An SL(4) web basis from hourglass plabic graphs

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Based on joint work with *Christian Gaetz, Oliver Pechenik, Stephan Pfannerer, and Jessica Striker* (submitted)

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Slides: https://www.jpswanson.org/talks/2023_USC_webs.pdf

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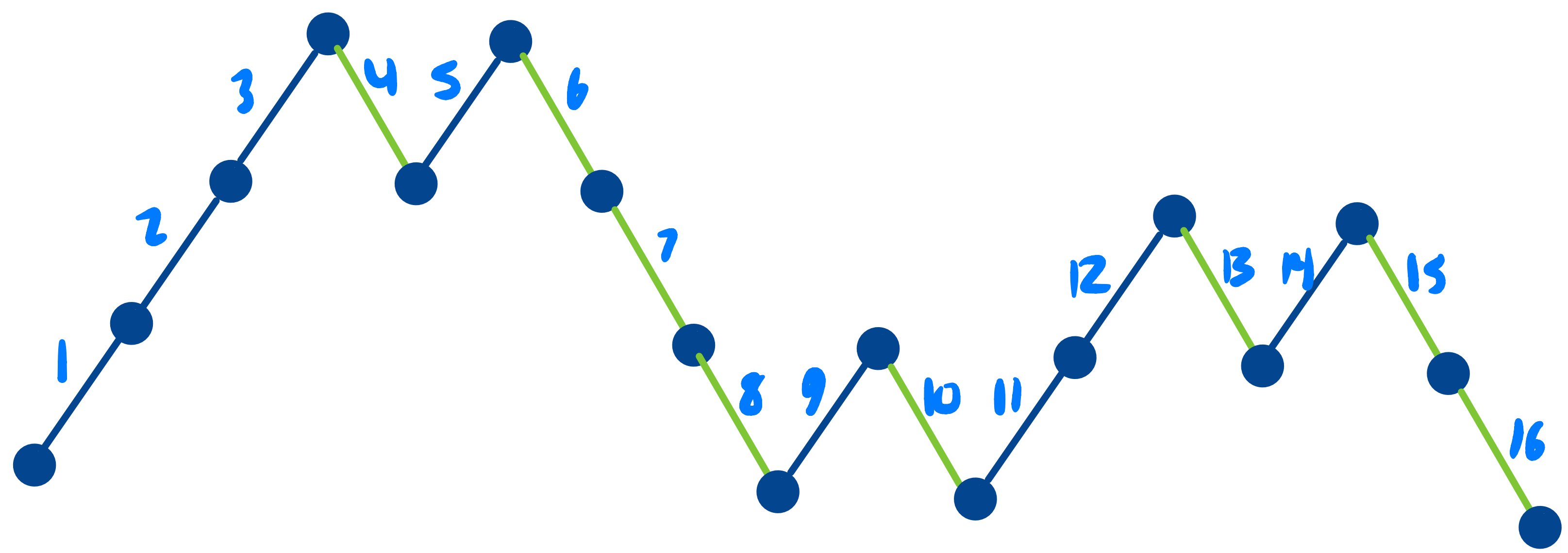
Outline

- Catalan combinatorics, 2-row webs
- 3-row web bases, plabic graphs, growth rules
- (New!) 4-row web bases, hourglass plabic graphs

Catalan objects

Some Catalan bijections:



Dyck path

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

$\longmapsto$ ((()()))()(()()) $\longmapsto$ 1112122212112122

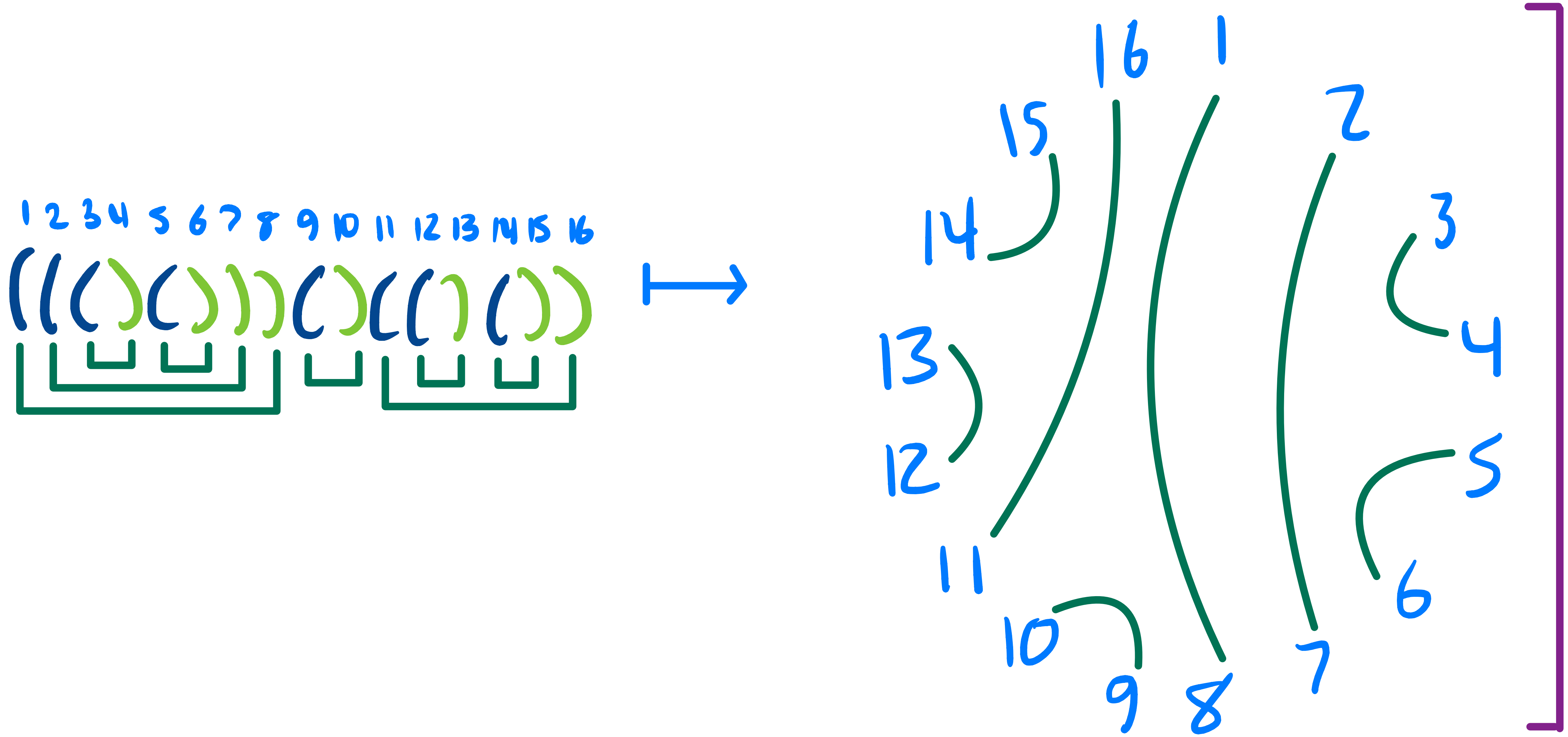
lattice word

$\longmapsto$

1	2	3	5	9	11	12	14
4	6	7	8	10	13	15	16

Standard Young tableau

Catalan objects



Non-crossing matching ("spiny picture"!)

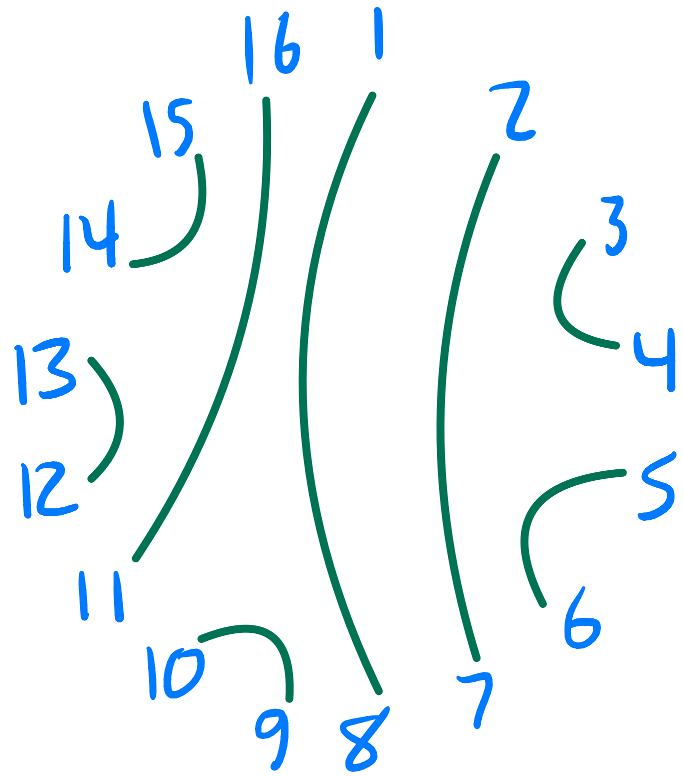
Catalan objects

$\boxed{\text{Thm}}$ The bijection $NCM(2n) \xrightarrow{\sim} SYT(2\times n)$ sends rotation to promotion, reflection to evacuation.

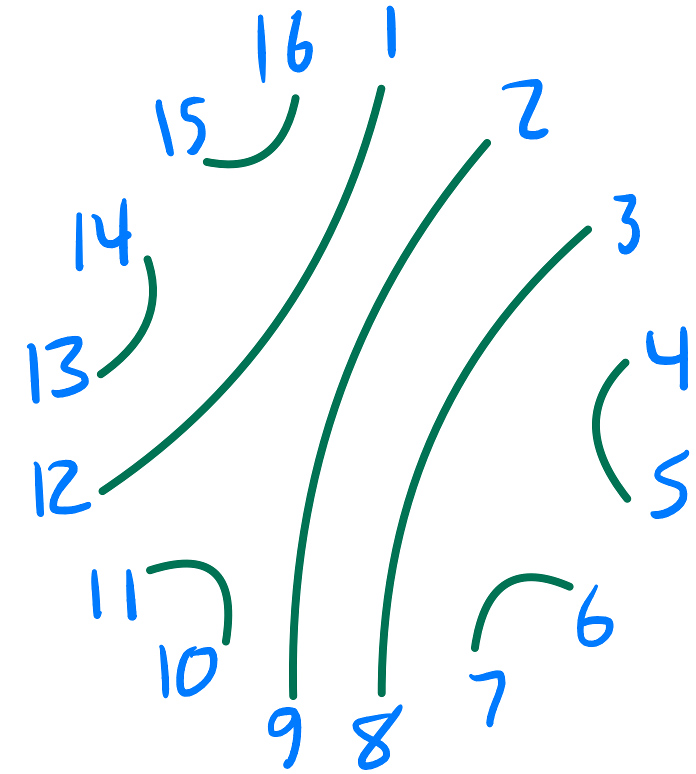
- "Hidden" dihedral action on $SYT(2\times n)$!

Catalan objects

Ex



Rotate $\longmapsto$



1	2	3	5	9	11	12	14
4	6	7	8	10	13	15	16

Promote $\longmapsto$

1	2	3	4	6	10	13	15
5	7	8	9	11	12	14	16

SL_2-Invariants

Let $V = \mathbb{C}^2$, $V_i \in \{V, V^*\}$.

Q What are the SL_2-invariants of $V_1 \otimes \cdots \otimes V_n$?

That is, identify $\mathrm{Hom}_{SL_2}(V_1 \otimes \cdots \otimes V_n, \mathbb{C}) \subset \mathbb{C}[x_{ij}; y_{kl}]$.

Ex $V \otimes V$: $\det\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = x_{11}x_{22} - x_{12}x_{21}$ is unique invariant

$\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$ $\quad \det(g \cdot X) = \det(g)\det(X) = \det(X)$

Ex $V^* \otimes V$: pairing $\langle y, x \rangle = x_1 y_1 + x_2 y_2$ is invariant

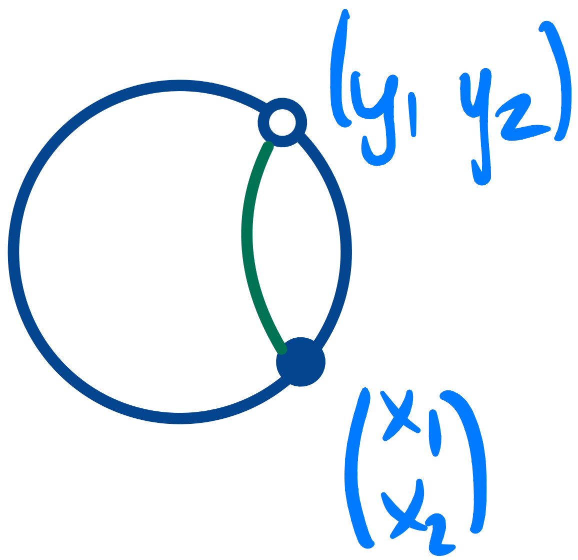
$(y_1\ y_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\quad \langle y g^{-1}, g x \rangle = \langle y, x \rangle$

SL_2-Tensor diagrams

- Encode morphisms/invariants in diagrams: $V \longleftrightarrow \bullet$

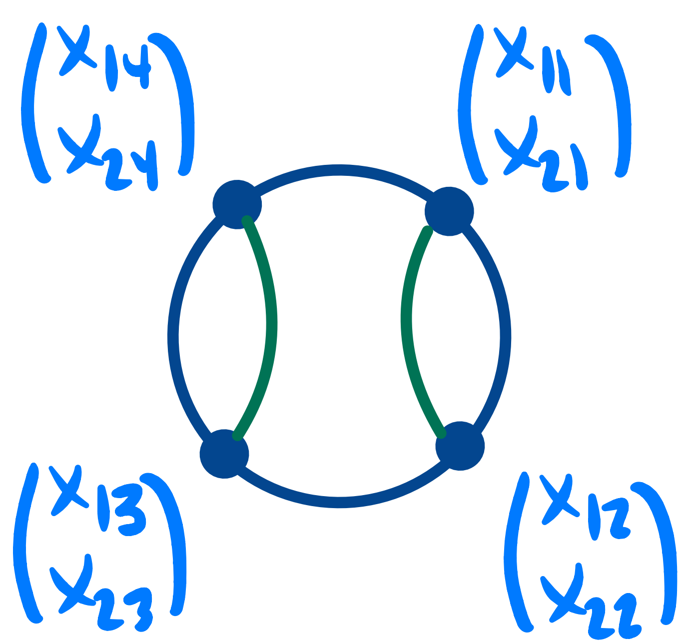
  $V^* \longleftrightarrow \circ$

Ex



$$= x_1y_1 + x_2y_2 \in \mathrm{Inv}(V^* \otimes V)$$

Ex



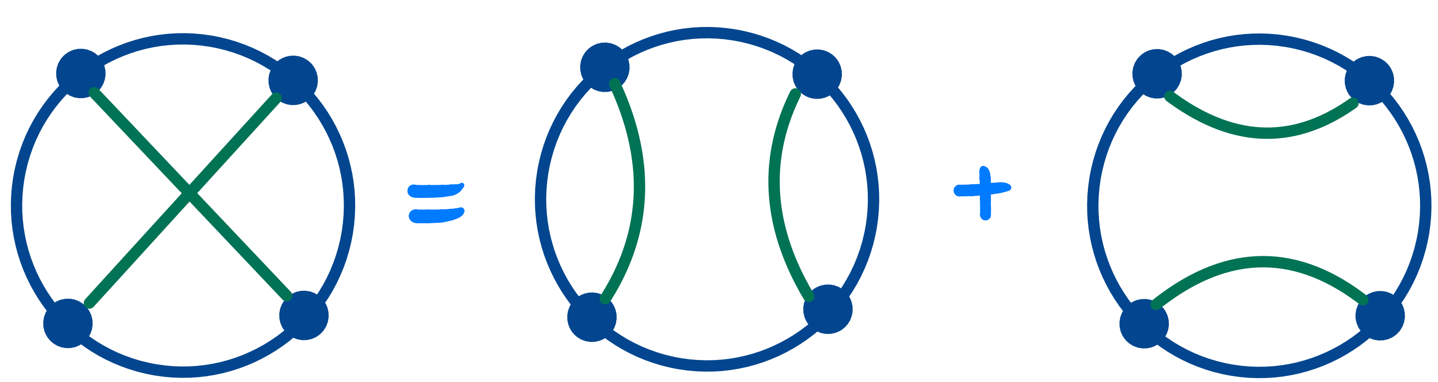
$$= \det\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \cdot \det\begin{pmatrix} x_{13} & x_{14} \\ x_{23} & x_{24} \end{pmatrix} \in \mathrm{Inv}(V^{\otimes 4})$$

