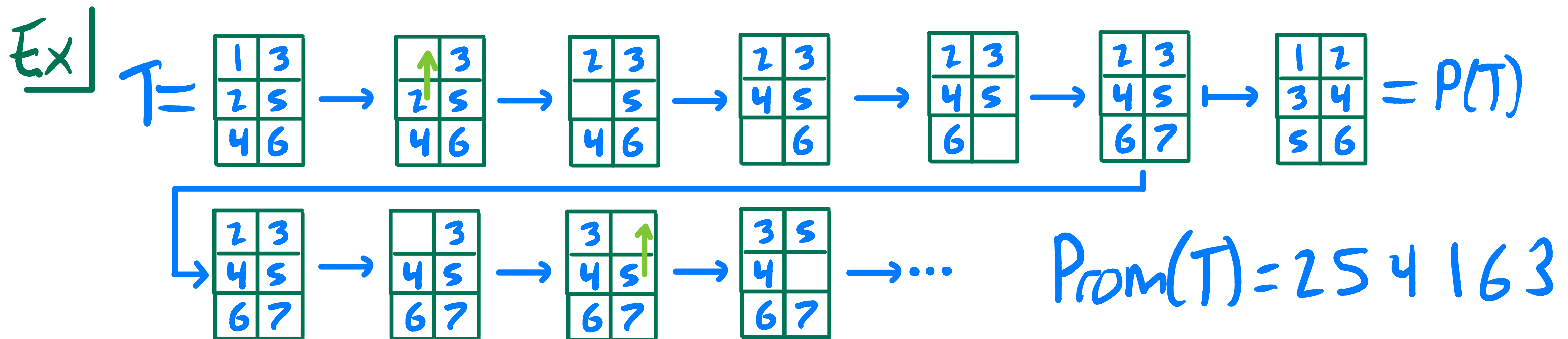
Thm (Postnikov)

Two reduced plabic graphs have the same Trip if and only if they are connected by a sequence of moves.

Promotion permutations

Obs (Hopkins-Rubey) 3-row basis webs are reduced plabic graphs. What are their Trip's?

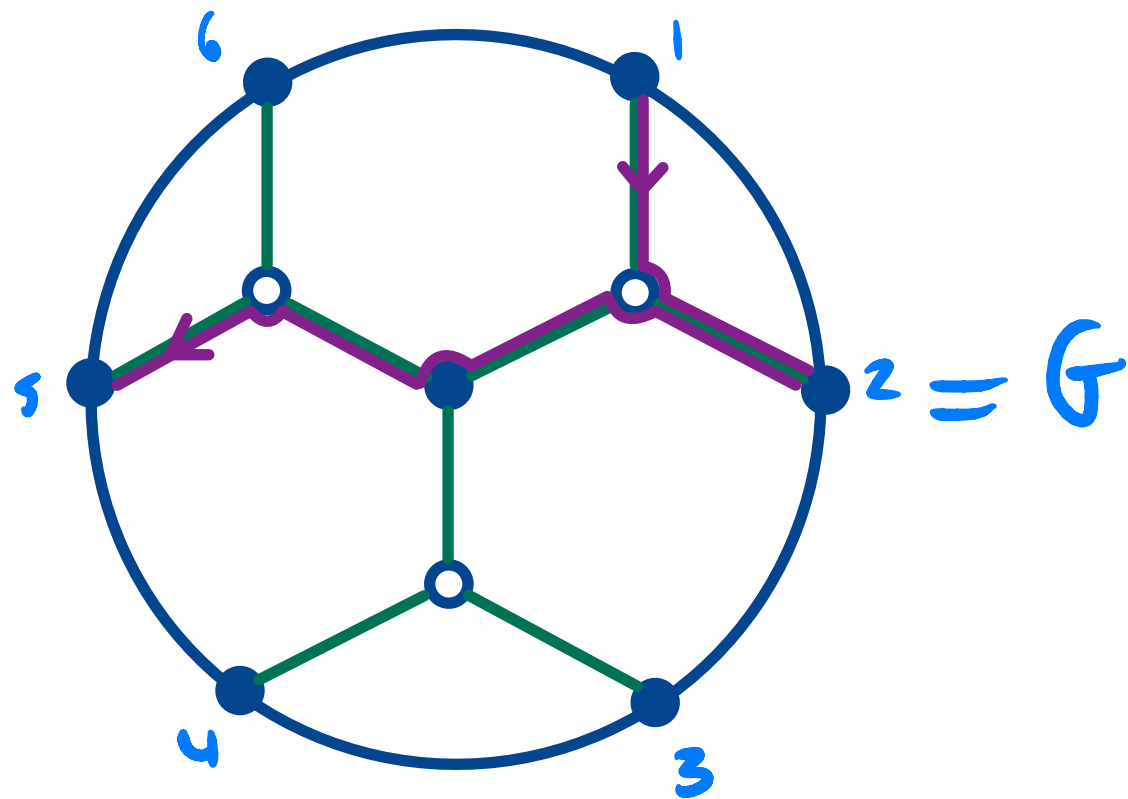
Def The promotion permutation of $TESPT(3 \times \frac{n}{3})$ trades what enters the top row when computing $P^n(T)$:



Promotion and trip permutations

Thm (Hopkins-Rubey) The bijection between \mathcal{S}_3 basis webs and $\text{SRT}(3 \times \frac{n}{3})$ sends Trip to Prom.

Ex

$$T = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline \end{array}$$


$$\text{Prom}(T) = 254163 = (125634) = \text{Trip}(G)$$

Promotion matrices

• $\text{prom}: \text{SVT}(3 \times \frac{n}{3}) \rightarrow S_n$ is injective, but

$\text{prom}: \text{SVT}(4 \times \frac{n}{4}) \rightarrow S_n$ is not.

• Fix: let prom_i record which number enters row i

Thm $T \in \text{SVT}(r \times \frac{n}{r})$ is uniquely determined by the sequence of promotion permutations $\text{prom}_1, \dots, \text{prom}_{r-1}$.

Moreover, $\text{prom}_i^{-1} = \text{prom}_{r-i}$.

Multiple trips

- Given a planar graph (or similar), let trip_i take the i th left at \circ
 i th right at \bullet .
- Idea: if r -valent, $\text{trip}_i = \text{trip}_{r-i}$.

Main theorem

Thm $\{ [W] \mid W \text{ is a top fully reduced hourglass plabic graph} \}$

is an SL_4 web basis. Furthermore, we have a bijection

$$SKT(4 \times \frac{n}{4}) \rightarrow TFRHPG(n)$$

$$T \mapsto G$$

$$\text{prom}_i(T) = \text{trip}_i(G)$$

$$(i=1, \dots, r-1)$$



Main theorem

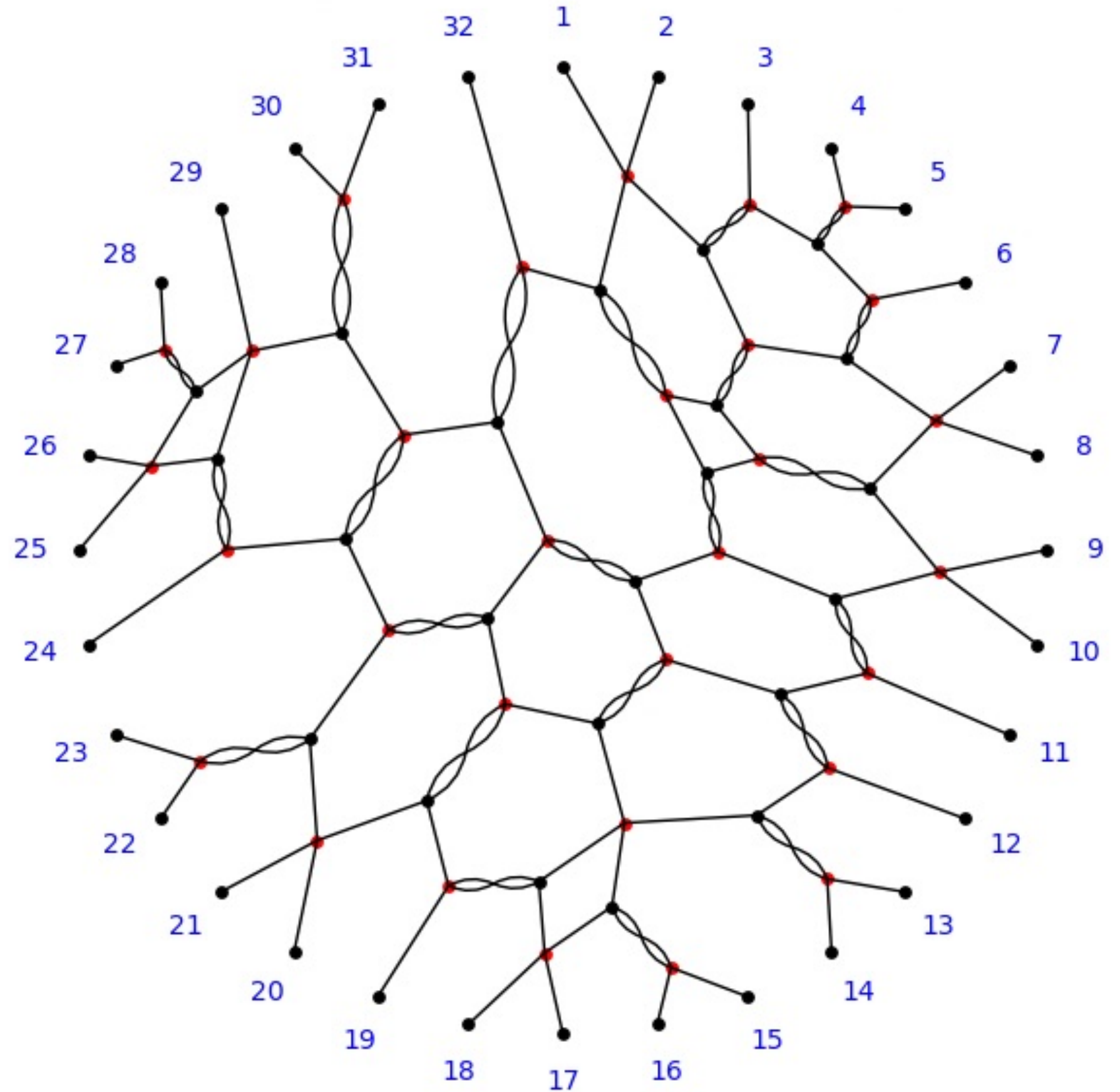
While $[W]$ is a collection of webs, they represent the same tensor invariant.

