Tanisaki Witness Relations

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Outline

I. Higher coinvariant algebras

II. Tanisaki witness relations

Coinvariant Algebras

Thm (Newton) $\mathbb{Q}[x_1, \ldots, x_n]^{S_n} = \mathbb{Q}[e_1, \ldots, e_n]$ where $e_i = \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k}$ (elementary symmetric polynomial)

and $\sigma(x_i) = x_{\sigma(i)}$

Thm (Hilbert) $\langle \mathbb{Q}[x_1, \ldots, x_n]_+^{S_n} \rangle = \langle e_1, \ldots, e_n \rangle$

Def The coinvariant algebra of S_n is

$$R_n = \frac{\mathbb{Q}[x_1, \ldots, x_n]}{\langle e_1, \ldots, e_n \rangle}$$

Coinvariant Algebras

Thm (Borel) $R_n \cong H^*(Fl_n)$

(singular cohomology over H^*; complete flag manifold under Fl_n)

Thm (Chevalley) $R_n \cong \mathbb{Q}S_n$

$\Rightarrow \dim R_n = n!$

Thm (Artin) $\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} : 0 \leq \alpha_i < i\}$ descends to a basis for R_n

$\Rightarrow \mathrm{Hilb}(R_n; q) = [n]_q!$

Thm (Lusztig-Stanley) $\mathrm{grFrob}(R_n; q) = \sum_{T \in SYT(n)} q^{\mathrm{maj}(T)} s_{sh(T)}$

Diagonal Coinvariant Algebras

Def (Garsia–Haiman '90's)

The <u>diagonal</u> coinvariant algebra of S_n is

$$DR_n = \frac{\mathbb{Q}[\underline{x}_n, \underline{y}_n]}{\langle \mathbb{Q}[\underline{x}_n, \underline{y}_n]^{S_n}_+ \rangle}$$

where S_n acts diagonally:

$$\sigma(x_i) = x_{\sigma(i)}, \quad \sigma(y_i) = y_{\sigma(i)}$$

Thm (Haiman)

$$\mathrm{GrFrob}(DR_n; q, t) = \nabla e_n$$

- Hilbert schemes
- $n!$ conjecture
- Macdonald poly's

Super Coinvariant Algebras

Def (Zabrocki '19)

The super diagonal coinvariants are

$$SDR_n = \mathbb{Q}[\underline{x}_n, \underline{y}_n, \underline{\theta}_n] / \langle \mathbb{Q}[\underline{x}_n, \underline{y}_n, \underline{\theta}_n]^{S_n}_+ \rangle$$

where $x_i y_j = x_j y_i$, $x_i \theta_j = \theta_j x_i$, $y_i \theta_j = \theta_j y_i$, and $\theta_i \theta_j = -\theta_j \theta_i$.

(anti-commute)

Conj (Zabrocki '19)

$$\mathrm{GrFrob}(SDR_n; q, t, z) = \sum_{k=0}^{n-1} z^k \Delta'_{e_{n-k-1}}(e_n)$$

Representation-theoretic model for Delta Conjecture

Super Coinvariant Algebras

- SDR_n is hard!
- From now on, focus on $\boxed{t=0}$ case.

Super Coinvariant Algebras

- **Superspace** is $\underbrace{\mathbb{Q}[x_1, \ldots, x_n, \theta_1, \ldots, \theta_n]}_{\mathrm{Sym}(x_1, \ldots, x_n) \otimes \Lambda(\theta_1, \ldots, \theta_n)}$ where $\theta_i \theta_j = -\theta_j \theta_i$ <u>anti-commute</u> (and $x_i \theta_j = \theta_j x_i$, $x_i x_j = x_j x_i$)

- S_n acts **diagonally**: $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(\theta_i) = \theta_{\sigma(i)}$

- Think of θ variables as differential forms <u>$\theta_i = dx_i$</u>,

$$\theta_i \theta_j = dx_i \wedge dx_j$$

Super Coinvariant Algebras

- The exterior derivative is

$$d = \sum_{i=1}^{n} \partial_{x_i} dx_i \in \mathrm{End}_{\mathbb{Q}}(\mathbb{Q}[\underline{x}_n, d\underline{x}_n])$$

<u>Thm</u> (Solomon)

$$\langle \mathbb{Q}[\underline{x}_n, d\underline{x}_n]^{S_n}_{+} \rangle = \langle e_1, \ldots, e_n, dp_1, \ldots, dp_n \rangle$$

Super Coinvariant Algebras

Def The **super coinvariant algebra** of S_n is

$$SR_n = \mathbb{Q}[\underline{x}_n, \underline{\theta}_n] / \langle \mathbb{Q}[\underline{x}_n, \underline{\theta}_n]_+^{S_n} \rangle.$$

