Promotion permutations

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Based on joint work with *Christian Gaetz, Oliver Pechenik, Stephan Pfannerer, and Jessica Striker* (submitted)

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Slides: https://www.jpswanson.org/talks/2024_SoCalDM_prom.pdf

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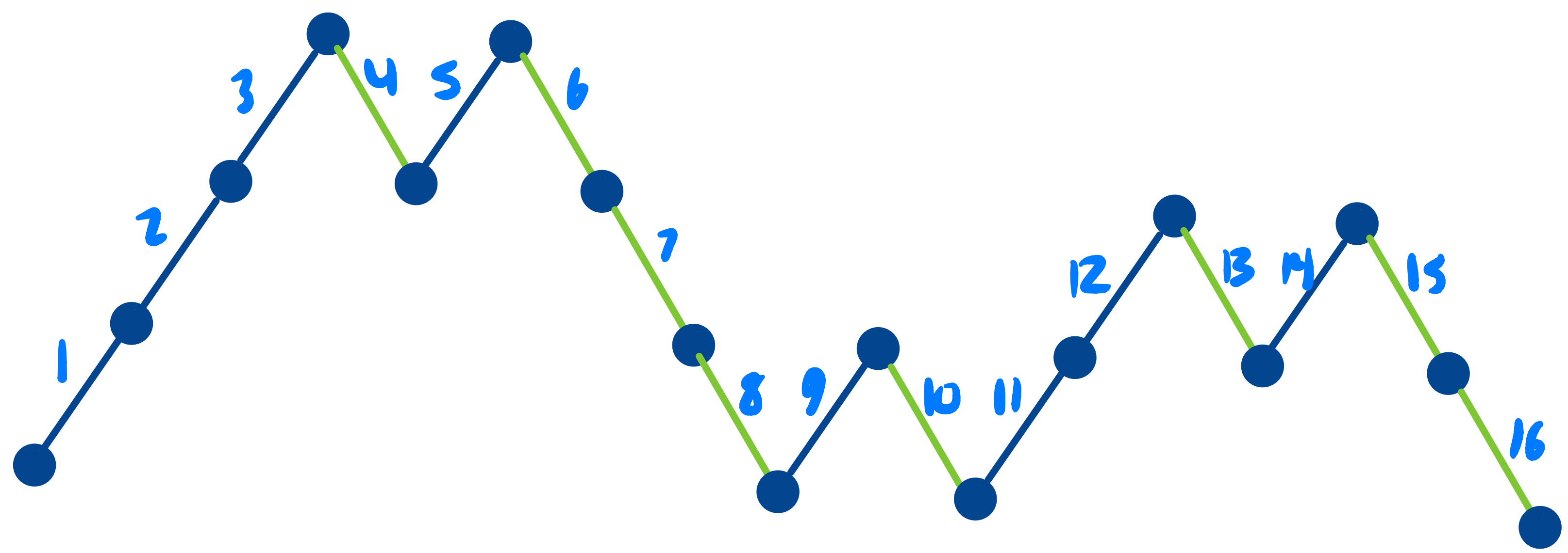
Outline

- Catalan and tableaux combinatorics
- Dihedral models and webs
- Promotion permutations

Catalan objects

Some Catalan bijections:



Dyck path

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

↦ ((()()))()(()())