SL_2-Tensor diagrams

- Encode morphisms/invariants in diagrams: $V \longleftrightarrow \bullet$

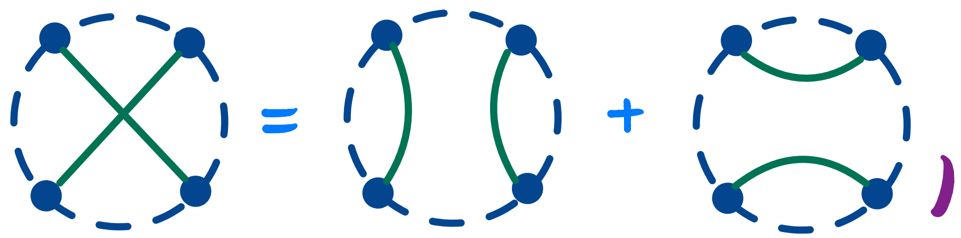
  $V^* \longleftrightarrow \circ$

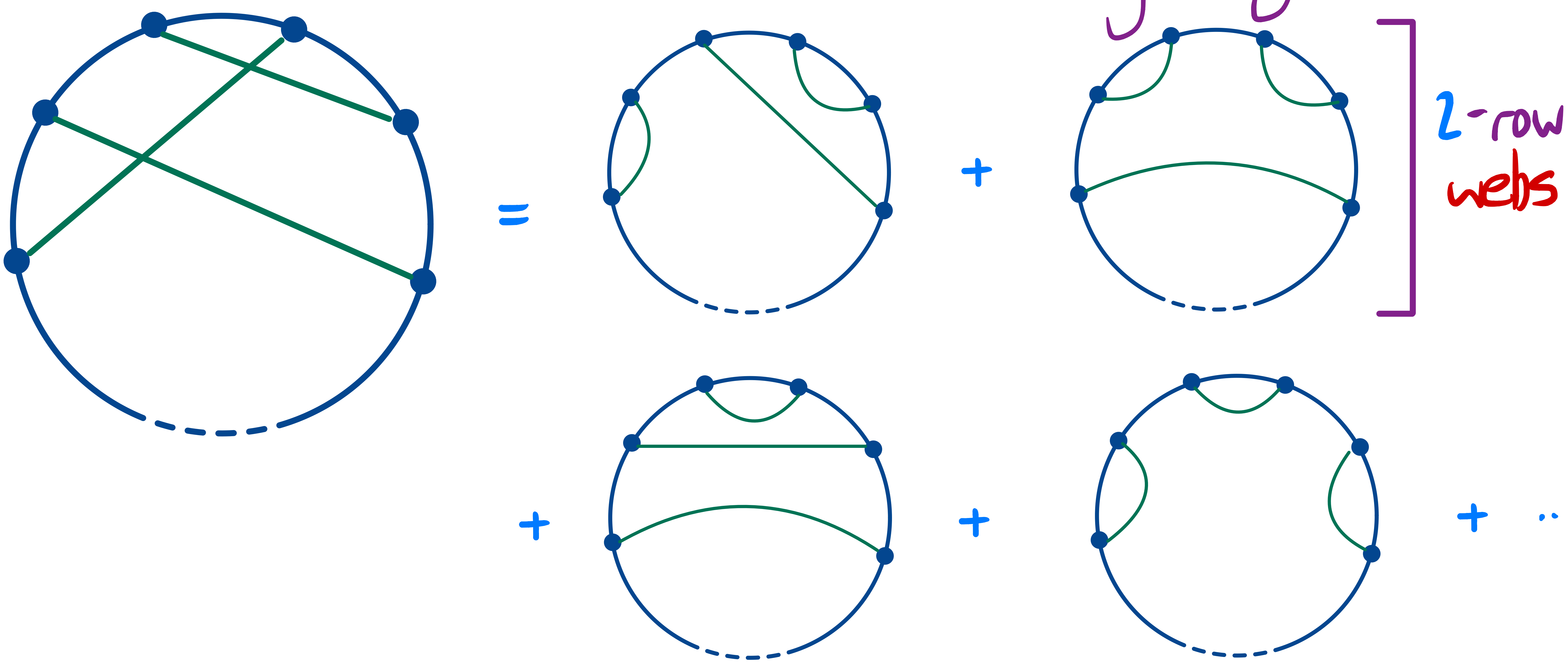
Ex | Plücker relations:



$$(x_{11}x_{23} - x_{21}x_{13})(x_{12}x_{24} - x_{22}x_{14})$$
$$=$$
$$(x_{11}x_{22} - x_{21}x_{12})(x_{13}x_{24} - x_{23}x_{14})$$
$$+$$
$$(x_{11}x_{24} - x_{21}x_{14})(x_{12}x_{23} - x_{22}x_{13})$$

Temperly-Lieb basis

• Using , can reduce any matching diagram to a linear combination of matching diagrams:



Temperley-Lieb basis

Thm The noncrossing 2-row webs are a basis for

$$\mathrm{Inv}_{SL_2}(V_1 \otimes \cdots \otimes V_n) \qquad (V_i \in \{V, V^*\})$$

Pf • <u>Spanning</u>: diagrams span by classical invariant theory, noncrossing by uncrossing rule.

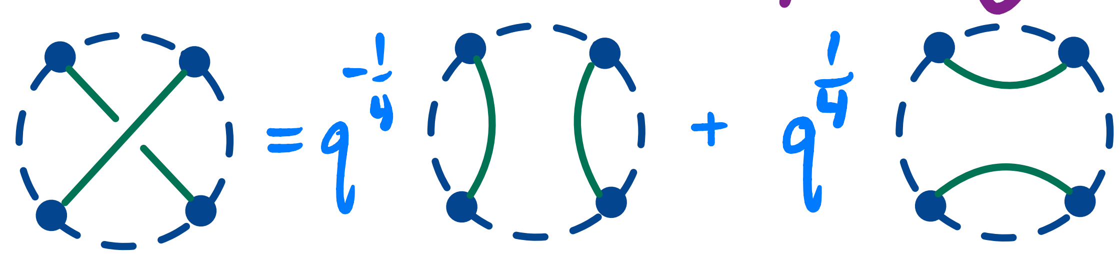
• <u>Independence</u>: by Pieri rule,

$$\dim \mathrm{Inv}_{SL_2}(V^n) = \#SYT(2 \times \tfrac{n}{2})$$

Quantum Link Invariants

The quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ is an algebra deforming the (universal enveloping algebra of) $\mathfrak{sl}_2$.

- Sending $q \to 1$ recovers the usual case.
- $\mathrm{Inv}(V_1 \otimes \cdots \otimes V_n)$ still has a Temperly-Lieb web basis:



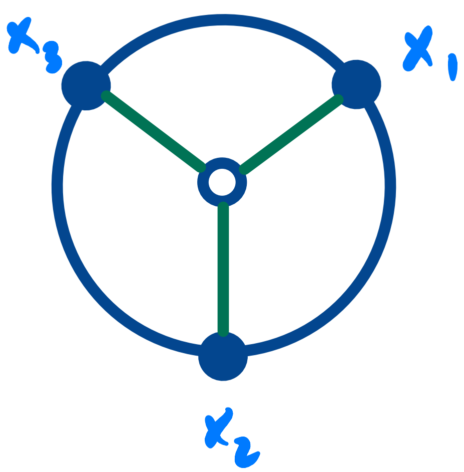
- Projections of knots/links/tangles become polynomials in q; refined version of Jones polynomial

SL_3-Tensor diagrams

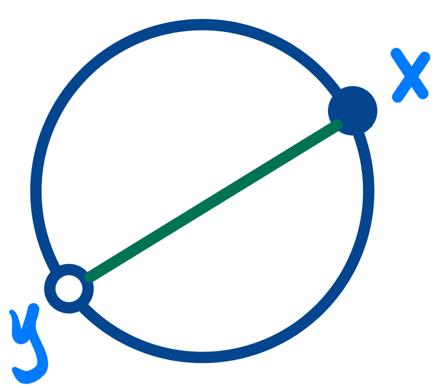
- Let $V = \mathbb{C}^3$, $V_i \in \{V, V^*\}$.

Q] Is there a web basis for $\mathrm{Inv}_{SL_3}(V_1 \otimes \cdots \otimes V_n)$?

Ex]



$$= \det\begin{pmatrix} | & | & | \\ x_1 & x_2 & x_3 \\ | & | & | \end{pmatrix}$$



$$= \langle y, x \rangle$$

There's more! $\longrightarrow$

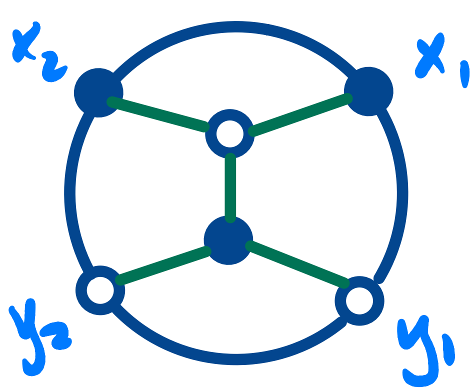
SL_3-Tensor diagrams

- Let $V = \mathbb{C}^3$, $V_i \in \{V, V^*\}$.

$\boxed{Q}$ Is there a web basis for $\mathrm{Inv}_{SL_3}(V_1 \otimes \cdots \otimes V_n)$?

$\boxed{\text{Note}}$ $\Lambda^2 V \cong V^*$ since $\Lambda^3 V = \det V = 1$ over SL_3.

$\Lambda^2 V^* \cong V$

$\boxed{\text{Ex}}$ 

$= \det\begin{pmatrix} | & | & | \\ x_1 & x_2 & y_1 \wedge y_2 \\ | & | & | \end{pmatrix}$

SL_3-Tensor diagrams

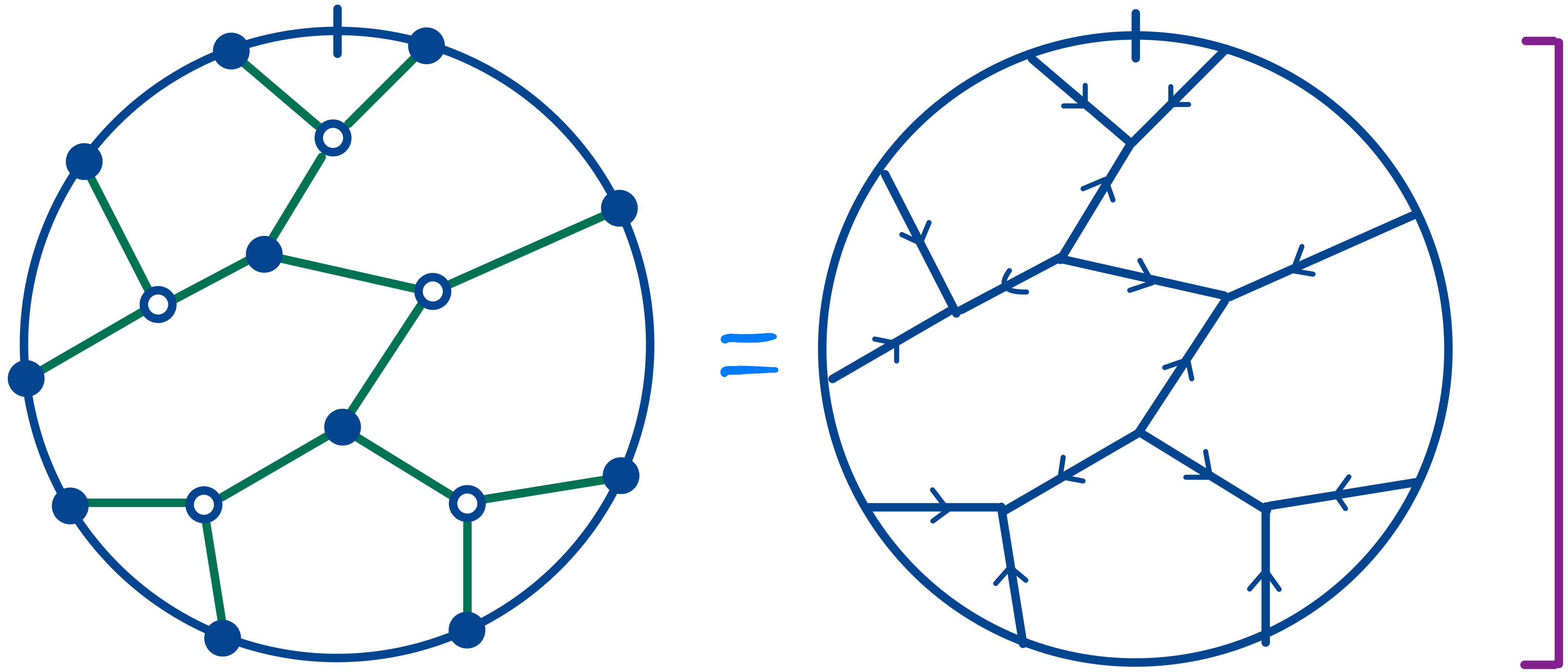
Def 3-row webs are

- planar graphs embedded in a disk
- trivalent interior vertices,
  univalent boundary vertices
- bipartite
- marked "initial" boundary vertex

They encode $\mathrm{Inv}_{SL_3}(V_1 \otimes \cdots \otimes V_n)$.