Hence we have a rotation-invariant basis of tensor invariants, encoded by webs!

4-row webs (new!)

Ex

1	3	4	7	8	17	19	23
2	5	6	9	14	18	21	24
10	12	13	15	16	25	26	28
11	20	22	27	29	30	31	32



4-row webs (new!)

Def | A 4-row web is...

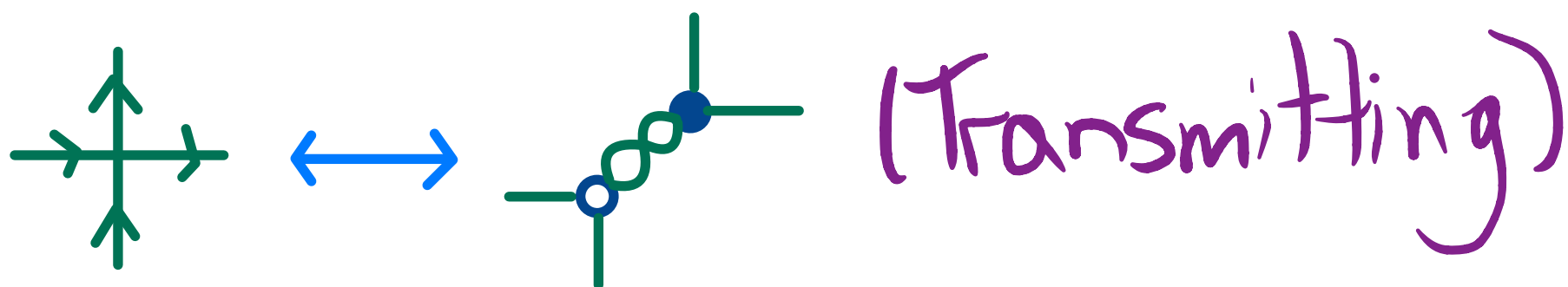
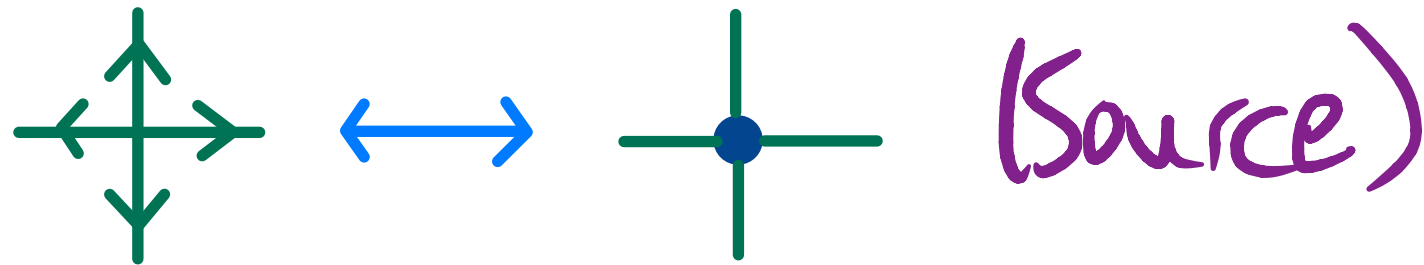
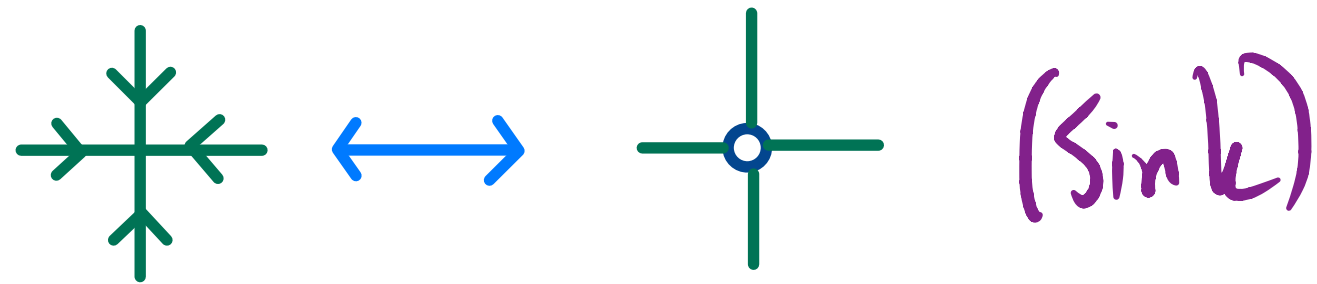
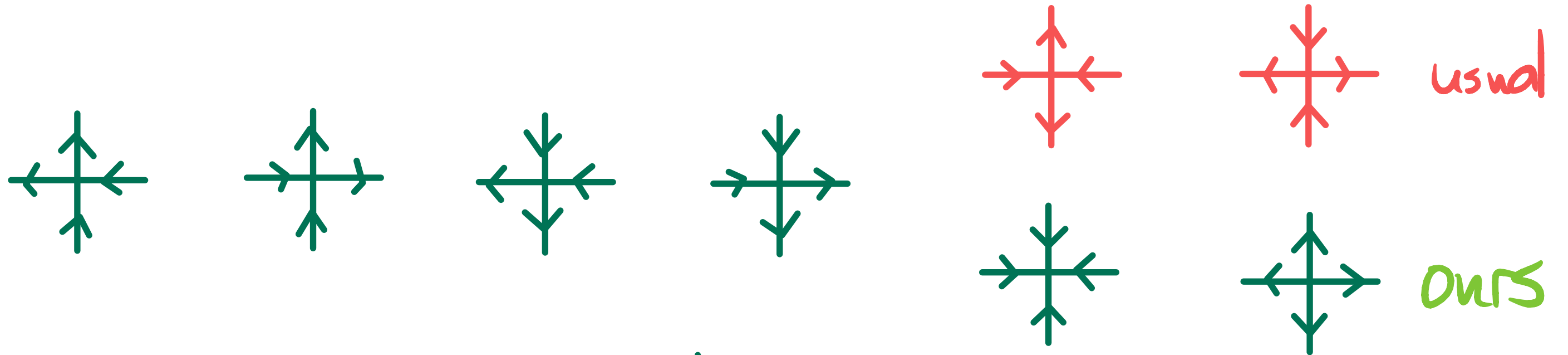
- Planar-embedded graph in disk, allowing