↦ 1112122212112122

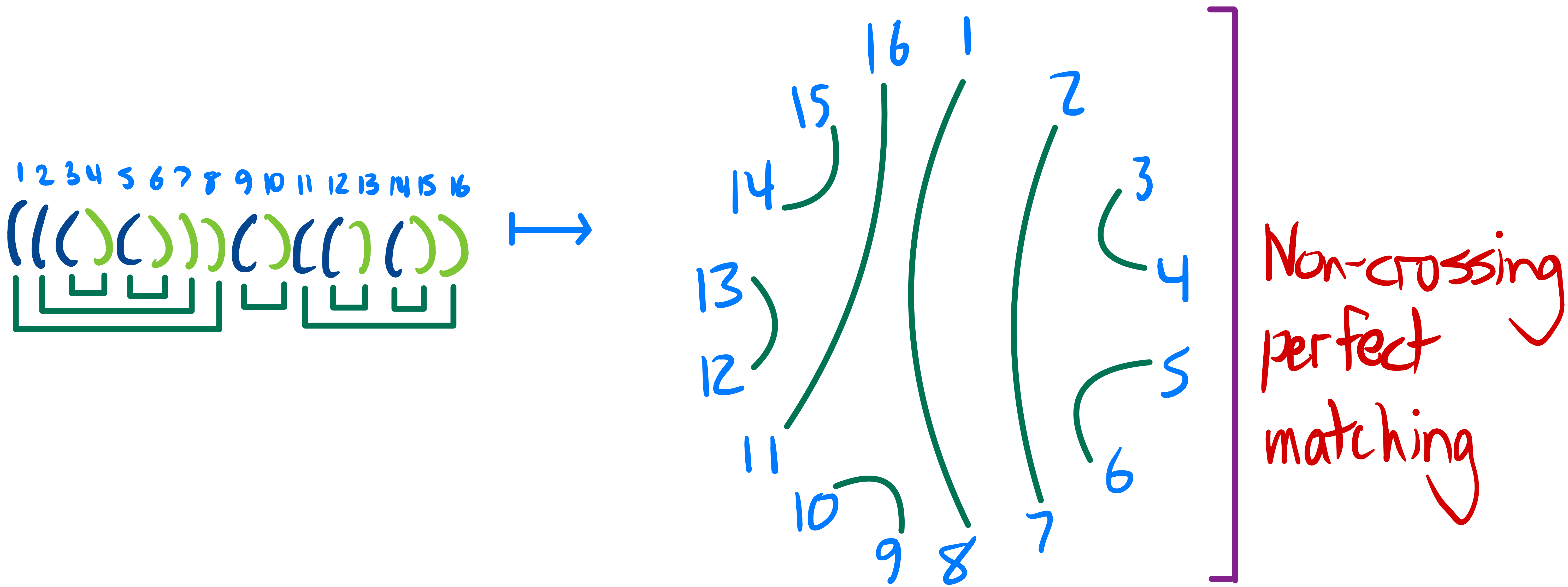
lattice word

↦

1	2	3	5	9	11	12	14
4	6	7	8	10	13	15	16

Standard Young tableau

Catalan objects

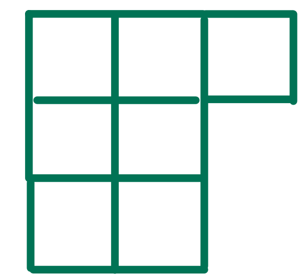


Q: There is a natural dihedral action on NCM's.
Is there an <u>intrinsic</u> description on tableaux?

Tableaux combinatorics

Def A **partition** of n is a list $\lambda = (\lambda_1, \dots, \lambda_k)$ s.t. $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 0$, $\lambda_1 + \dots + \lambda_k = n$.

Ex $\lambda = (3,2,2)$ has **diagram**



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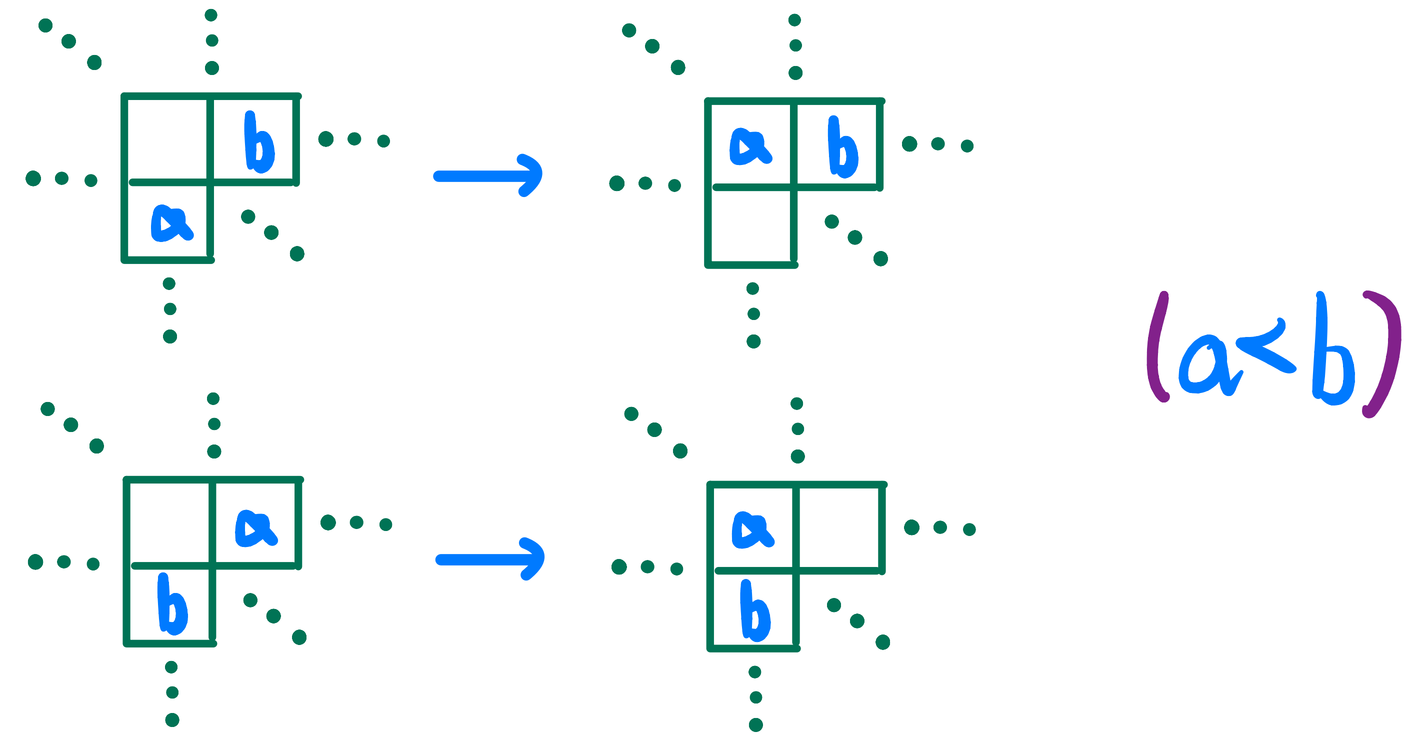
Def A **standard Young tableau** of shape λ is a filling of the diagram of λ with $1, 2, \dots, n$ increasing along rows and down columns.