SL_3-Tensor diagrams

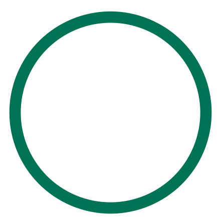
Ex

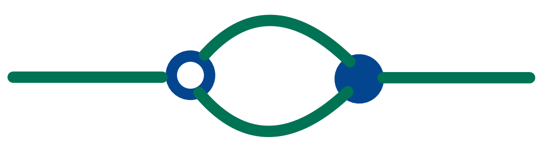


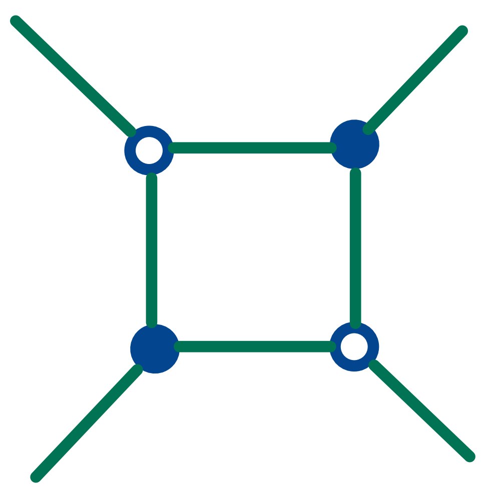
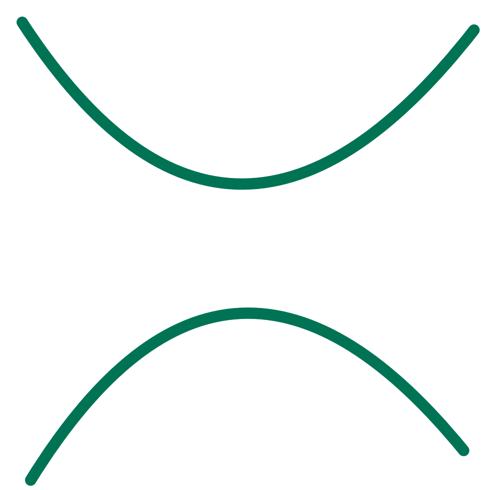
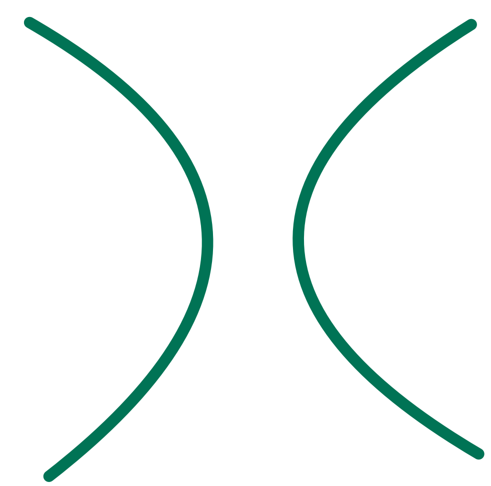
"Spinny picture"!

SL_3-Web basis

Thm (Kuperberg) The generating SL_3-web relations are

 $= 3$

 $= 2 \cdot$

 $=$  $+$ 

SL_3-Web basis

Thm (Kuperberg; Kuperberg–Khovanov)

Call an SL_3-web non-elliptic if it is connected and it has no internal 2-faces or 4-Faces.

The non-elliptic webs form a basis of

$$\mathrm{Inv}_{SL_3}(V_1 \otimes \cdots \otimes V_n). \qquad (V_i \in \{V, V^*\})$$

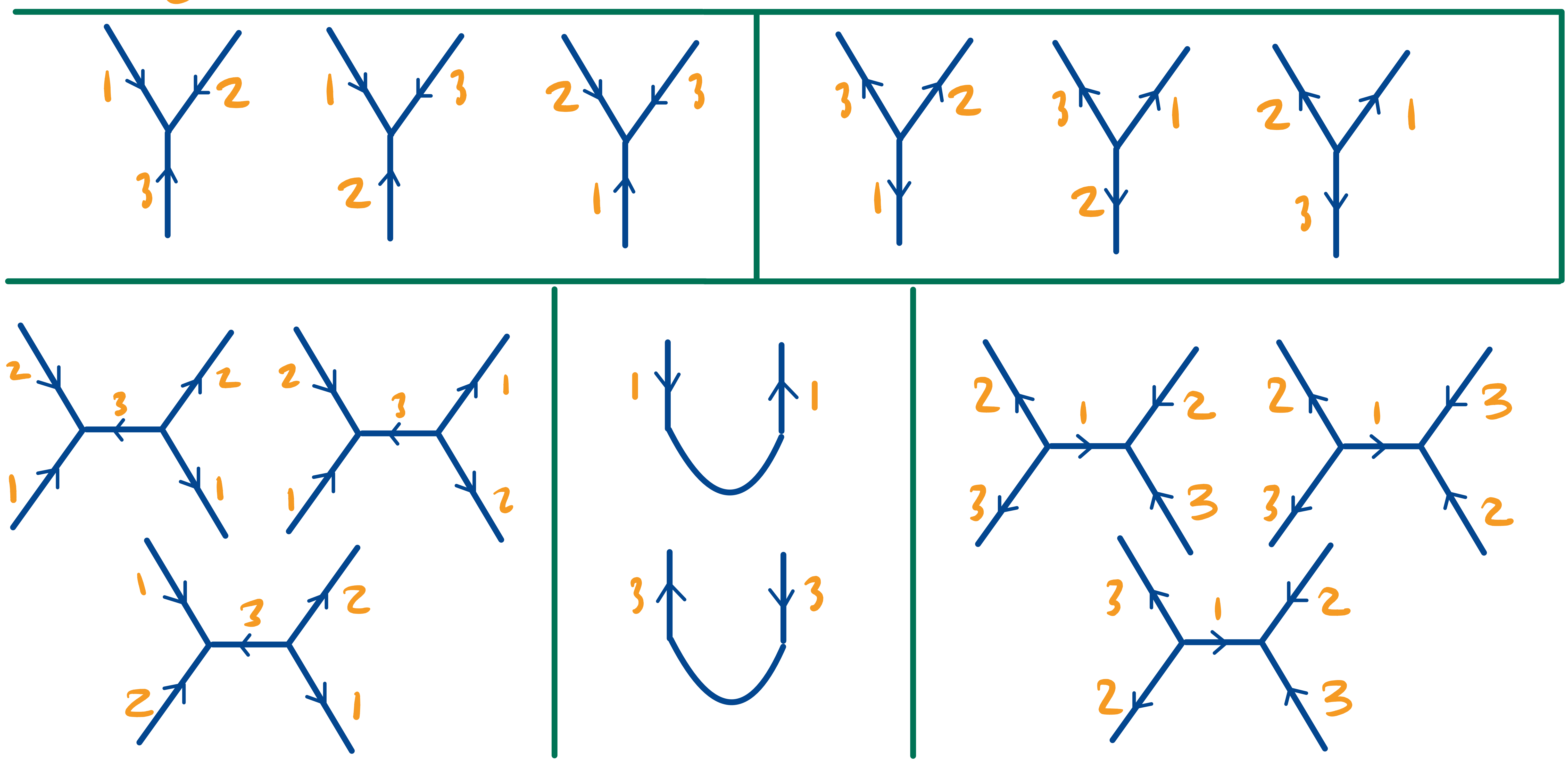
SL_3-Web basis

$\boxed{\text{Pf}}$ • **Spanning**: similar to SL_2 case.

• **Independence**: bijection to $SYT(3 \times \frac{n}{3})$ using growth rules. (Other descriptions of this bijection have since been found.)

SL_3-Web basis

Kuperberg-Khovanov growth rules:



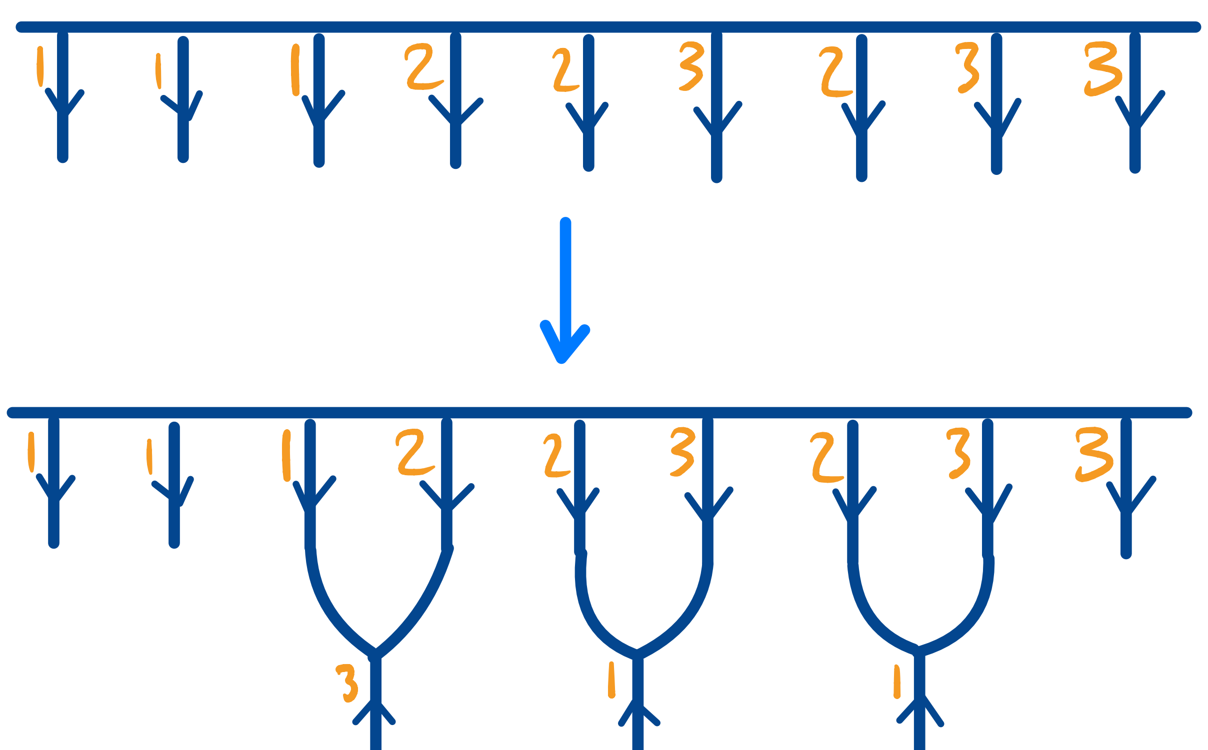
SL_3-Web basis

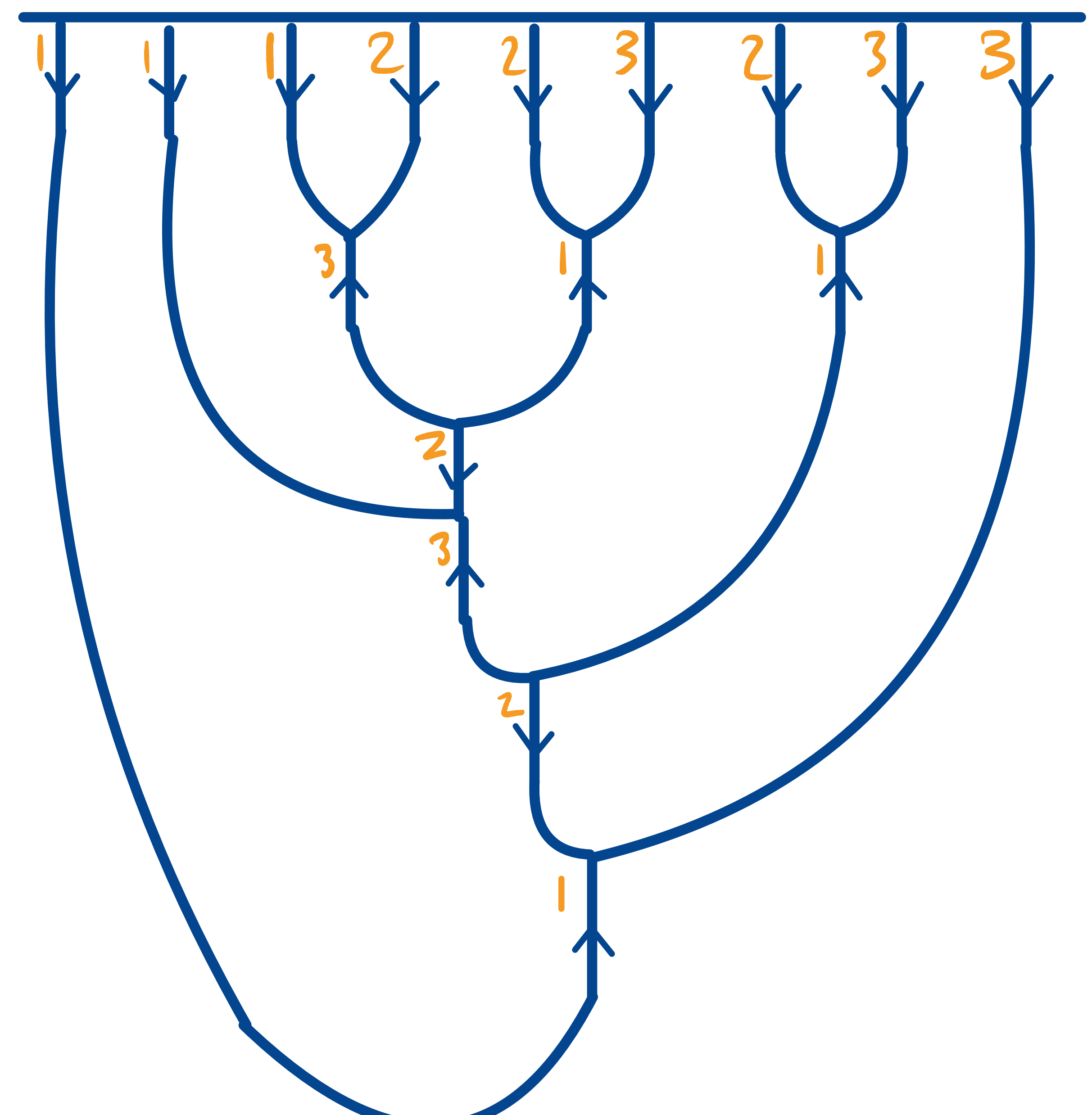
Ex

$T =$

1	2	3
4	5	7
6	8	9

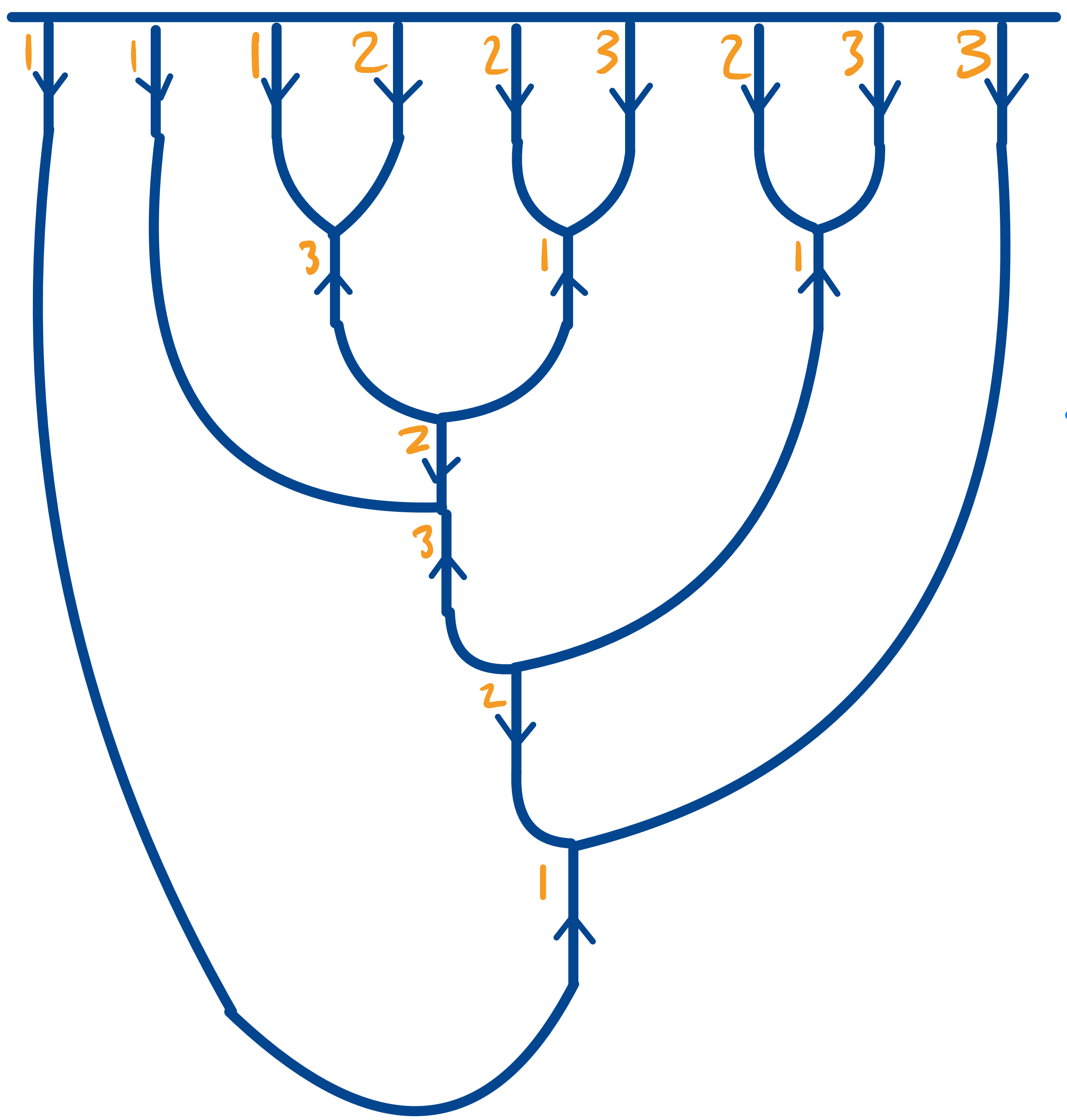
$\longrightarrow$ 111223233

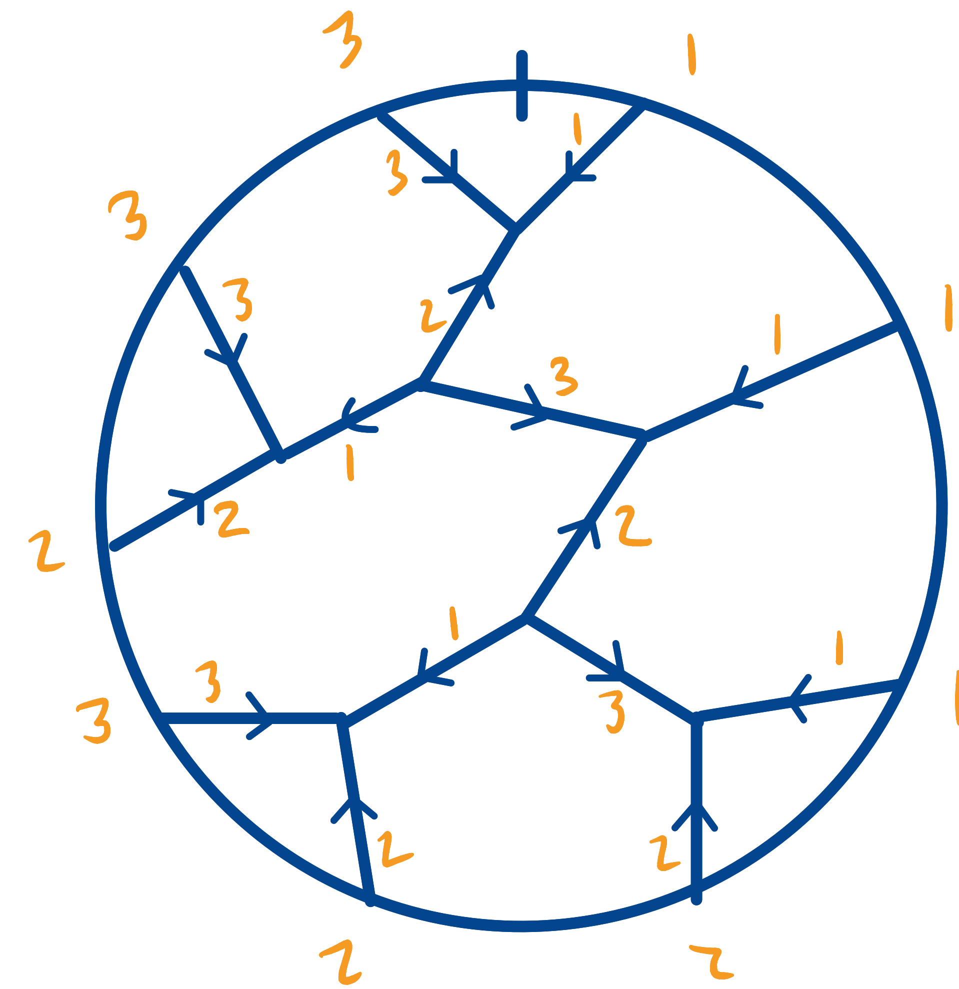




SL_3-Web basis

$(T \longrightarrow 111223233)$





Now just erase labels!

SL_3-Web basis

Thm (Petersen-Pylyavskyy-Rhoades)
The bijection from non-elliptic webs to $SYT(3 \times \frac{n}{3})$
sends rotation to promotion
reflection to evacuation.

- "Hidden" dihedral action on $SYT(3 \times m)$!

SL_3-Web basis

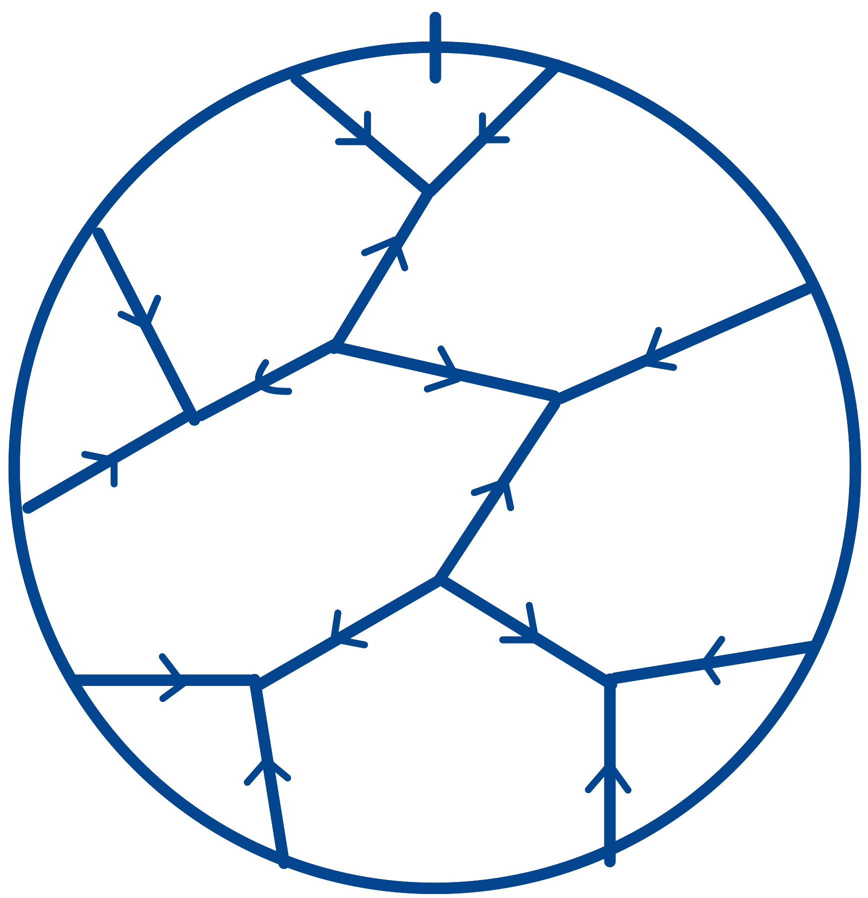
Ex

1	2	3
4	5	7
6	8	9

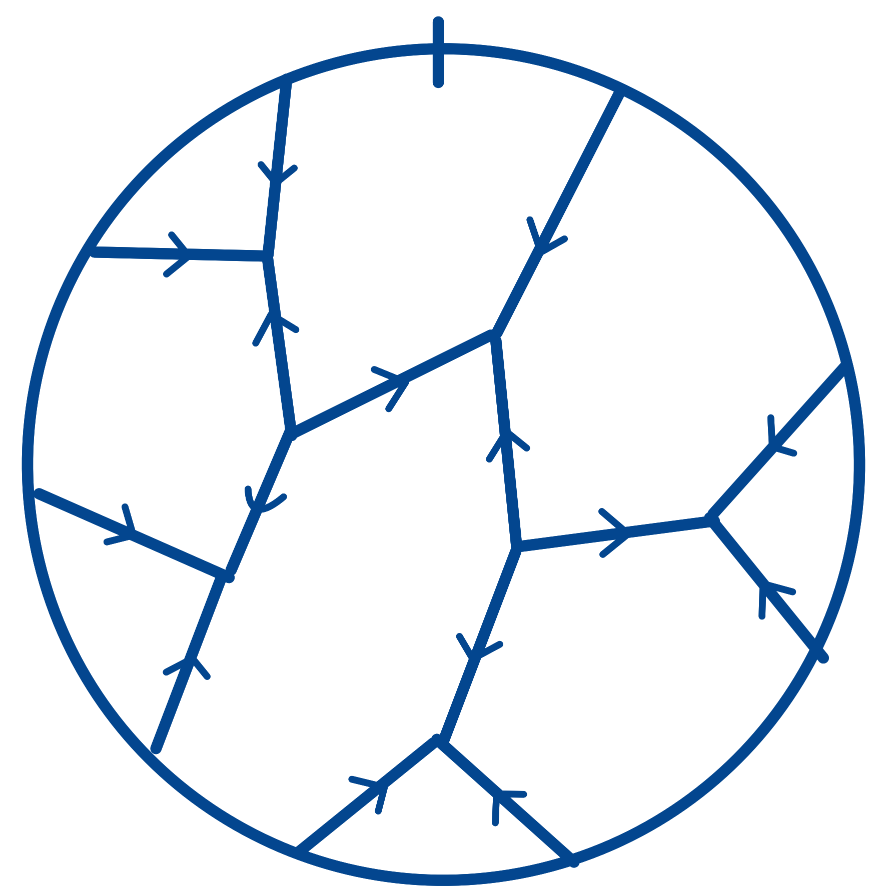
$\xmapsto{\text{Prom}}$

1	2	6
3	4	8
5	7	9

$\downarrow$



$\xmapsto{\text{Rotation}}$



SL_3-Web basis

Applications

- Quantum link invariants
  - SL_3 link polynomials, foams, ...
- Cluster algebras
  - Cluster structures on $\mathbb{C}[x_{ij}, y_{kl}]^{SL_3}$
- Enumerative combinatorics
  - promotion, evacuation, cyclic sieving
- Dimer models
- Representation theory

The web basis problem

Problem (Khovanov–Kuperberg '96)

Give a web basis* for $\mathrm{Inv}_{SL_r}(V_1 \otimes \cdots \otimes V_n)$ for $r \geq 4$.

*with desirable properties for use in applications:

- testability
- reduction rules
- <u>rotation invariance</u>

The web basis problem

Next: our solution for $r=4$!

- Unifies $r=2,3,4$
- Many pieces work for general r (TBD!)
- Introduces new plabic graphs with multiple trip's
- Combinatorially beautiful,

e.g. connects ASM's and PP's in a certain sense

Plabic graphs

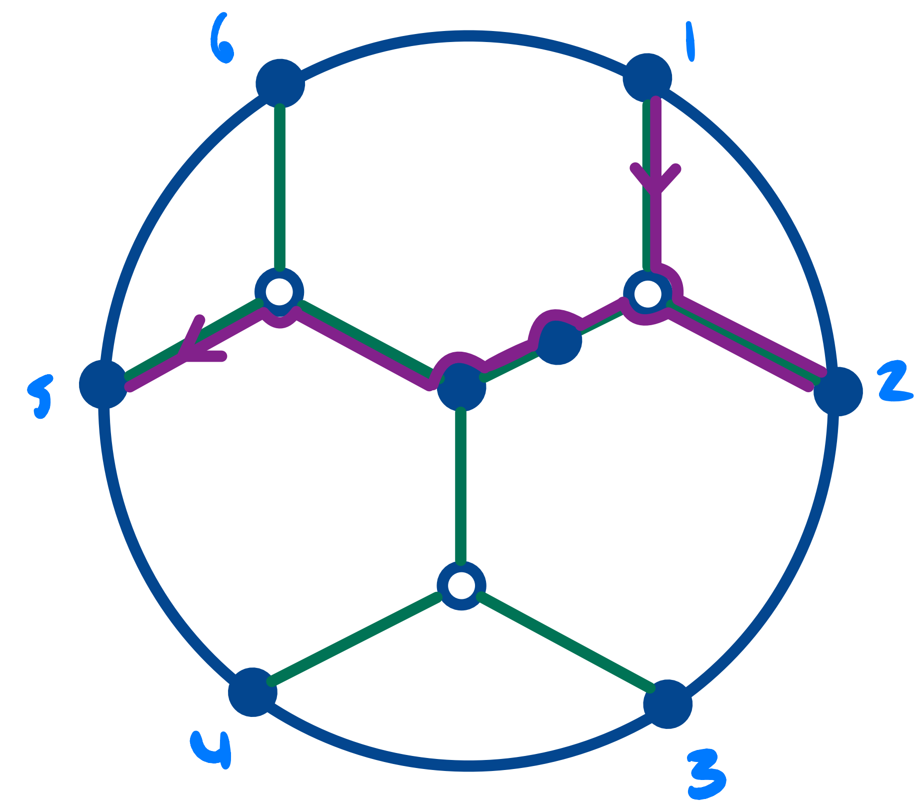
- Postnikov introduced plabic graphs to parametrize cells in $Gr_{\geq 0}(k,n)$. "planar bicolored"
- They have trip permutations:

Rules of the road

Left at ○

Right at ●

Ex



$1 \mapsto 2$

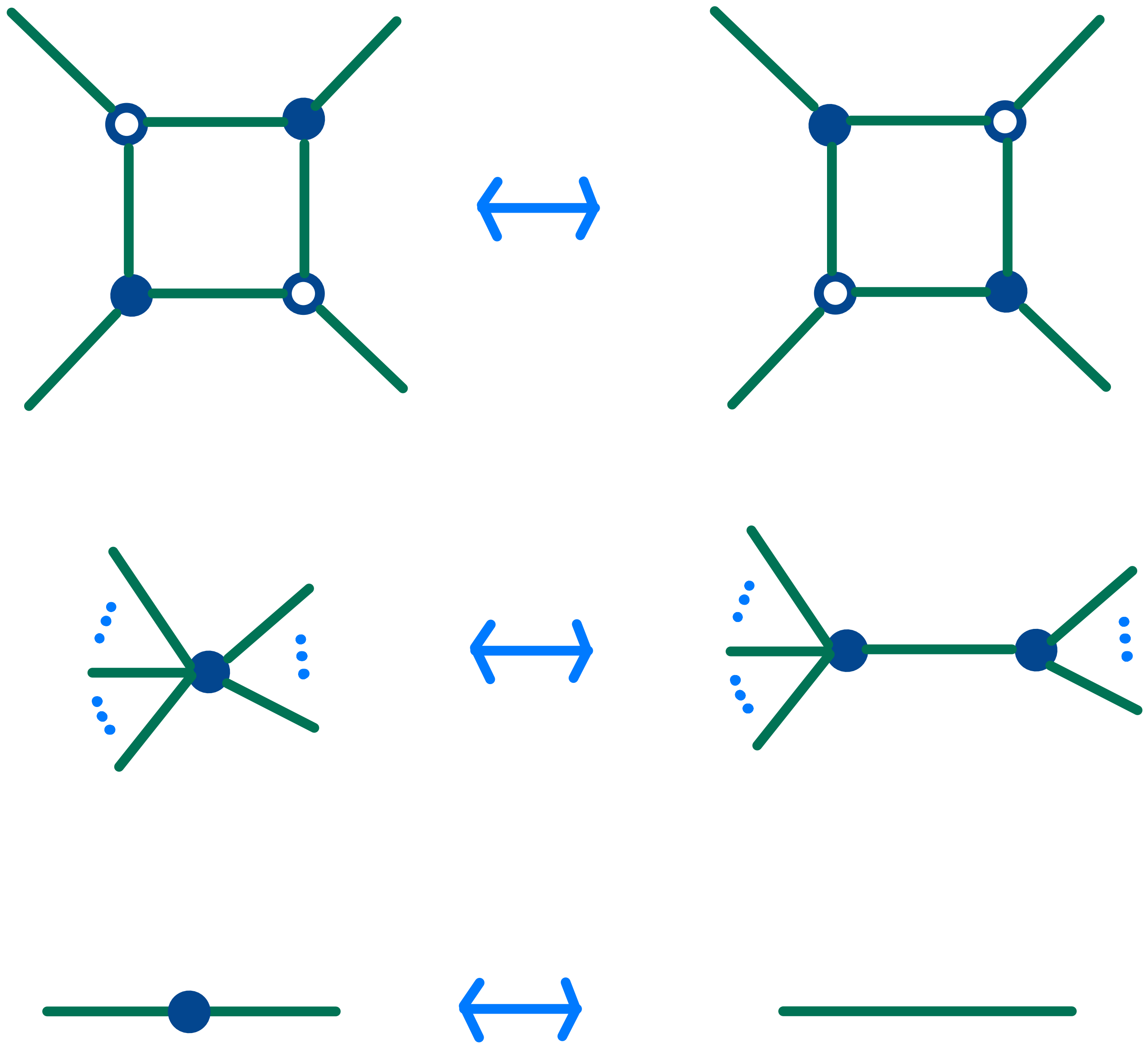
$2 \mapsto 5$

$\vdots$

Trip $= (1\,2\,5\,6\,3\,4)$

Plabic graphs

- Moves preserve Trip:



Thm (Postnikov)

Two reduced plabic graphs have the same Trip if and only if they are connected by a sequence of moves.