hourglass edges



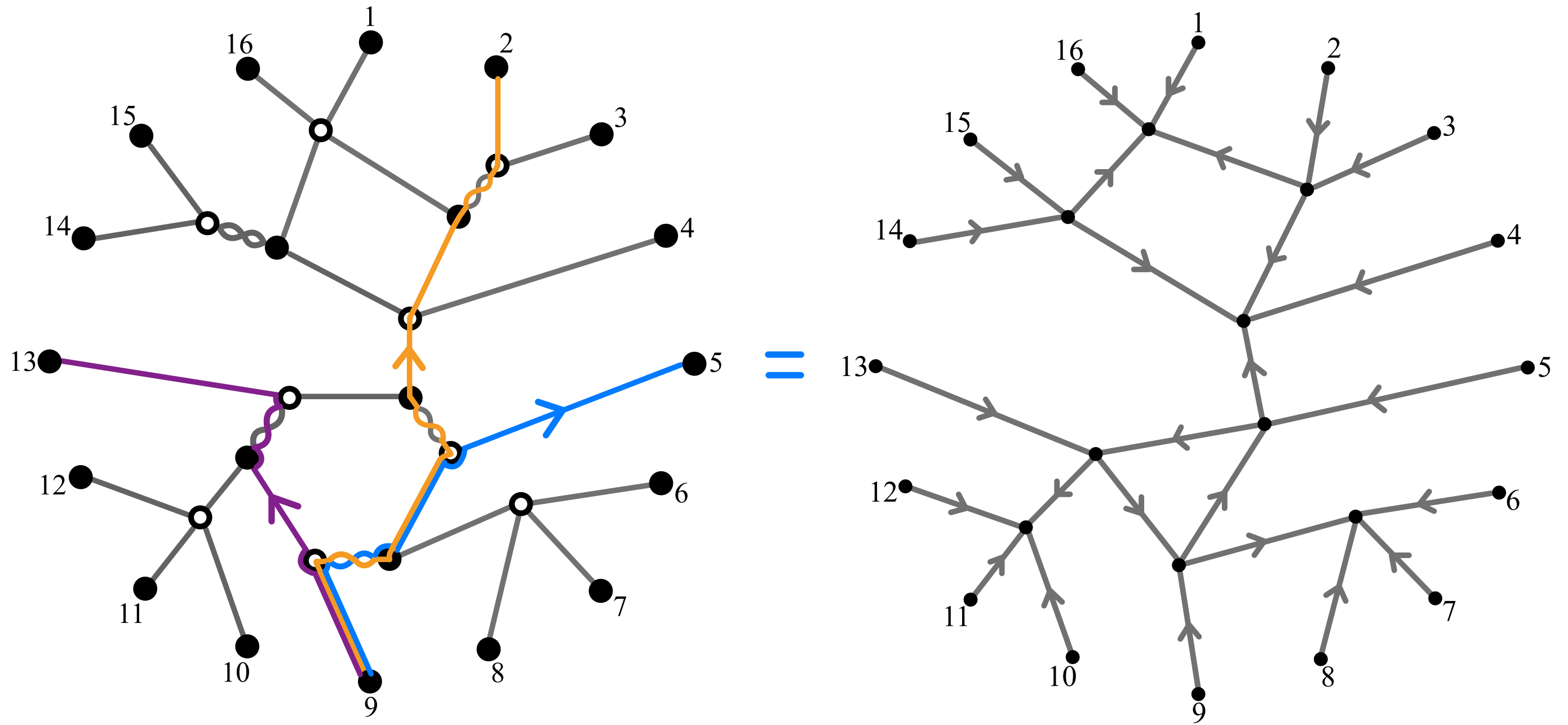
- 4-valent interior vertices, univalent boundary
- Bipartite, marked "initial" outer vertex

6-vertex model



6-vertex model

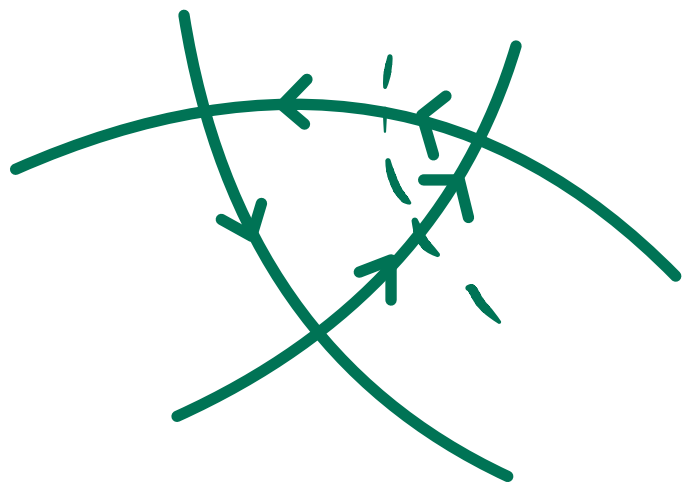
Ex



Top fully reduced

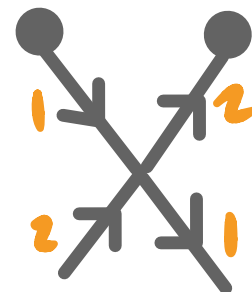
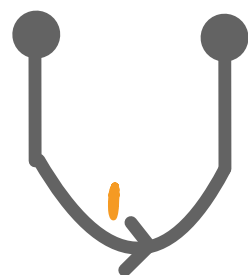
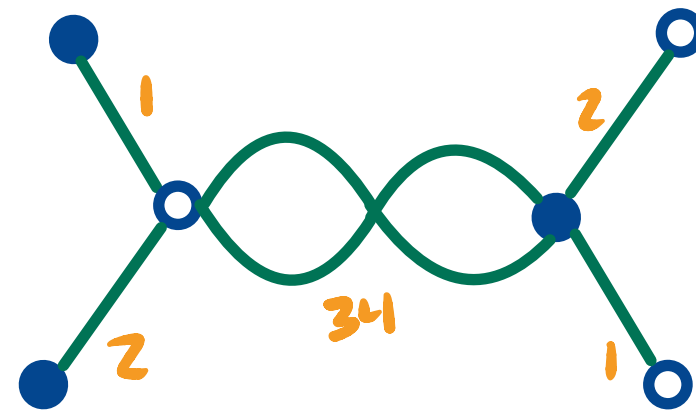
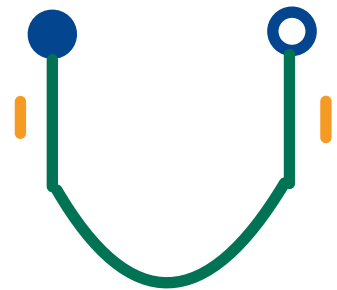
Def A 4-row basis web is top fully reduced if all triangles* in the 6-vertex configuration are oriented counterclockwise.