Ex $T = \begin{array}{|c|c|c|} \hline 1 & 3 & 7 \\ \hline 2 & 5 \\ \cline{1-2} 4 & 6 \\ \cline{1-2} \end{array} \in SYT(\lambda)$

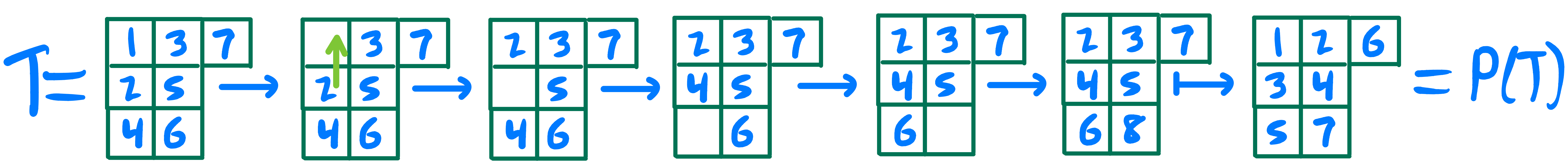
Tableaux combinatorics

Def (~Schützenberger '77) Promotion on $T \in SYT(\lambda)$:

1. Delete 1
2. Slide!
3. Fill hole with $n+1$, decrement all by 1, yields $P(T) \in SYT(\lambda)$



Ex



Tableaux combinatorics

Def (Schützenberger) Evacuation on $T \in SYT(\lambda)$:

1. Delete 1, slide; delete 2, slide; eventually get $\emptyset$
2. Record sequence of holes in new tableaux $\mathcal{E}(T)$

Ex

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 7 \\ \hline 2 & 5 & \\ \cline{1-2} 4 & 6 & \\ \cline{1-2}\end{array} \to \begin{array}{|c|c|c|}\hline 2 & 3 & 7 \\ \hline 4 & 5 & \\ \cline{1-2} 6 & 1 & \\ \cline{1-2}\end{array} \to \begin{array}{|c|c|c|}\hline 3 & 5 & 7 \\ \hline 4 & 2 & \\ \cline{1-2} 6 & 1 & \\ \cline{1-2}\end{array} \to \begin{array}{|c|c|c|}\hline 4 & 5 & 7 \\ \hline 6 & 2 & \\ \cline{1-2} 3 & 1 & \\ \cline{1-2}\end{array} \to \begin{array}{|c|c|c|}\hline 5 & 7 & 4 \\ \hline 6 & 2 & \\ \cline{1-2} 3 & 1 & \\ \cline{1-2}\end{array} \to \begin{array}{|c|c|c|}\hline 6 & 7 & 4 \\ \hline 5 & 2 & \\ \cline{1-2} 3 & 1 & \\ \cline{1-2}\end{array} \to \begin{array}{|c|c|c|}\hline 7 & 6 & 4 \\ \hline 5 & 2 & \\ \cline{1-2} 3 & 1 & \\ \cline{1-2}\end{array} \to \begin{array}{|c|c|c|}\hline 7 & 6 & 4 \\ \hline 5 & 2 & \\ \cline{1-2} 3 & 1 & \\ \cline{1-2}\end{array}$$

$$\mapsto \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 & \\ \cline{1-2} 5 & 7 & \\ \cline{1-2}\end{array} = \mathcal{E}(T)$$

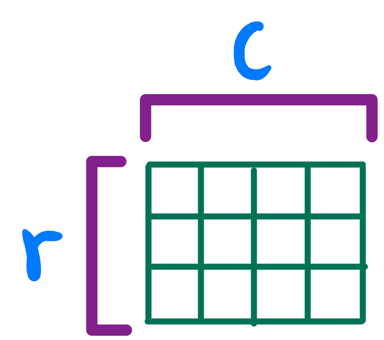
Tableaux combinatorics

Thm P and $\mathcal{E}$ are bijections on SYT. Moreover:

a) $\mathcal{E}^2 = id$

b) $\mathcal{E} P \mathcal{E} = P^{-1}$

Recall $Dih_{2n} = \langle r, s \mid r^n = s^2 = 1, srs = r^{-1} \rangle$

Thm If $\lambda = (c^r) =$ 

$=$ "$r \times c$" is rectangular, then $P^n = id$.