Promotion permutations

Obs (Hopkins–Rubey) 3-row basis webs are reduced plabic graphs. What are their Trip's?

Def The promotion permutation of $T \in SYT(3 \times \frac{n}{3})$ tracks what enters the top row when computing $P^n(T)$:

Ex

$$T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|}\hline \uparrow & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \ & 5 \\ \hline 4 & 6 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & 5 \\ \hline \ & 6 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & 5 \\ \hline 6 & \ \\ \hline \end{array} \rightarrow \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & 5 \\ \hline 6 & 7 \\ \hline \end{array} \mapsto \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array} = P(T)$$

$$\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & 5 \\ \hline 6 & 7 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|}\hline \ & 3 \\ \hline 4 & 5 \\ \hline 6 & 7 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|}\hline 3 & \uparrow \\ \hline 4 & 5 \\ \hline 6 & 7 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|}\hline 3 & 5 \\ \hline 4 & \ \\ \hline 6 & 7 \\ \hline \end{array} \rightarrow \cdots$$

$$\mathrm{Prom}(T) = 2\,5\,4\,1\,6\,3$$

Promotion and trip permutations

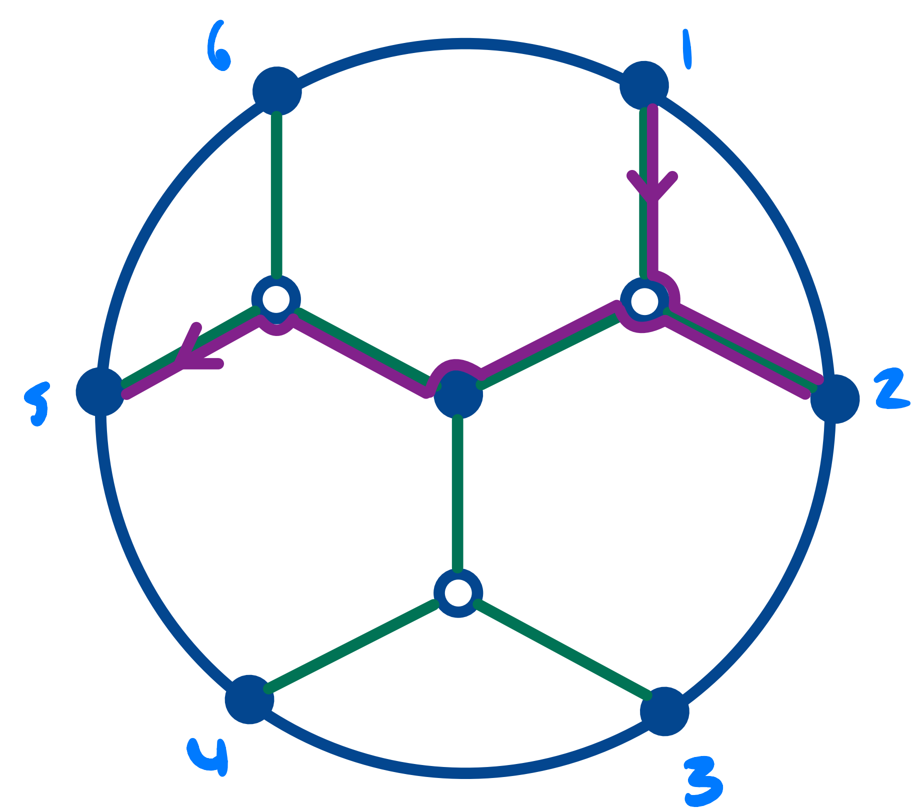
Thm (Hopkins-Rubey) The bijection between SL_3 basis webs and $SYT(3 \times \frac{n}{3})$ sends Trip to Prom.

Ex

$T =$

1	3
2	5
4	6

$\longleftrightarrow$



$= G$

$$\mathrm{Prom}(T) = 254163 = (125634) = \mathrm{Trip}(G)$$

Promotion matrices

- $\mathrm{prom}: SYT(3 \times \frac{n}{3}) \to S_n$ is injective, but
  $\mathrm{prom}: SYT(4 \times \frac{n}{4}) \to S_n$ is <u>not</u>.

- <u>Fix:</u> let prom_i record which number enters row i

  $\underline{\text{Thm}}|$ $T \in SYT(r \times \frac{n}{r})$ is uniquely determined by the sequence of promotion permutations $\mathrm{prom}_1, \ldots, \mathrm{prom}_{r-1}$.
  Moreover, $\mathrm{prom}_i^{-1} = \mathrm{prom}_{r-i}$.

Multiple trips

- Given a plabic graph (or similar), let
  trip_i: take the ith left at ∘
  ith right at •

- <u>Idea</u>: if r-valent, $\text{trip}_i^{-1} = \text{trip}_{r-i}$.

Main theorem

Thm $\{[W] \mid W$ is a top fully reduced hourglass plabic graph$\}$

is an SL_4 web basis. Furthermore, we have a bijection

$$SYT(4 \times \tfrac{n}{4}) \to TFRHPG(n)$$

$$T \mapsto G$$

$$\mathrm{prom}_i(T) = \mathrm{trip}_i(G) \qquad (i = 1, \ldots, r-1)$$

Main theorem

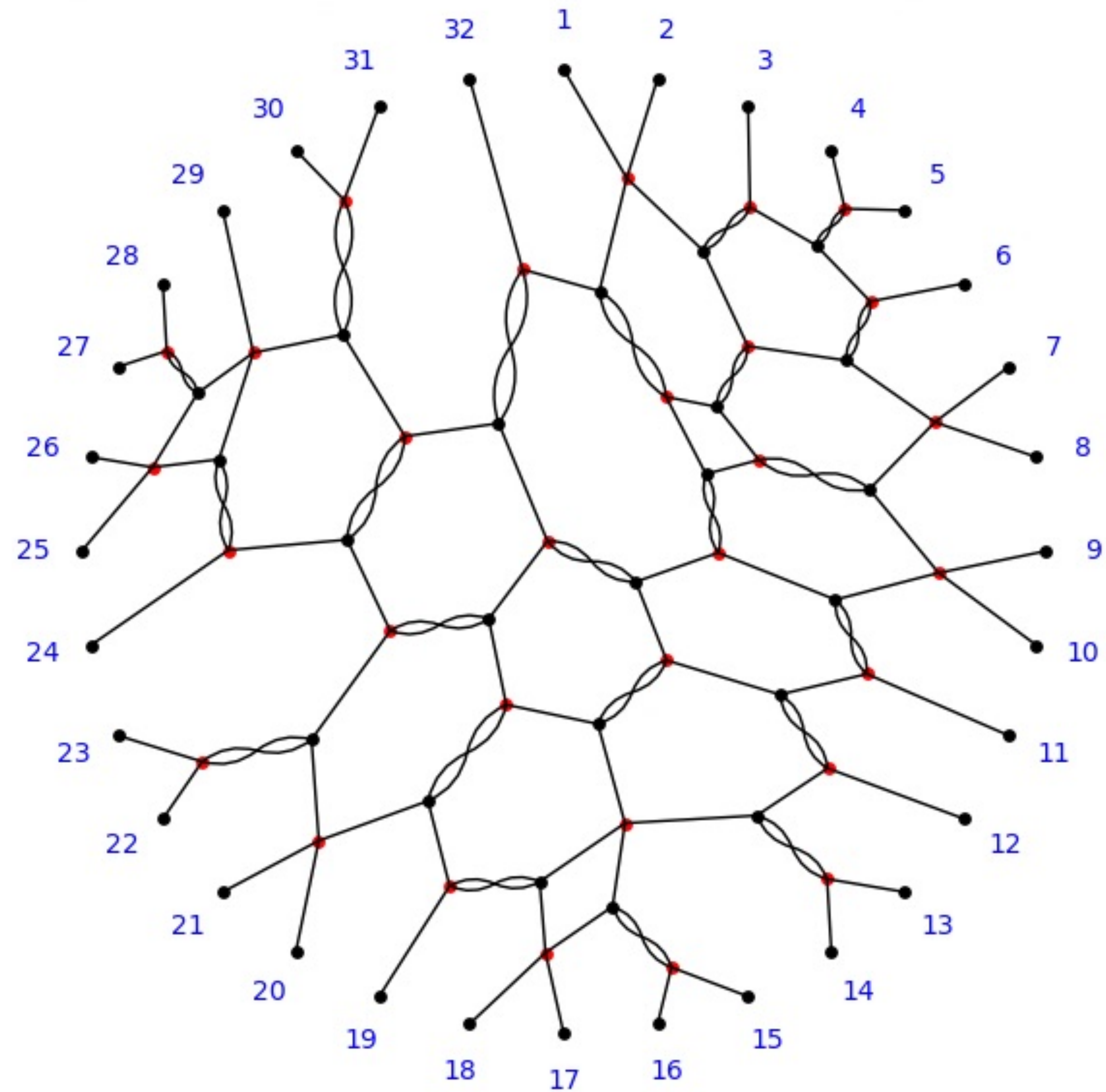
While $[W]$ is a collection of webs, they represent the same tensor invariant.

$$\boxed{\text{Hence we have a rotation-invariant basis of tensor invariants, encoded by webs!}}$$

4-row webs (new!)

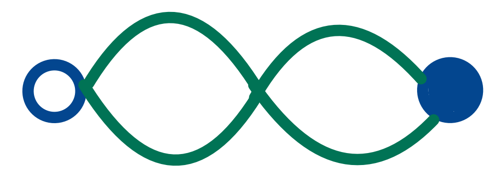
Ex

1	3	4	7	8	17	19	23
2	5	6	9	14	18	21	24
10	12	13	15	16	25	26	28
11	20	22	27	29	30	31	32



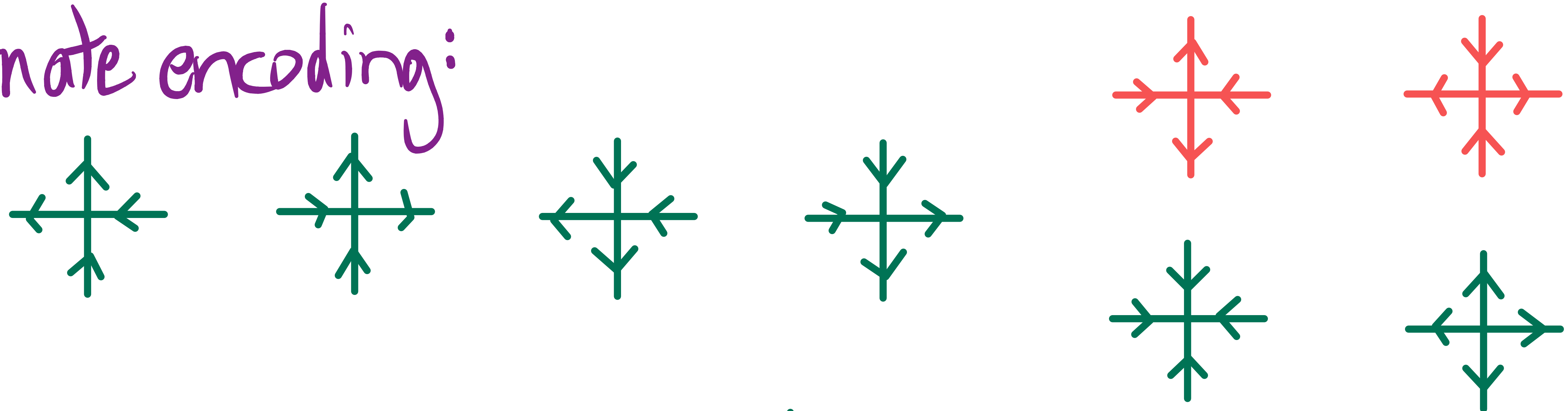
4-row webs (new!)

Def A 4-row web is...