*even "big" triangles

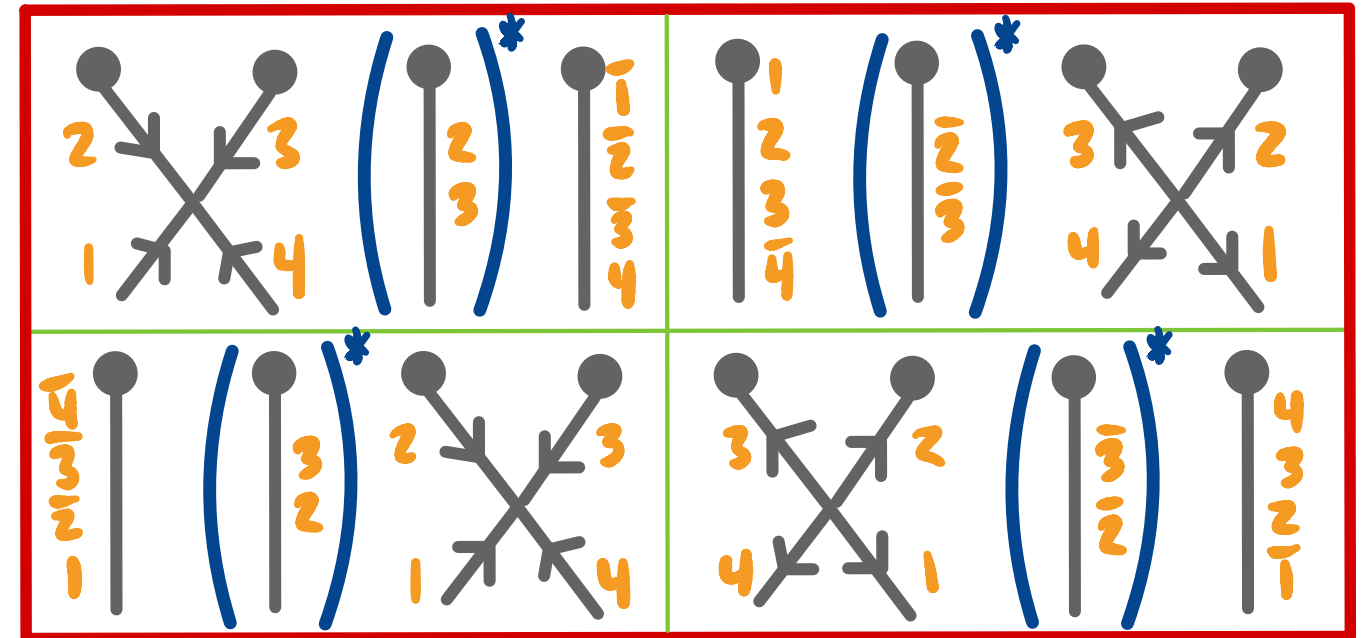
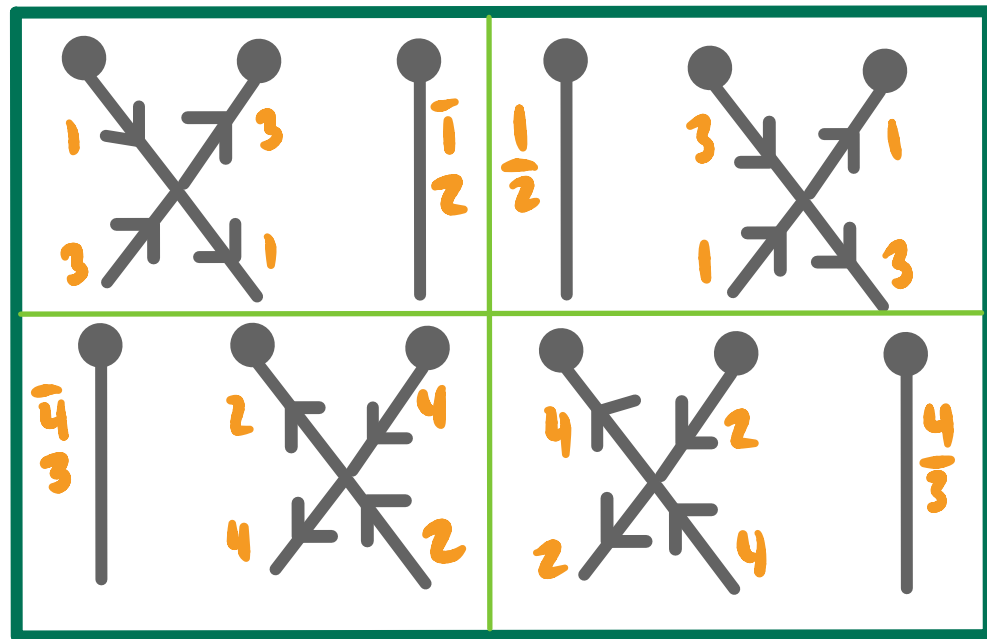
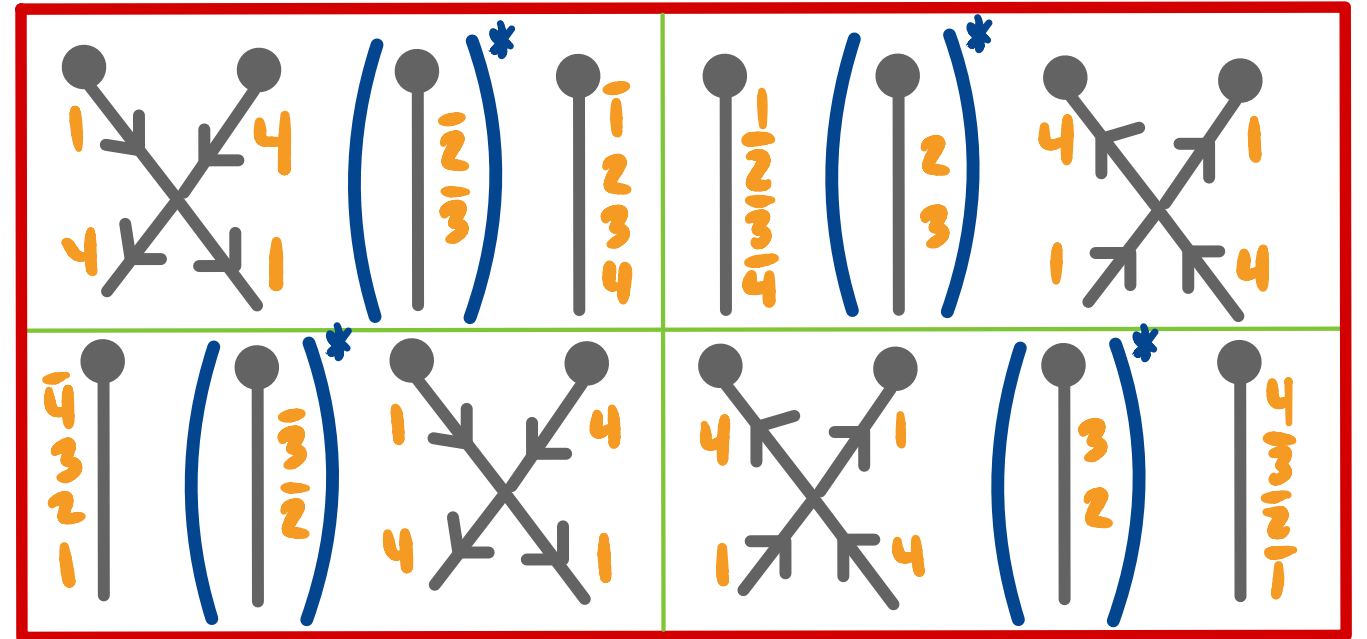
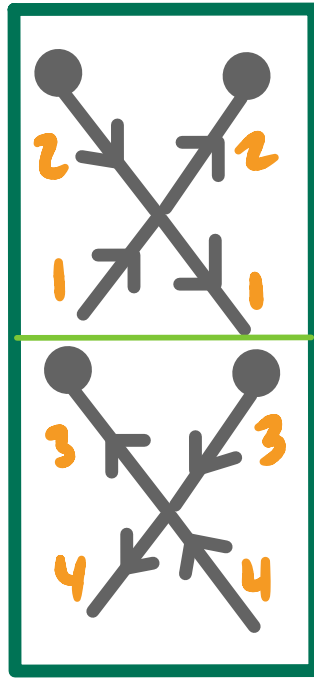
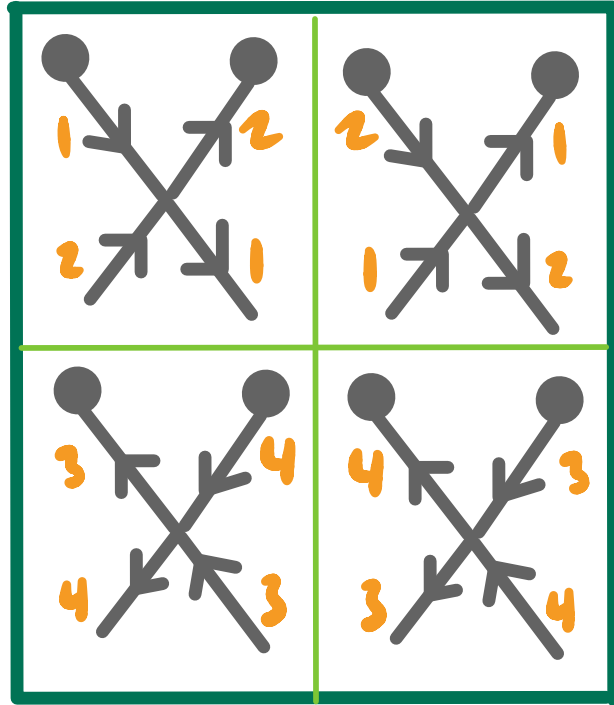
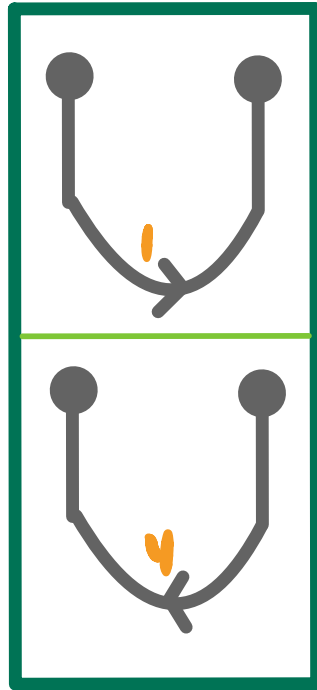