$(n = rc)$

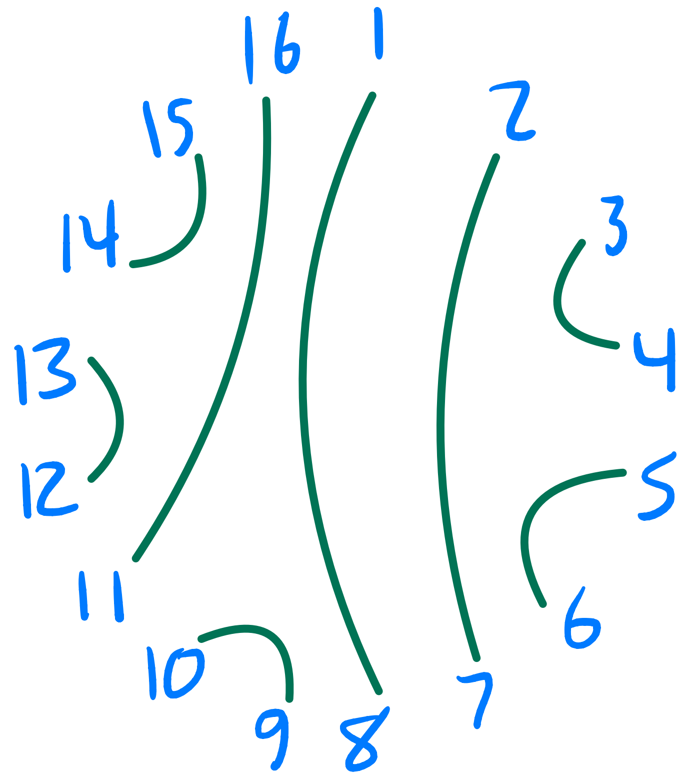
Catalan objects

Thm The bijection $NCM(2c) \xrightarrow{\sim} SYT(2 \times c)$ sends rotation to promotion, reflection to evacuation.

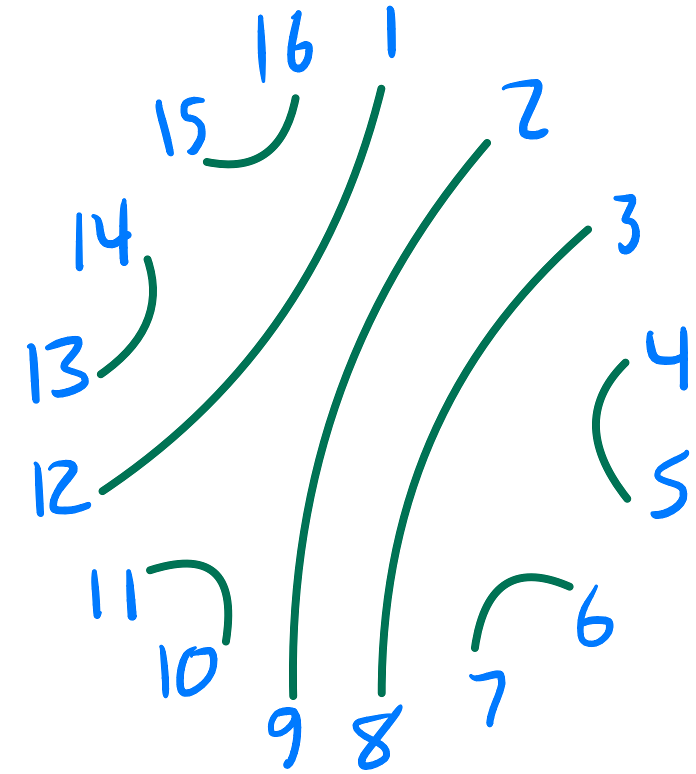
- Explains "hidden" dihedral action on $SYT(2 \times c)$!

Catalan objects

Ex



Rotate ⟼



↧ ↧

1	2	3	5	9	11	12	14
4	6	7	8	10	13	15	16

Promote ⟼

1	2	3	4	6	10	13	15
5	7	8	9	11	12	14	16

- Reflection through diameter between 1 and n gives evacuation

Dihedral models

Q Is there a natural combinatorial model for rectangular tableaux with $r > 2$ which explains the dihedral action?

A $(r=3)$ Kuperberg '96 introduced a "nonelliptic web basis" for $U_q(\mathfrak{sl}_3)$ indexed by $SYT(3 \times c)$.