Conj (Zabrocki '19; Haglund-Rhoades-Shimozono '18)

$$\mathrm{Hilb}(SR_n; q, z) = \sum_{k=1}^{n} [k]! \, S[n,k] \, z^{n-k}$$

$$\mathrm{GrFrob}(SR_n; q, z) = \sum_{\mu \vdash n} z^{n-\ell(\mu)} q^{\sum_{i=1}^{\ell(\mu)} (i-1)(\mu_i - 1)} \binom{\ell(\mu)}{m_1(\mu), \ldots, m_n(\mu)}_q \omega Q'_\mu(\underline{x}; q)$$

Operator Theorem

- Top-degree component of R_n is spanned by

$$\Delta_n = \prod_{1 \le i < j \le n} (x_j - x_i)$$

Thm (Steinberg)

$$\mathbb{Q}[\partial_{x_1}, \ldots, \partial_{x_n}]\Delta_n \xrightarrow{\sim} R_n$$ is a bijection!

Super Operator Theorem

Thm (S.-Wallach) Alternating component of SR_n has basis

$$\{d_I \Delta_n : I \subset [n-1]\}$$

where $d_I = d_{i_1} \cdots d_{i_k}$ for $d_i = \sum_{j=1}^{n} \partial_j^i dx_j$.

Thm (Rhoades-Wilson 23+; conjectured by S.-Wallach)

$$\mathbb{Q}[\partial_{x_1}, \ldots, \partial_{x_n}]\{d_I \Delta_n : I \subset [n-1]\}\Delta_n \xrightarrow{\sim} SR_n$$

is a bijection!

Flip Action

- Have 2^{n-1} "tent poles" $d_I \Delta_n$ which generate SR_n as an $\mathbb{Q}[x_1, \ldots, x_n]$-module under the flip action

$$g \cdot w = g(\partial_{x_1}, \ldots, \partial_{x_n}) w.$$

- HRS conjecture has 2^{n-1} total ${}_wQ'_\mu(\underline{x};q)$'s! (Strong comp's.)

  Q Is there a filtration of SR_n adding one generator at a time whose successive quotients prove the HRS formula?

Tanisaki Ideals

Thm (Tanisaki)

$$\operatorname{GrFrob}(\mathbb{Q}[\underline{x}_n]/I_\mu; q) = \operatorname{rev}_q \tilde{Q}'_\mu(\underline{x}; q)$$

where

- $I_\mu = \langle e_r(S) : |S| - d_{|S|}(\mu) < r \leq |S|, S \subset [n] \rangle$ is a Tanisaki ideal
- $e_r(S) = \sum_{\{i_1 < \cdots < i_r\} \subset S} x_{i_1} \cdots x_{i_r}$
- $d_k(\mu) = \mu'_n + \mu'_{n-1} + \cdots + \mu'_{n-k+1}$

Twists

- Let $SR_I = \mathbb{Q}[\partial_{x_1}, \ldots, \partial_{x_n}]\, d_I \Delta_n$.

- Suppose for illustration that $\operatorname{Ann} SR_I = I_\mu$. Then

$$\operatorname{GrFrob}(SR_I; q) = \underbrace{w}_{d_I\Delta_n \text{ is alternating}}\, q^{\overbrace{\binom{n}{2}}^{\deg \Delta_n} \overbrace{- i_1 - \cdots - i_k}^{d_I}}\, \operatorname{GrFrob}(\mathbb{Q}[\underline{x}_n]/I_\mu; \underbrace{q^{-1}}_{\text{flip lowers deg.}})$$

$$= w q^{\cdots} Q'_\mu(\underline{x}; q)$$

- Flip action can fully account for w and rev_q twists!

Filtration Approach

Q Is there a total order $I_1 < I_2 < \cdots$ on $2^{[n-1]}$ and a bijection $\Phi_n : 2^{[n-1]} \to \{\kappa \vDash n\}$ such that the successive filtration quotients

$$\sum_{j \leq m} SR_{I_j} \Big/ \sum_{j < m} SR_{I_j}$$

are annihilated precisely by the Tanisaki ideal $\mathcal{I}_{\Phi_n(I_m)}$?

- Would prove HRS formula, and give a basis!
- RW 23+ also proved Hilbert series formula!
  $\Rightarrow$ Just need "enough relations"!

Generic Tanisaki Witness Relations

Thm (S. '23) Let $I = \{i_1 < \cdots < i_k\} \subset [n-1]$. Then

$$\sum (-1)^d \partial_{e_{n-k-d}(\underline{n-1})} d_{j_1 \cdots j_k} \Delta_n = 0$$

"generic" Tanisaki witness relations

where the sum is over all subsets $J = \{j_1 < \cdots < j_k\} \subset [n-1]$ where

$$1 \leq i_1 \leq j_1 < i_2 \leq j_2 < \cdots < i_k \leq j_k < n$$

and

$$d = (j_1 - i_1) + \cdots + (j_k - i_k).$$

Generic Tanisaki Witness Relations

- Using a (non-trivial) bijection from [S.'23], solves 1-form case.