4-row growth rules

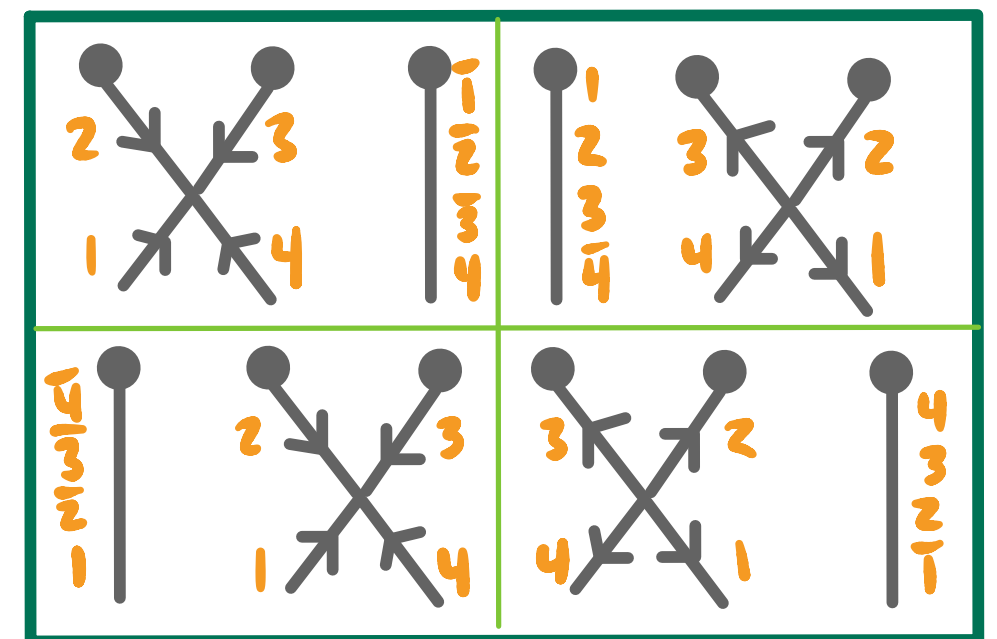
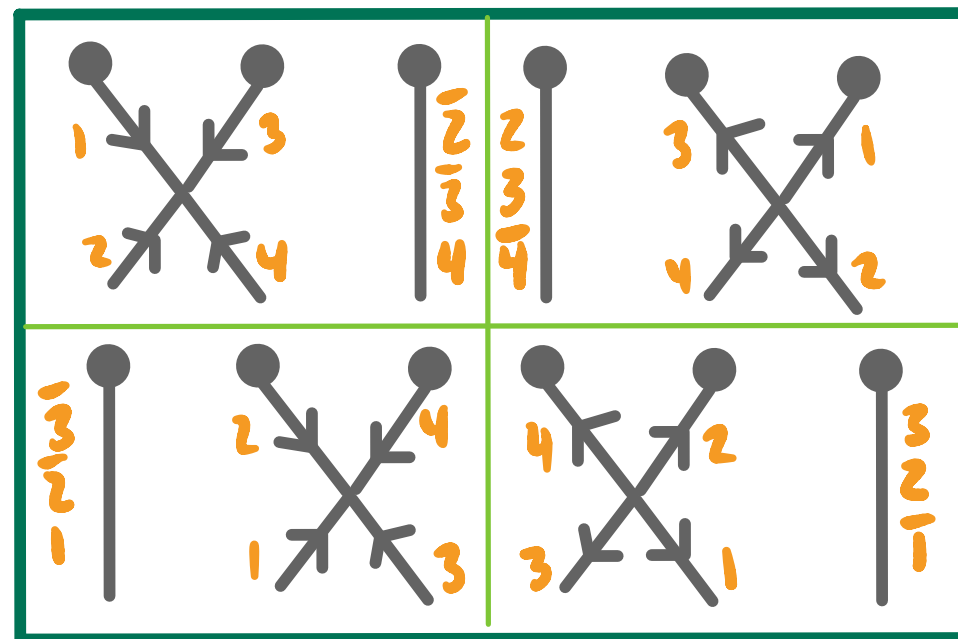
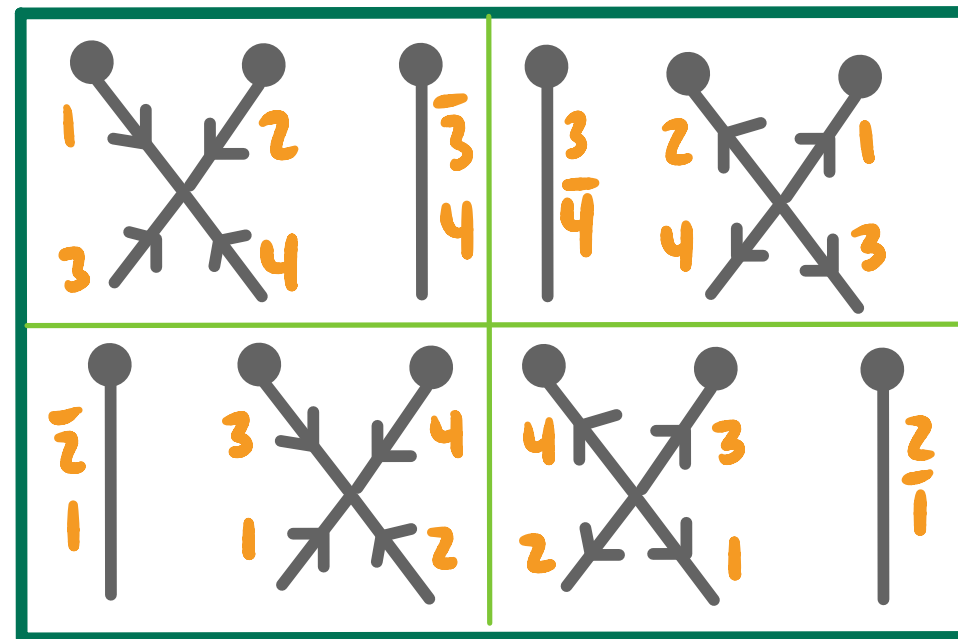
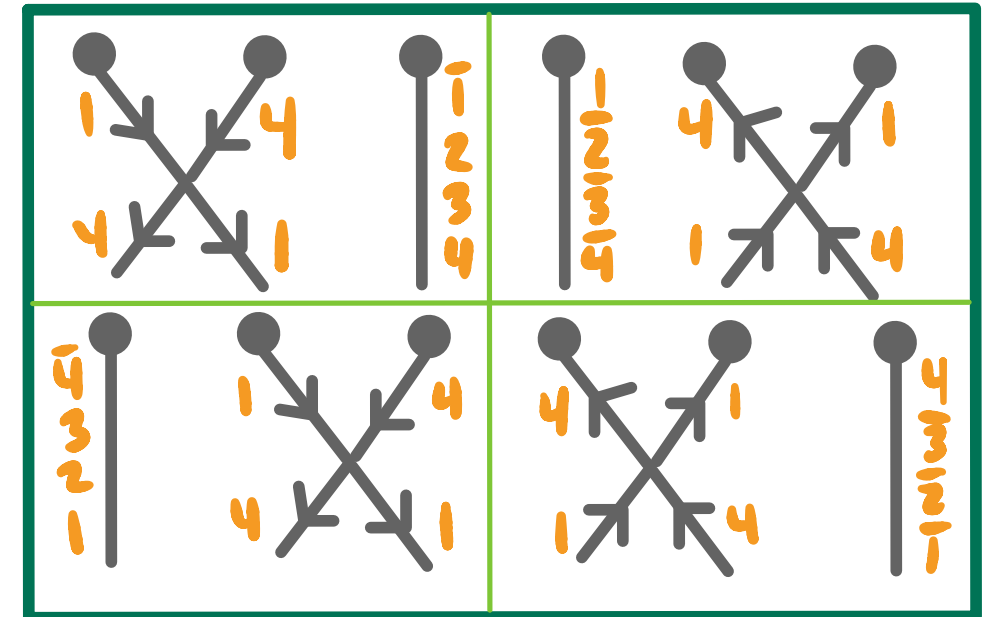
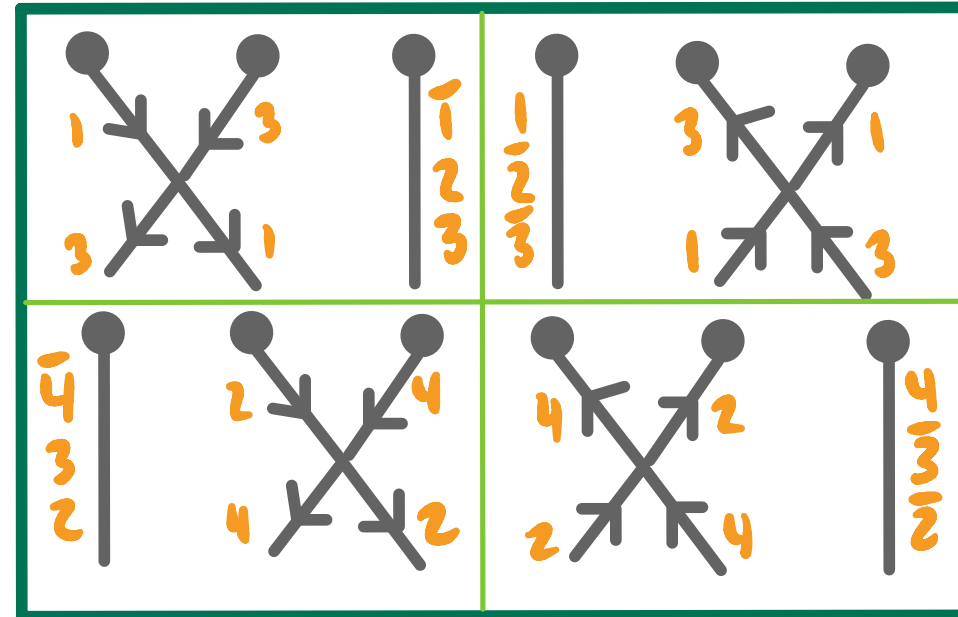
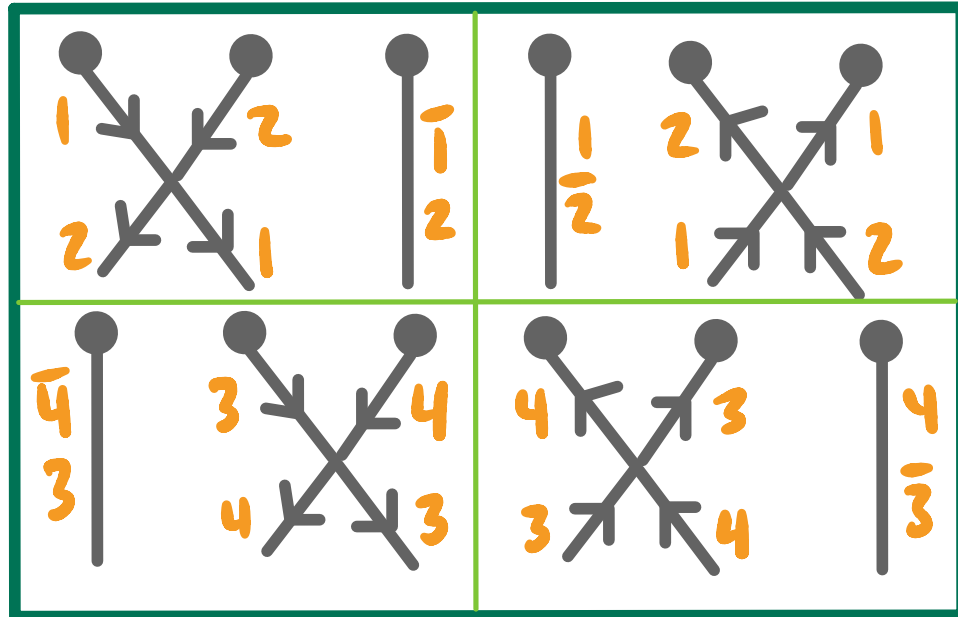
Some sample 4-row growth rules:



4-row growth rules



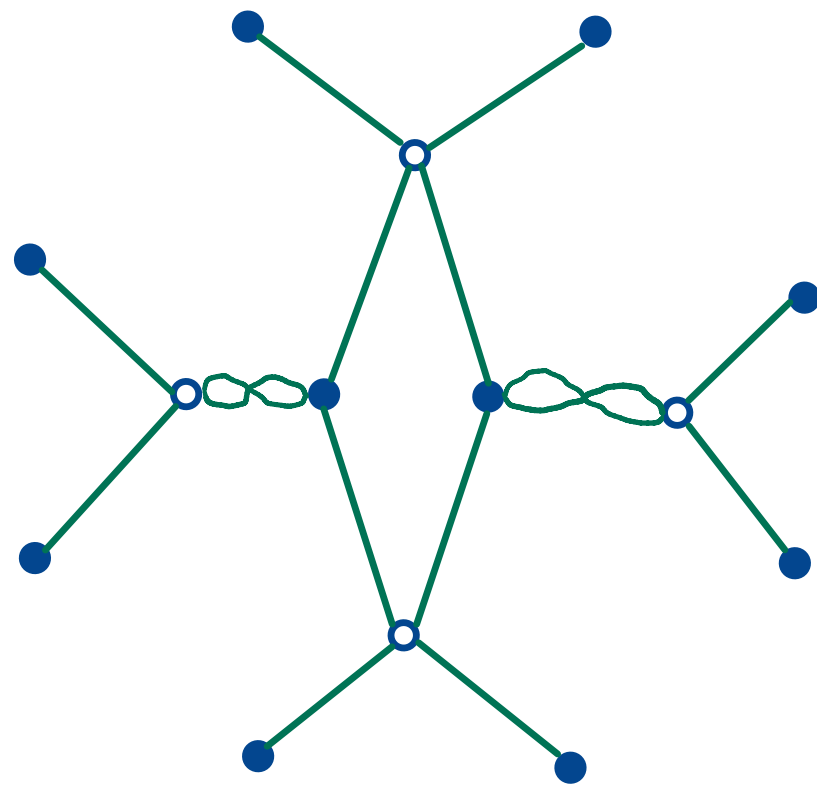
4-row growth rules



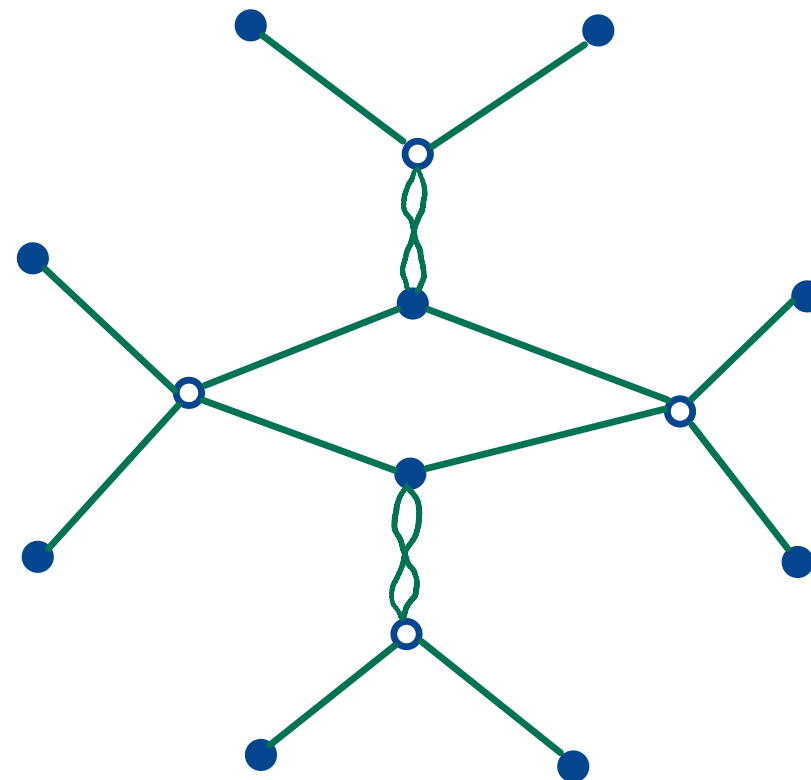
4-row moves

Thm Flip equivalence essentially arises from two moves:

1. Square moves: e.g.

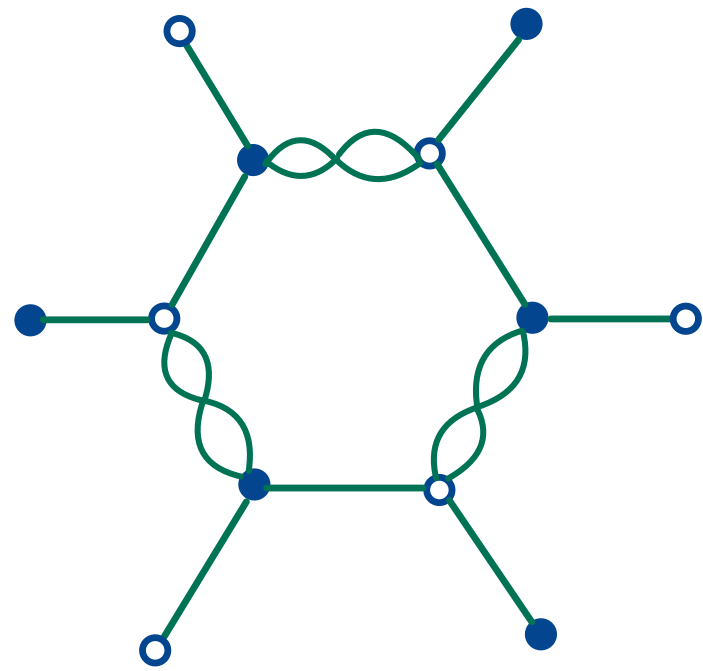


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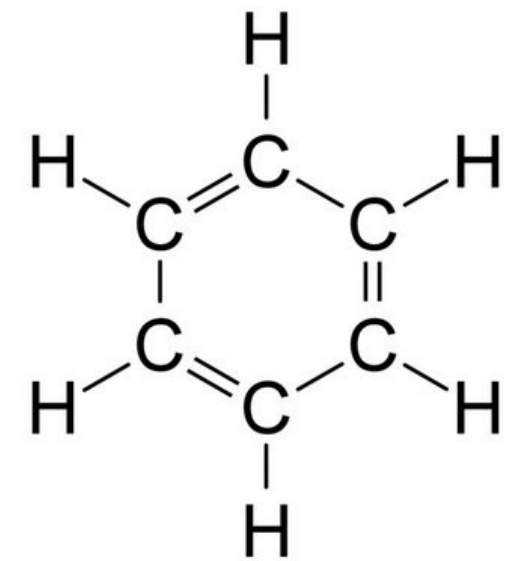
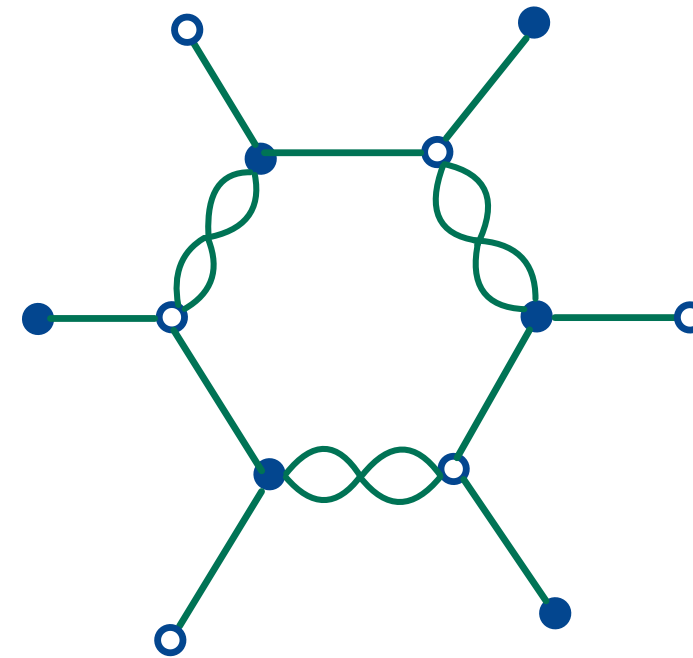


4-row moves

2. Benzene moves:



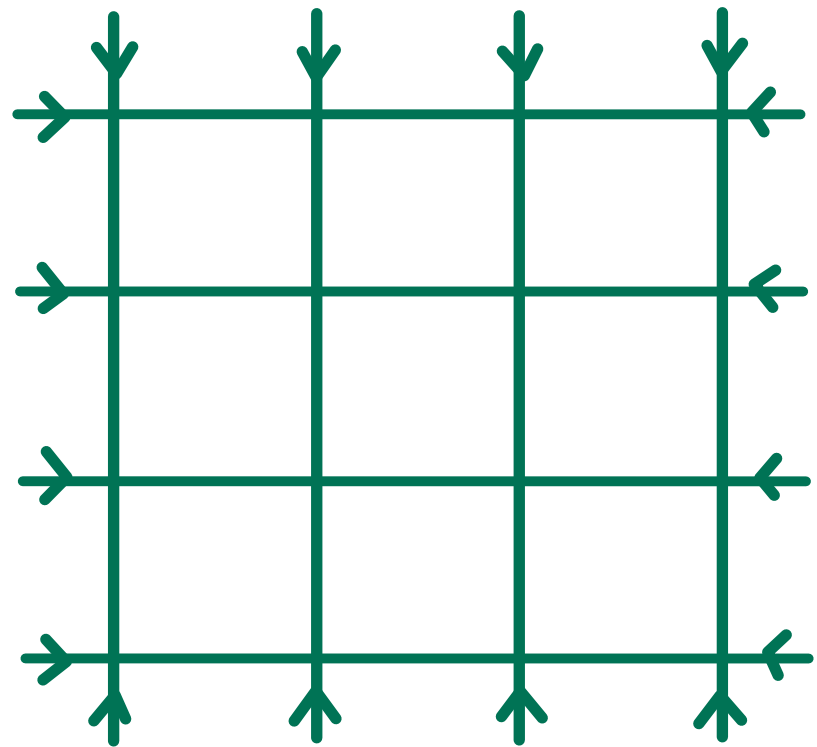
\approx



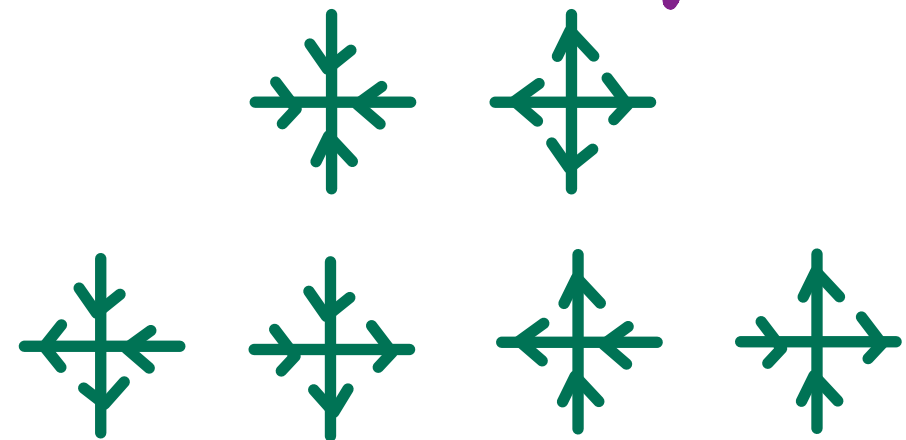
(Actual benzene)

ASM case

Thm | The tableau with lattice word $1^n 2^n 3^n 4^n$ has webs



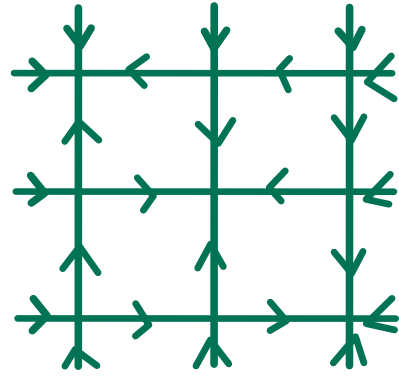
Completed in all ways using...



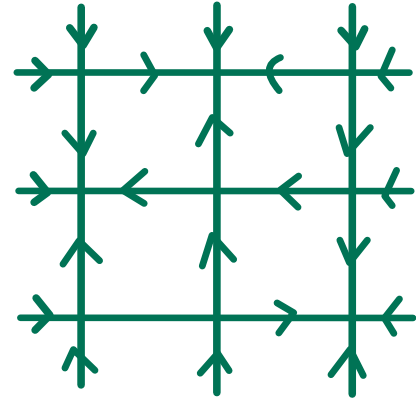
Note | These are naturally alternating sign matrices!