SL_3-Web basis

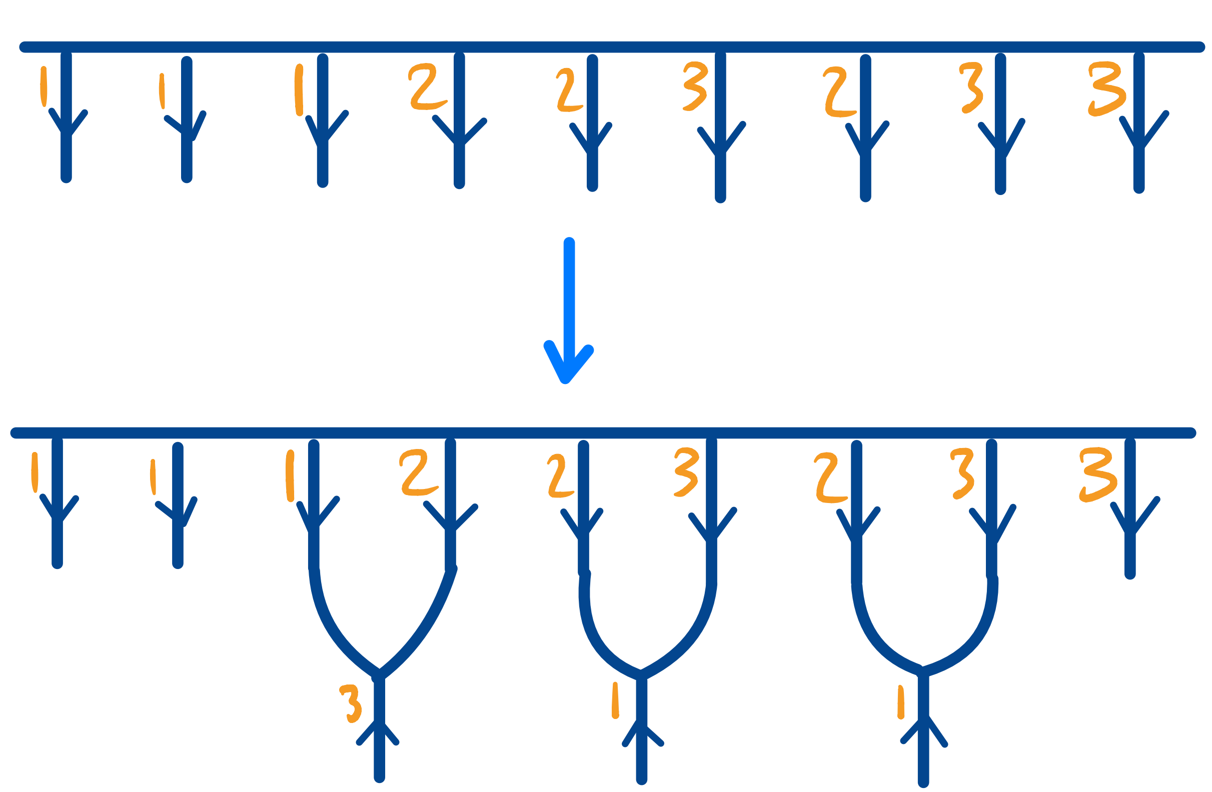
Ex

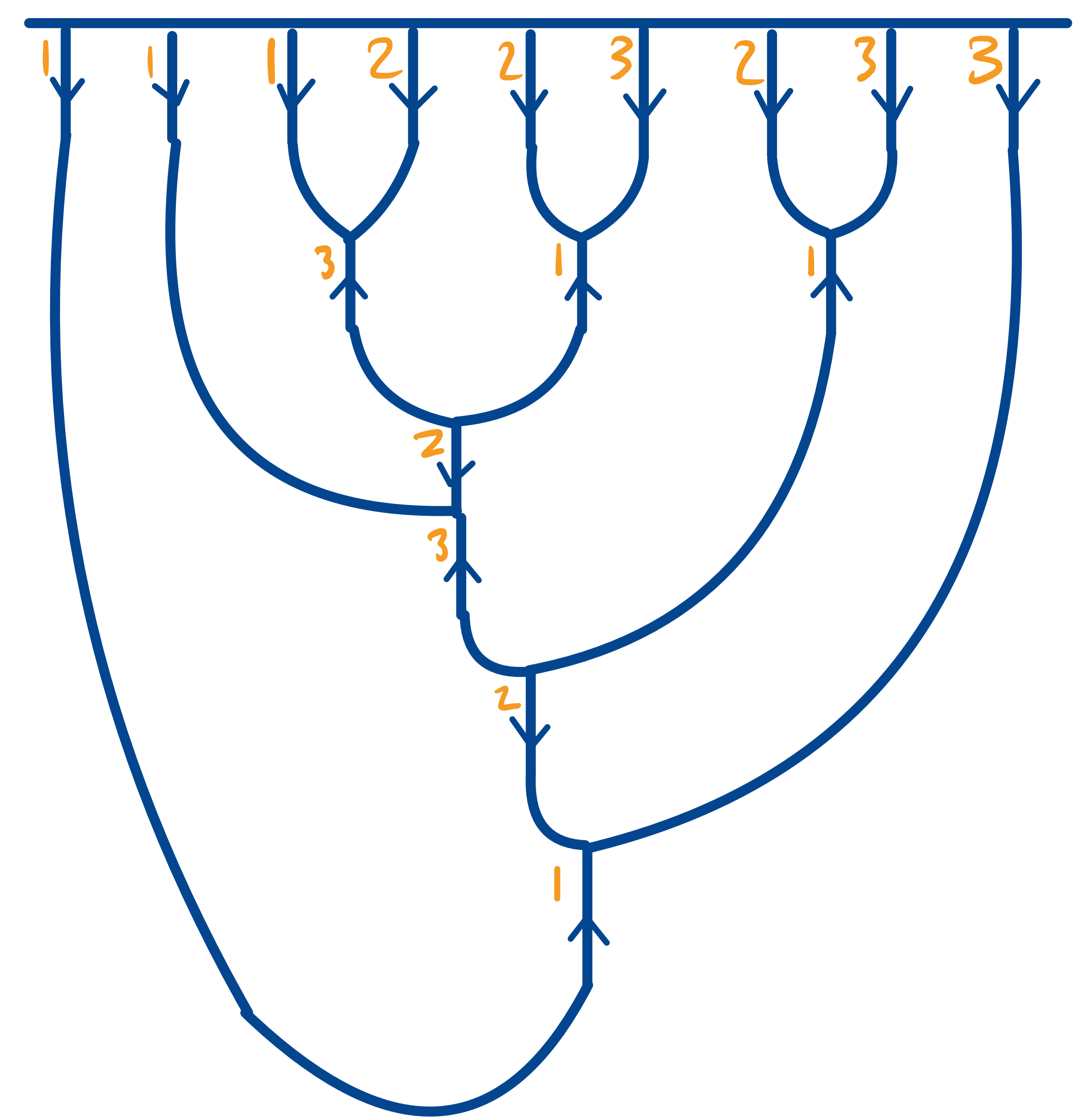
$T =$

1	2	3
4	5	7
6	8	9

→ 111223233