- Planar-embedded graph in disk, allowing hourglass edges 
- 4-valent interior vertices, univalent boundary
- Bipartite, marked "initial" outer vertex

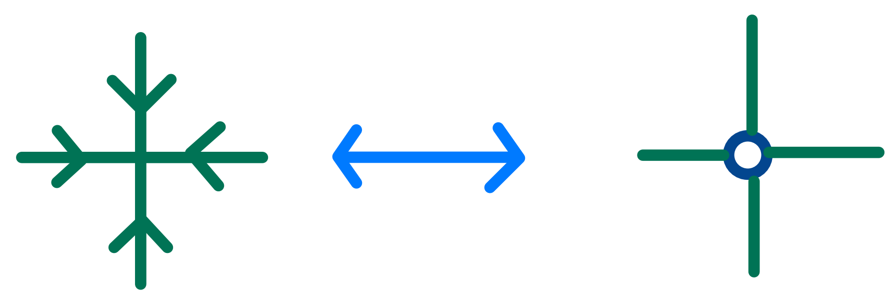
6-vertex model

Alternate encoding:

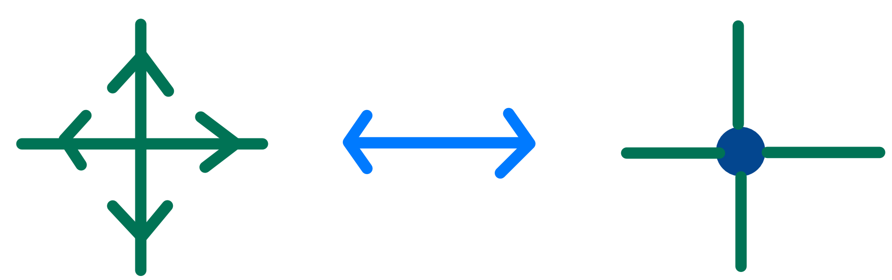


usual

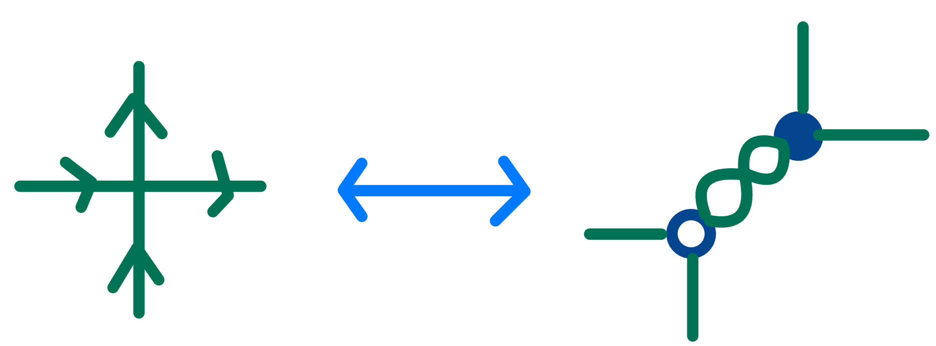
ours



(Sink)



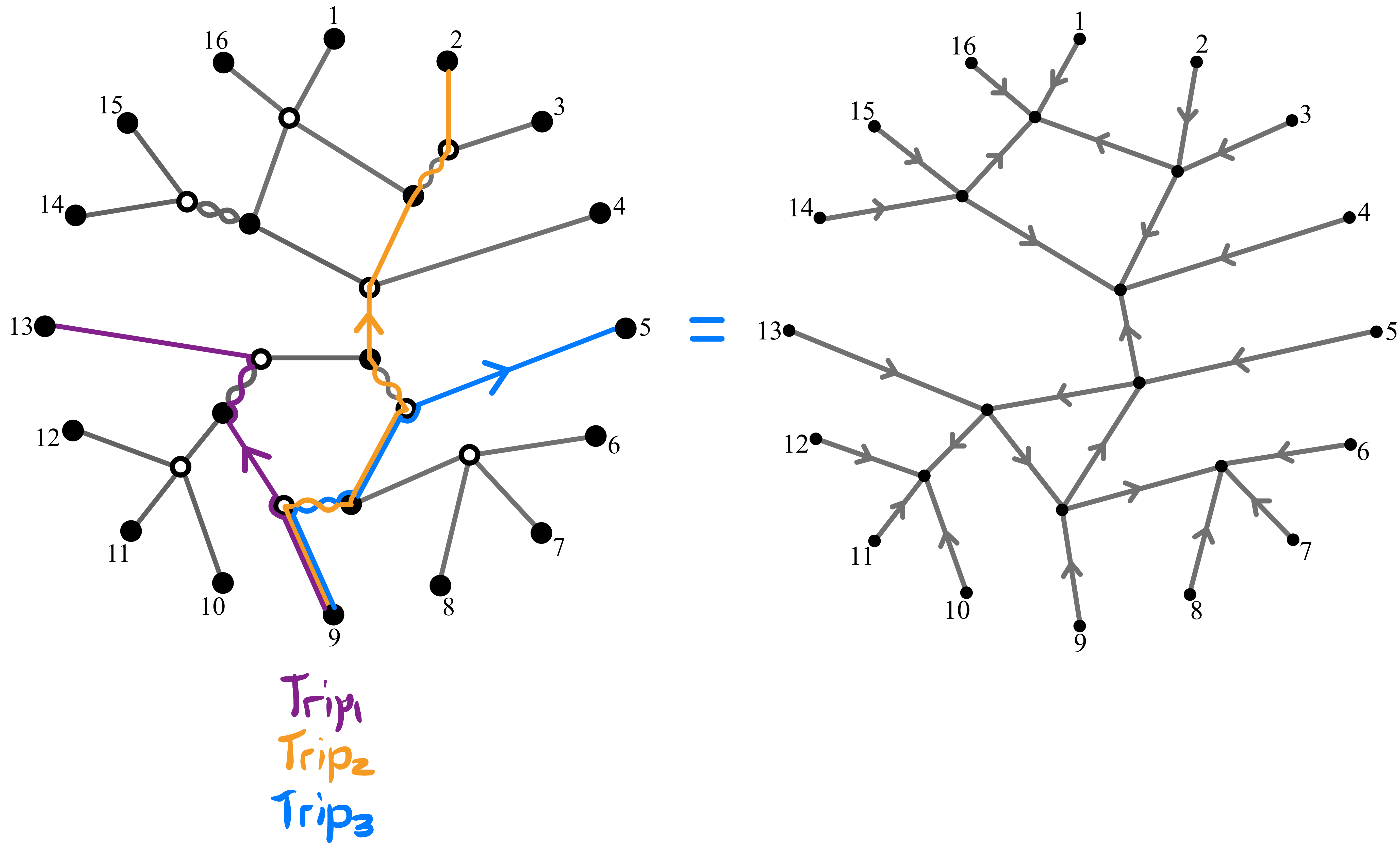
(Source)



(Transmitting)

6-vertex model

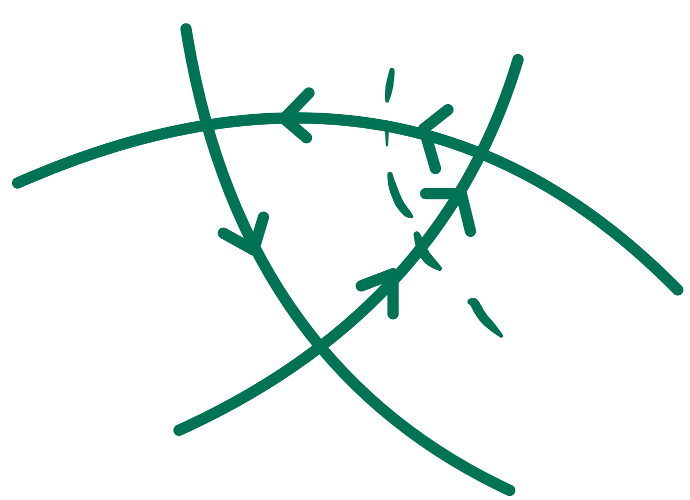
Ex



Top fully reduced

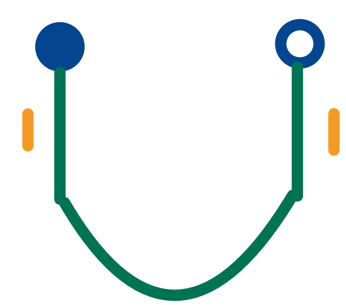
$\boxed{\text{Def}}$ A 4-row basis web is top fully reduced if all triangles* in the 6-vertex configuration are oriented counterclockwise.

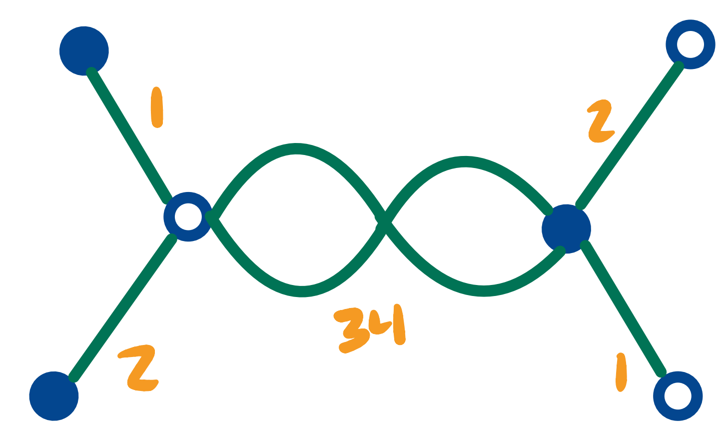
*even "big" triangles

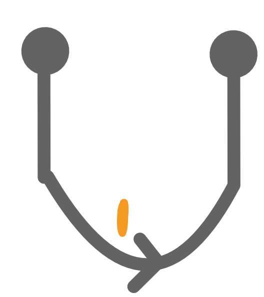


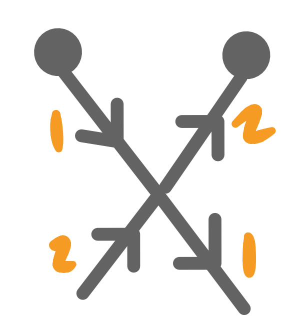
4-row growth rules

Some sample 4-row growth rules:

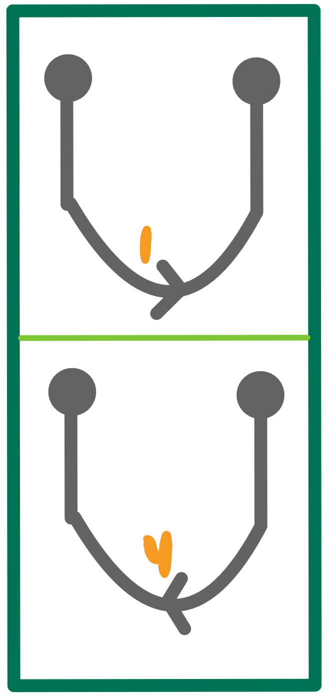


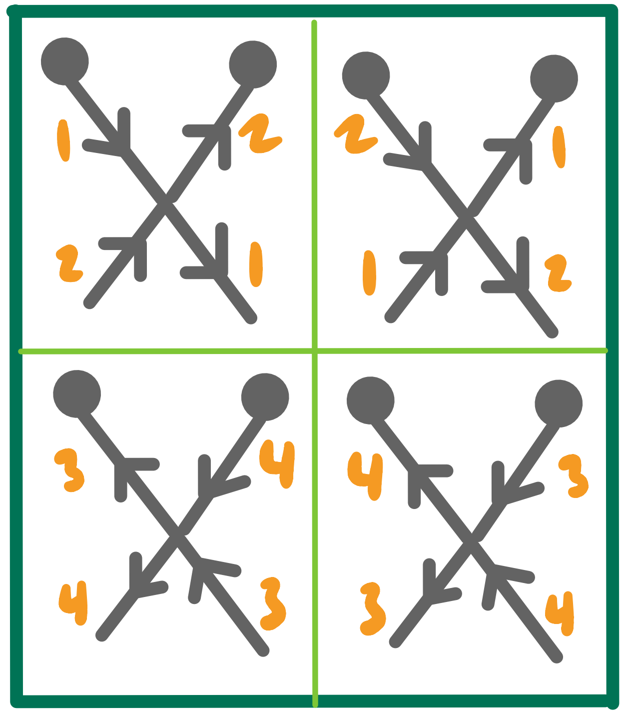


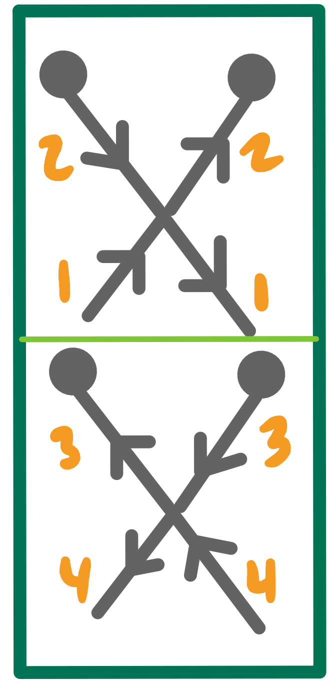


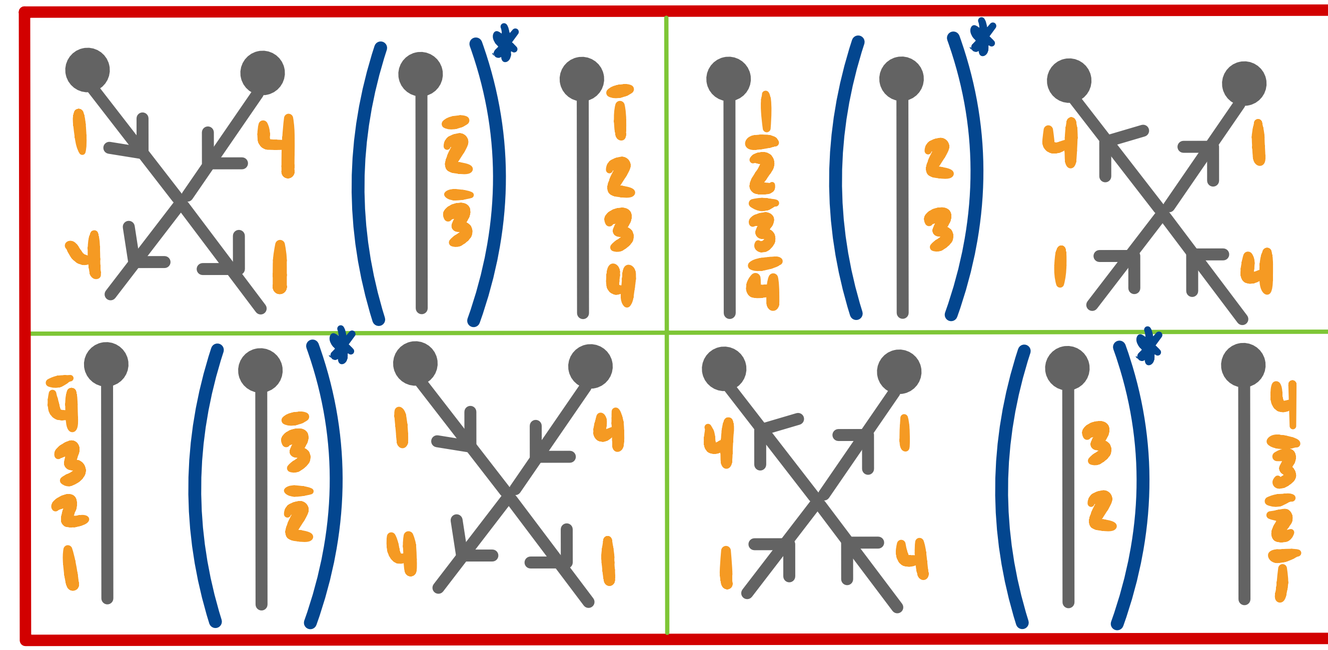


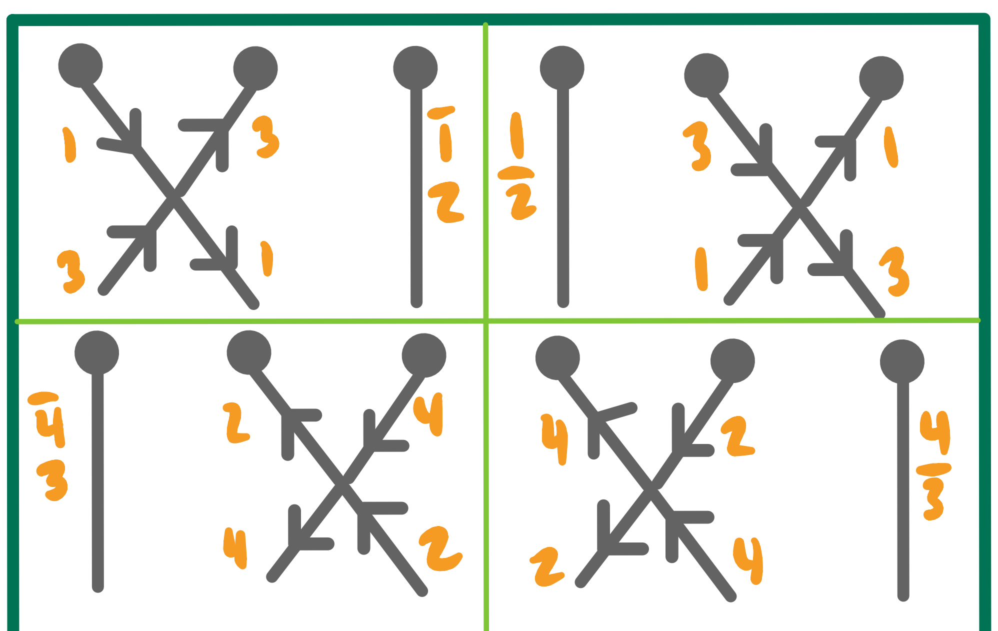
4-row growth rules

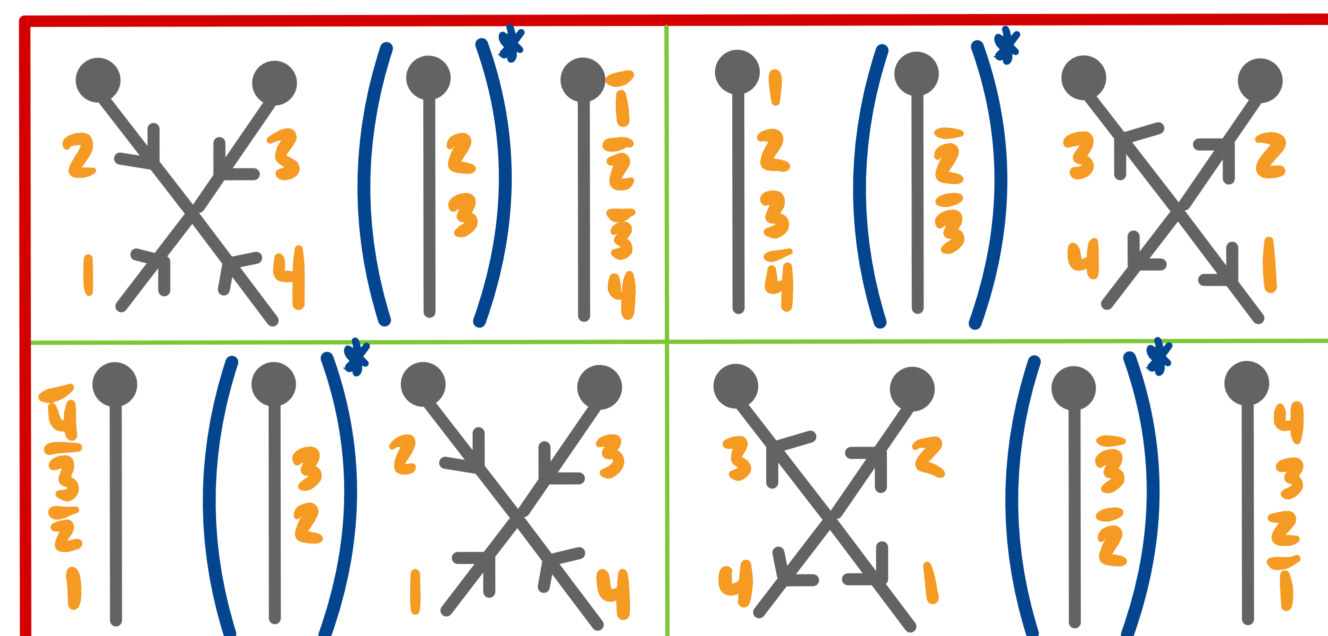




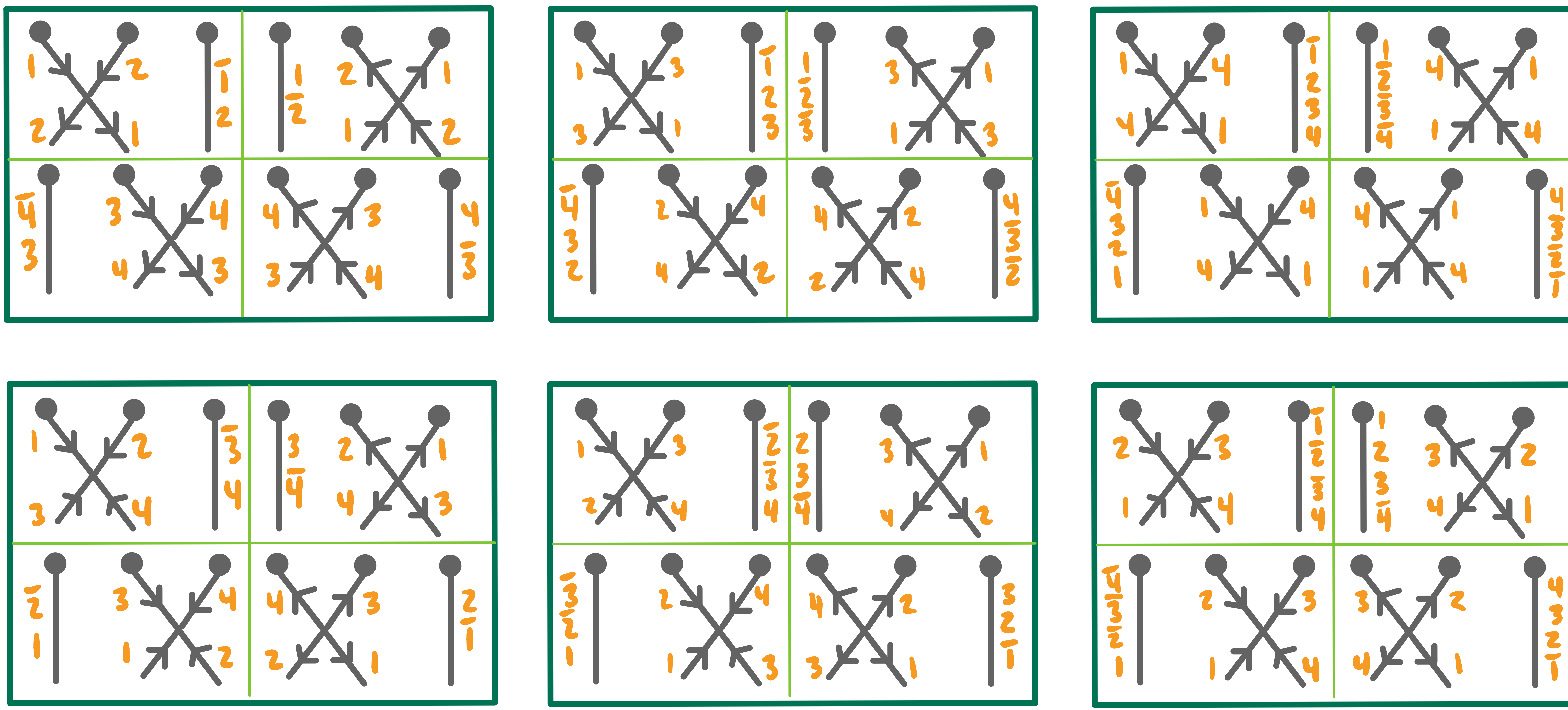








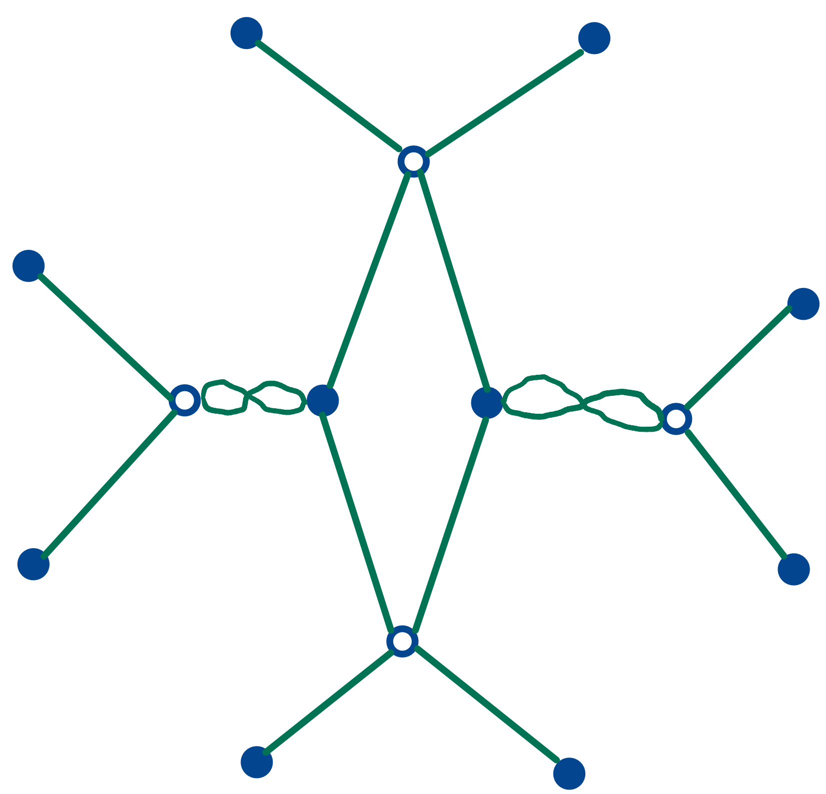
4-row growth rules



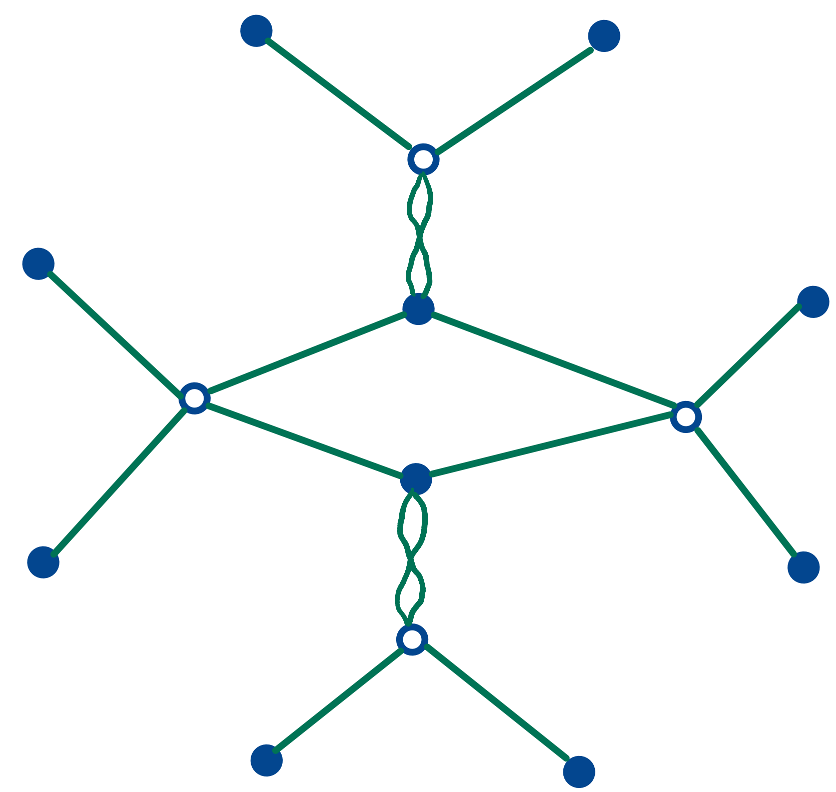
4-row moves

Thm Fip equivalence essentially arises from two moves:

1. Square moves: e.g.

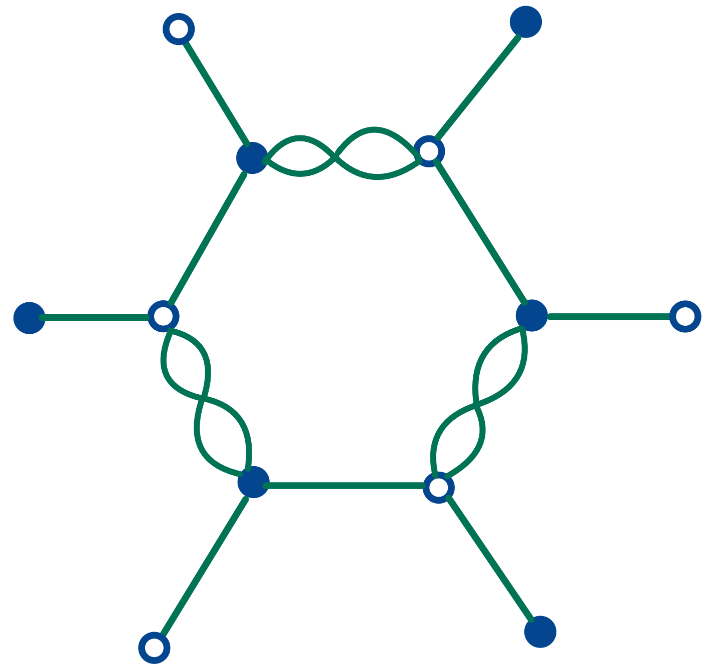


$\cong$

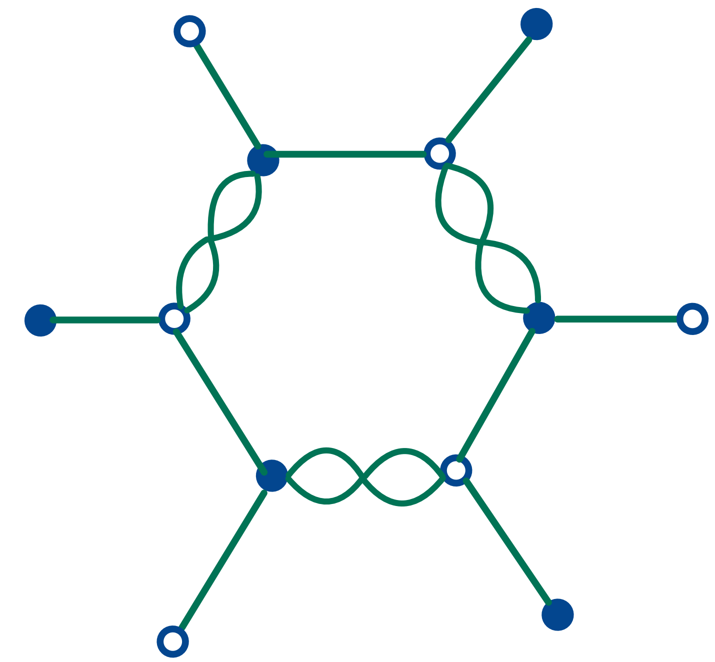


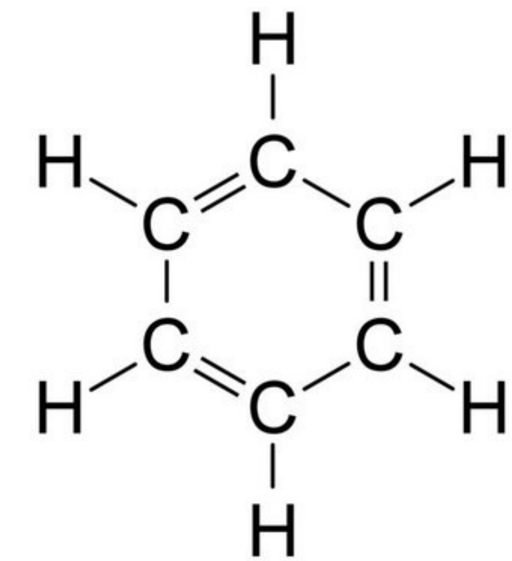
4-row moves

2. Benzene moves:



$\cong$

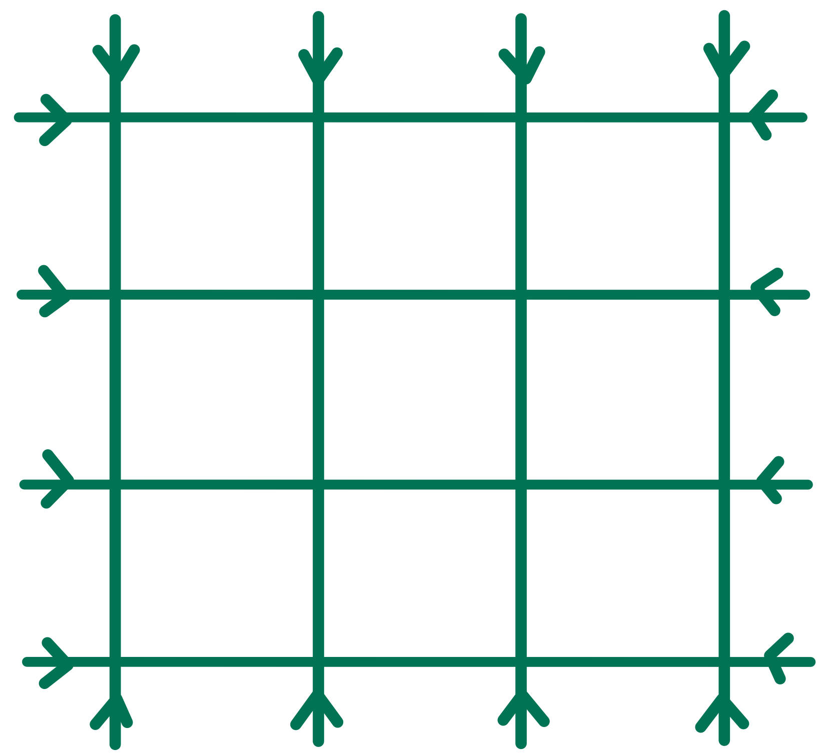




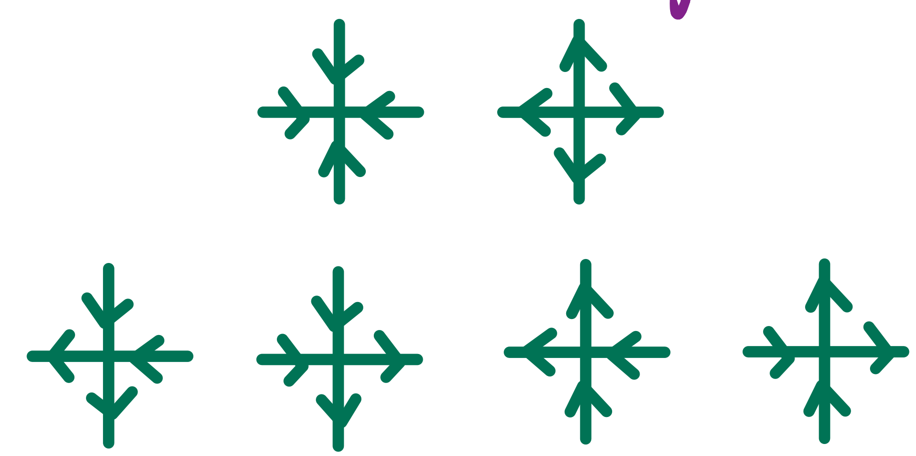
(Actual benzene)

ASM case

Thm The tableau with lattice word $1^n 2^n 3^n 4^n$ has webs



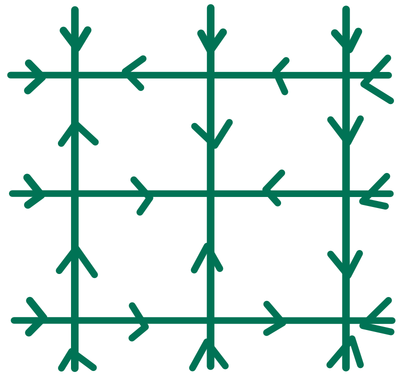
Completed in all ways using...



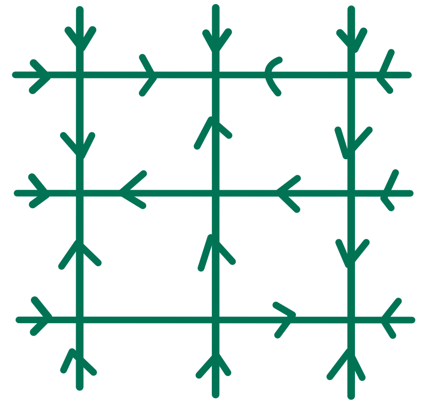
Note These are naturally alternating sign matrices!

ASM case

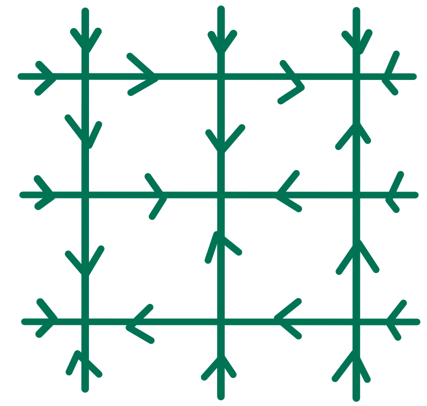
Ex



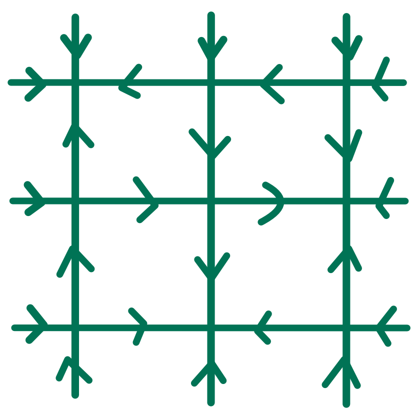
$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$



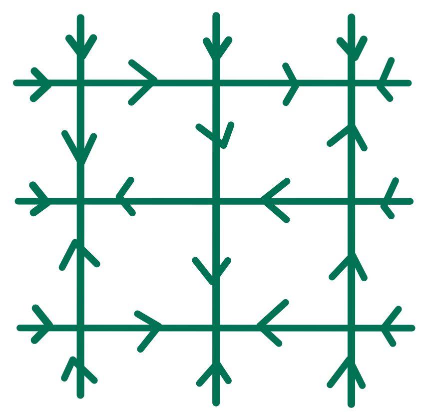
$$\begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{matrix}$$



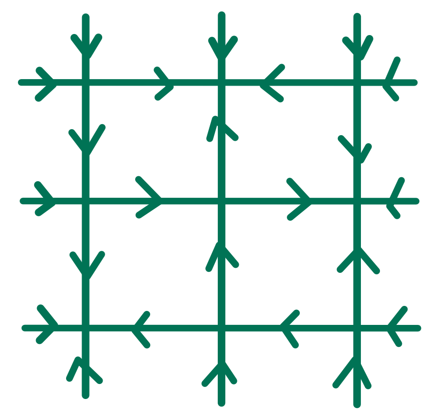
$$\begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix}$$



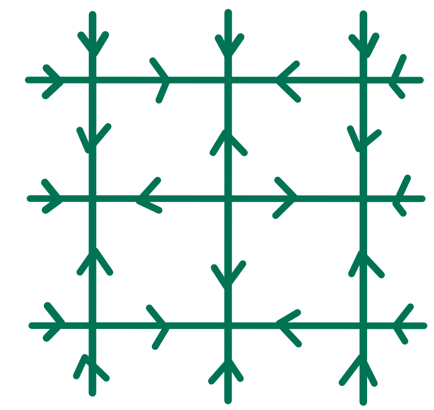
$$\begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}$$



$$\begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}$$



$$\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{matrix}$$



$$\begin{matrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{matrix}$$

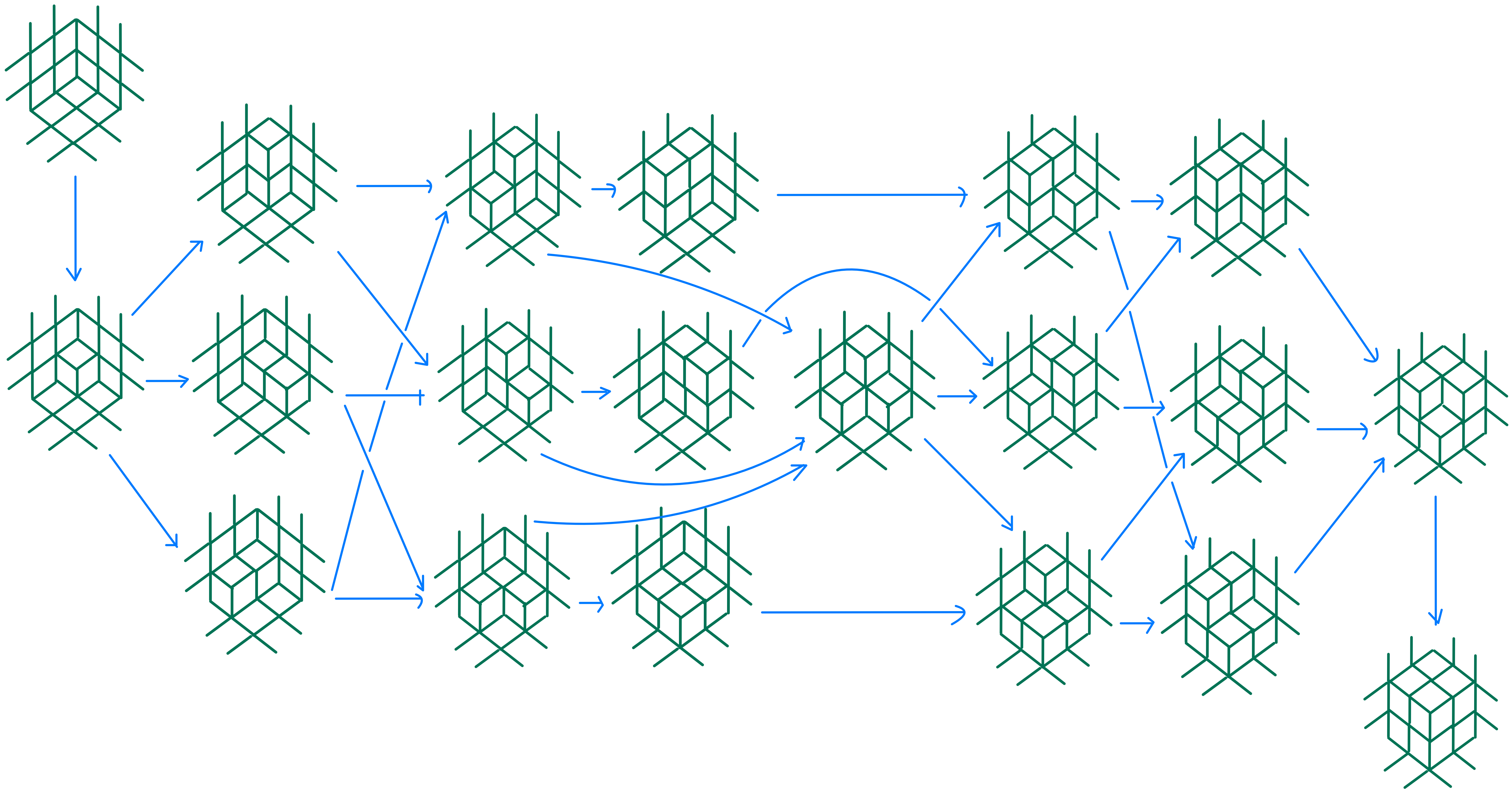
Plane partition case

Thm The tableau with lattice word $1^a \bar{4}^b 2^c \bar{1}^{a-c} \bar{2}^c 4^b \bar{1}^c$ has equivalence class $(a \geq c)$

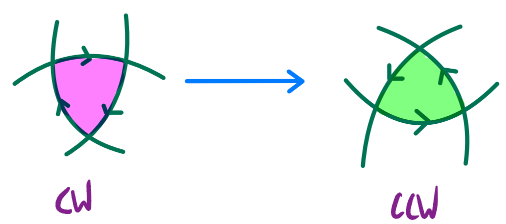
$$\left\{ \begin{array}{c} \text{plane partitions} \\ \text{in the } a \times b \times c \text{ box} \end{array} \right\}$$

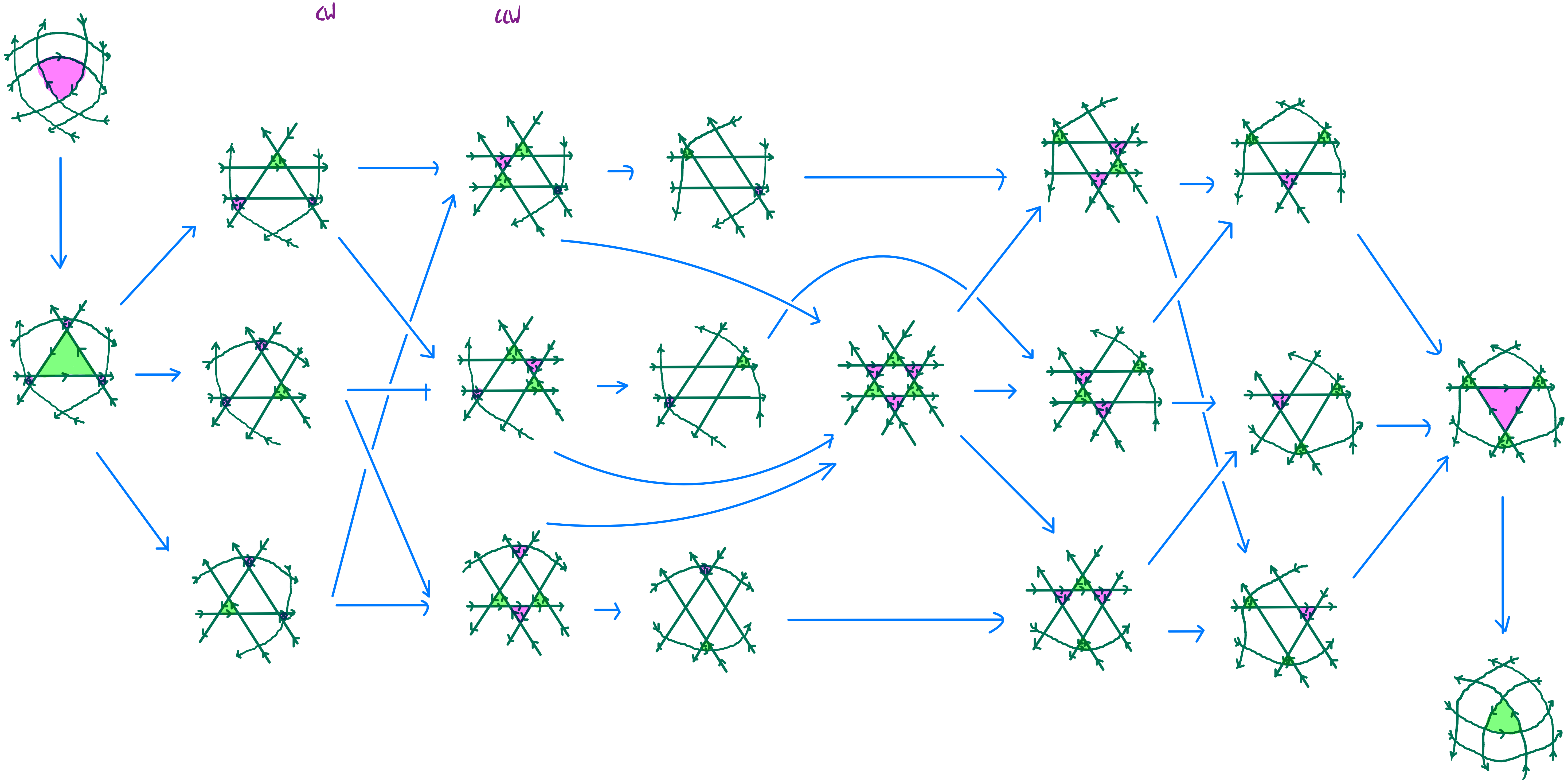
Plane partition case

Ex



Plane partition case





THANKS!