ASM case

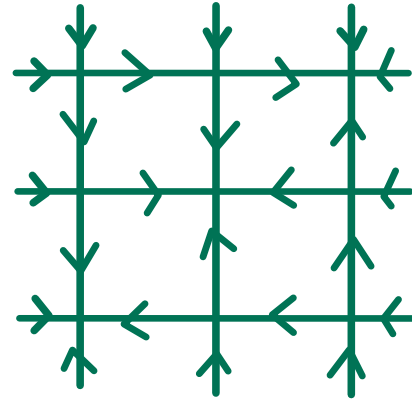
Ex



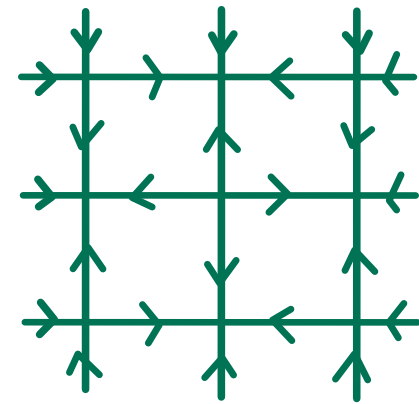
100
010
001



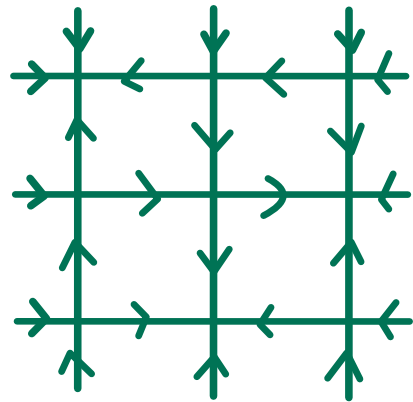
010
100
001



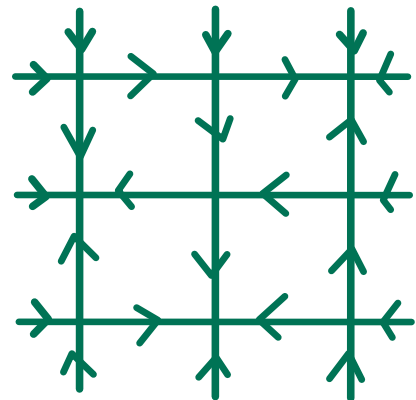
001
010
100



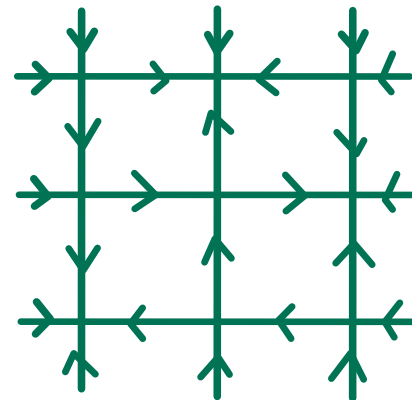
010
1-1-1
010



100
001
010



001
100
010



010
100
100

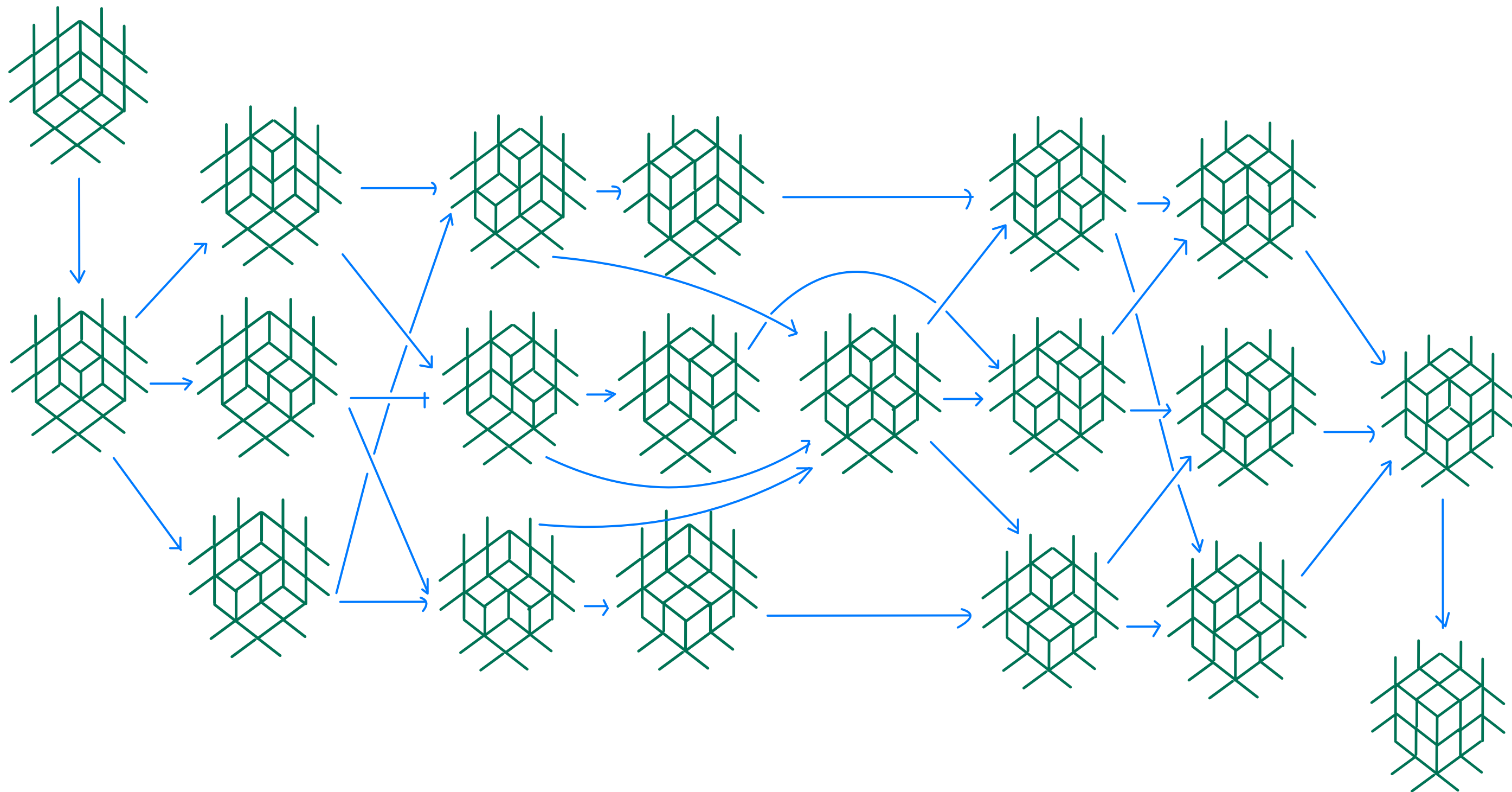
Plane partition case

Thm | The tableau with lattice word $1^a \bar{4}^b 2^c \bar{1}^{a-c} \bar{2}^c 4^b \bar{1}^c$
has equivalence class $(a \geq c)$

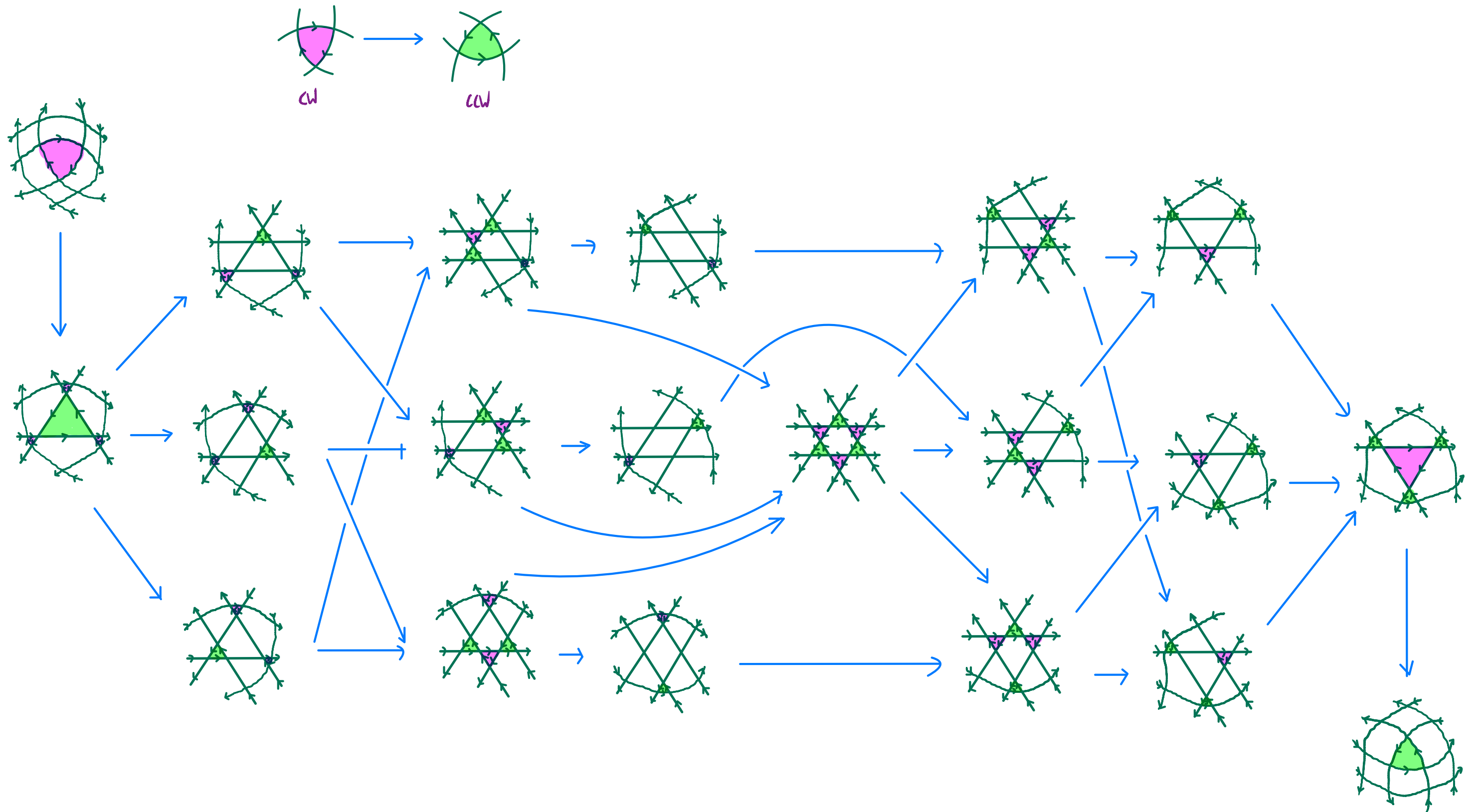
{ plane partitions
in the $a \times b \times c$ box }

Plane partition case

Ex



Plane partition case



THANKS!