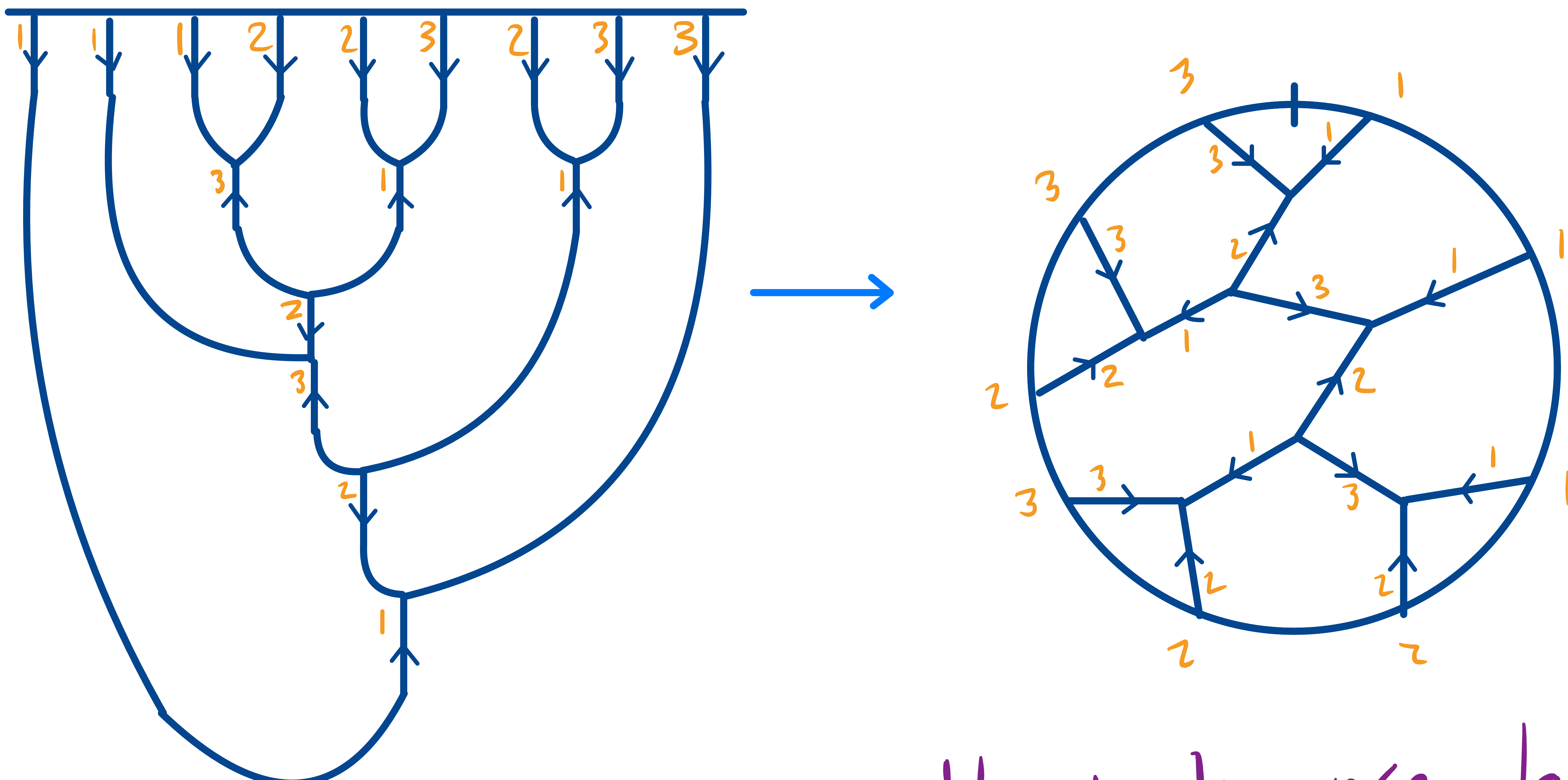
Khovanov-Kuperberg "growth algorithm"





SL_3-Web basis

$(T \rightarrow 111223233)$



Now just erase labels!