Ex $n=3$, $k=1$ has $\alpha \in \{(2,1),(1,2)\}$ with $I \in \{\{1\},\{2\}\}$.

Ideal $\mathcal{I}_{(2,1)}$ generated by $e_1(\underline{3}), e_2(\underline{3}), e_3(\underline{3})$ and $\mathfrak{S}_3 \cdot \underbrace{e_2(\underline{2})}_{x_1 x_2}$.

Have

$$\partial_{e_2(\underline{2})} d_{\{2\}} \Delta_3 = 0$$

$$\partial_{e_2(\underline{2})} d_{\{1\}} \Delta_3 = \partial_{e_1(\underline{2})} d_{\{2\}} \Delta_3 .$$

In fact, annihilators of $SR_{\{2\}}$ and $(SR_{\{1\}} + SR_{\{2\}})/SR_{\{2\}}$ are precisely $\mathcal{I}_{(2,1)}$.

Extreme Tanisaki Witness Relations

Thm (S. '23) Let $I=\{i_1<\cdots<i_k\}\subset[n-1]$ have $1\le s\le k$ s.t.

$i_1,\ldots,i_{k-s+1}\le n-k$ and $i_{k-s+1+j}\le n-s+j$ for $1\le j<s$.

Pick $0\le u\le s$. Then

$$\sum(-1)^d\Delta_s(j_{k-s+1},\ldots,j_k)\binom{d+u}{u}\partial_{e_{n-s-d(n-s+u)}}d_J\Delta_n=0$$

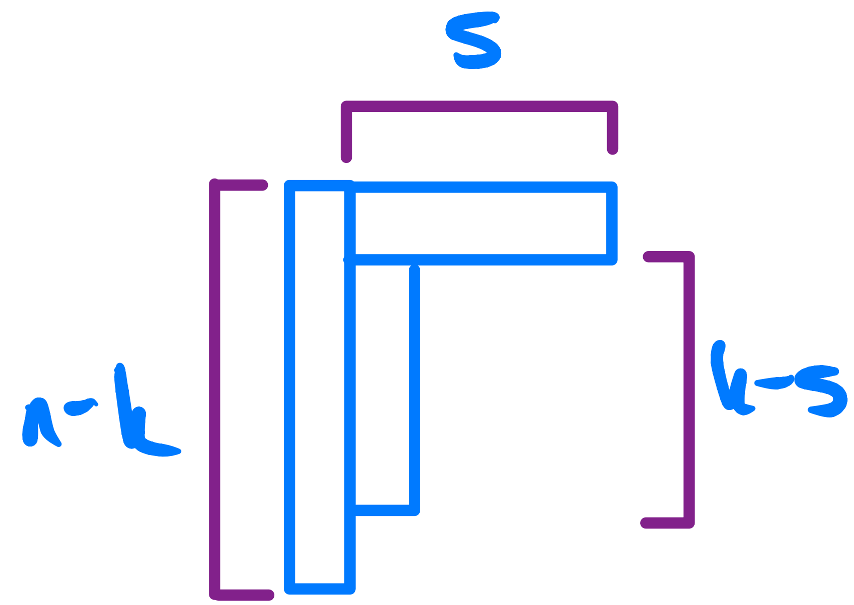
where the sum is over $J=\{j_1<\cdots<j_k\}\subset[n-1]$ s.t.

$$j_1=i_1,\ldots,j_{k-s}=i_{k-s}$$

$$d=(j_{k-s+1}-i_{k-s+1})+\cdots+(j_k-i_k)\ge 0.$$

Extreme Tanisaki Witness Relations

- Gives relations for

$\mu =$ 

- Both families "graded-positive" up to $(-1)^d$!

Ex

$$0 = 5\partial_{e_5(\underline{s})} d_{16}\Delta_7 - 4\partial_{e_4(\underline{s})} d_{26}\Delta_7 + 3\partial_{e_3(\underline{s})} d_{36}\Delta_7$$

$$- 2\partial_{e_2(\underline{s})} d_{46}\Delta_7 + \partial_{e_1(\underline{s})} d_{56}\Delta_7$$

$$+ 3\partial_{e_5(\underline{s})} d_{25}\Delta_7 - 2\partial_{e_4(\underline{s})} d_{35}\Delta_7 + \partial_{e_3(\underline{s})} d_{45}\Delta_7$$

$$+ \partial_{e_5(\underline{s})} d_{34}\Delta_7 .$$

Further Directions

- Main problem: find enough relations to complete the program!

Q) What is a combinatorial description for their coefficients?

Is there a geometric/algebraic/topological interpretation?

graded-positivity gives hope!

THANKS!