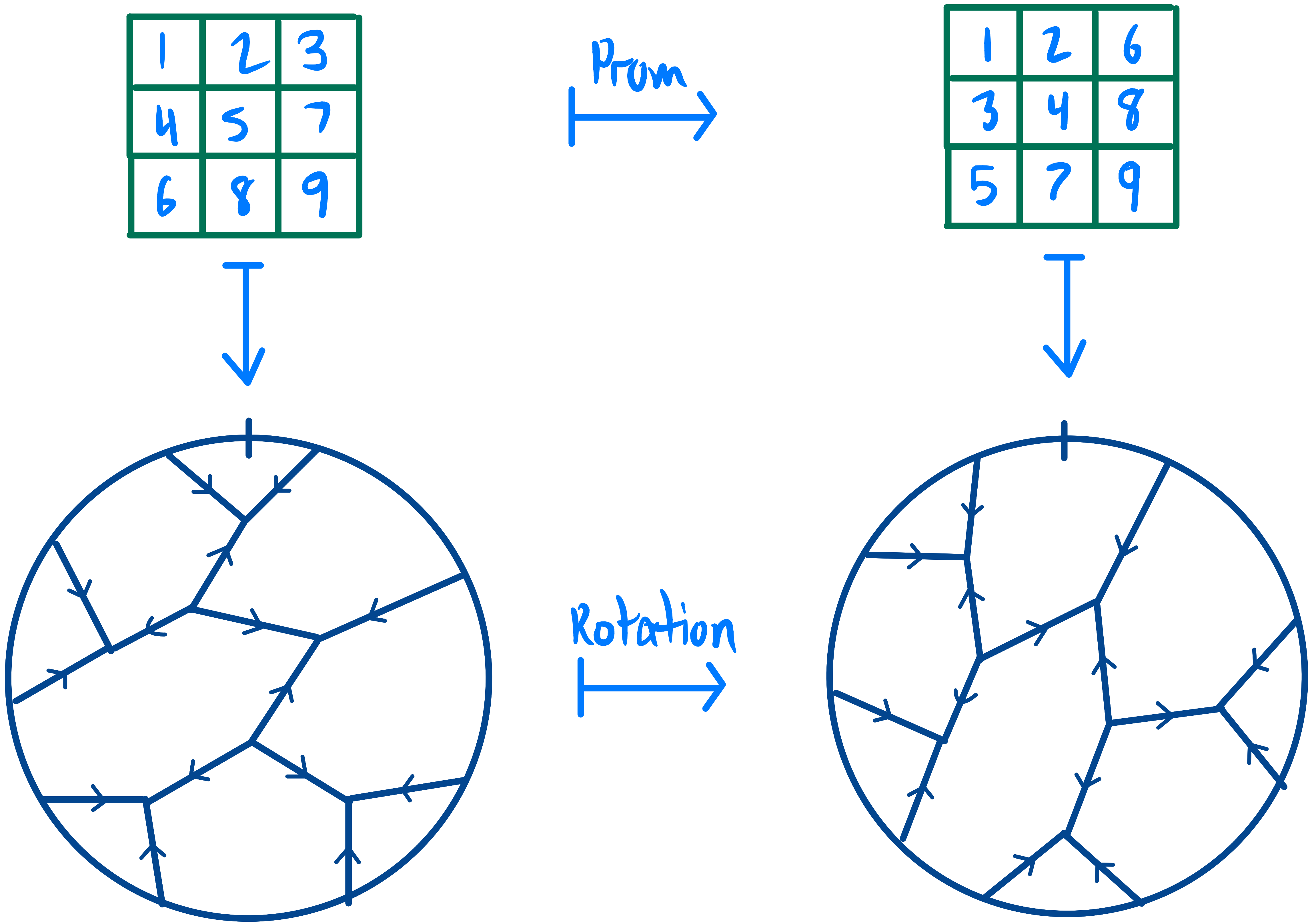
SL_3-Web basis

Thm (Petersen–Pylyavskyy–Rhoades '09, Patrias–Pechenik '21)
The bijection from non-elliptic webs to $SYT(3\times c)$ sends rotation to promotion
reflection to evacuation.

- "Hidden" dihedral action on $SYT(3\times c)$!

SL_3-Web basis

Ex



SL_4-Web basis

A (r=4) Gaetz–Pechenik–Pfannerer–Striker–S. '23.b introduced a "top fully reduced hourglass web basis" for $U_q(\mathfrak{sl}_4)$ indexed by $SYT(4 \times c)$.

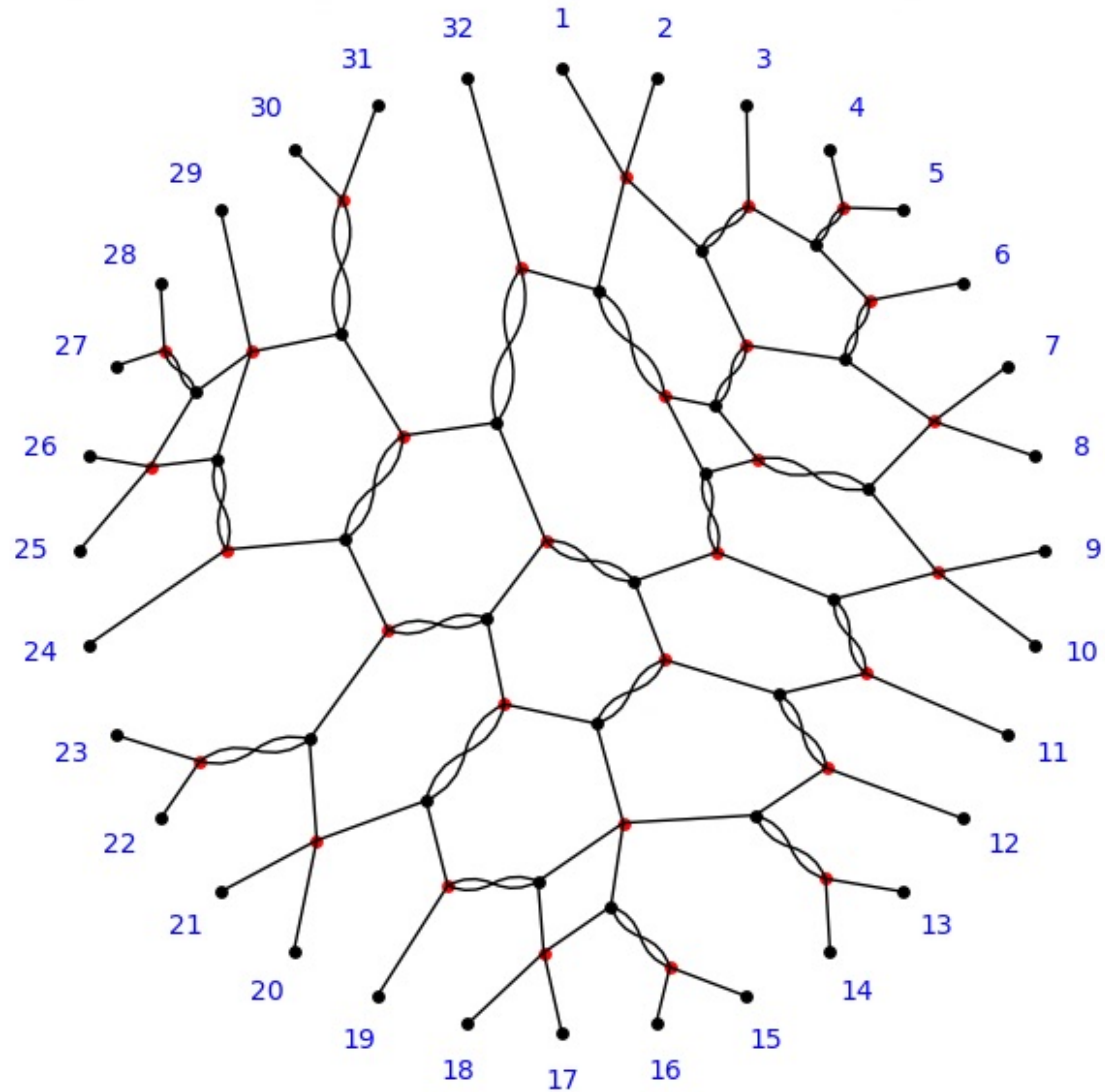
Thm (GPPSS '23.b) This bijection sends
rotation to promotion
reflection to evacuation.

SL_4-Web basis

Ex

1	3	4	7	8	17	19	23
2	5	6	9	14	18	21	24
10	12	13	15	16	25	26	28
11	20	22	27	29	30	31	32

$\longmapsto$



Promotion permutations

Obs (Hopkins-Rubey '21) 3-row basis webs are reduced plabic graphs in the sense of Postnikov '06.
Such graphs are entirely determined by their "trip permutations."

Q What are these Trip's?

Promotion permutations

Def (GPPSS '23.a building on Hopkins-Rubey)

The **promotion permutations** of $T \in SYT(r \times c)$ are

$$\text{prom}_\bullet(T) = (\text{prom}_1(T), \ldots, \text{prom}_{r-1}(T))$$

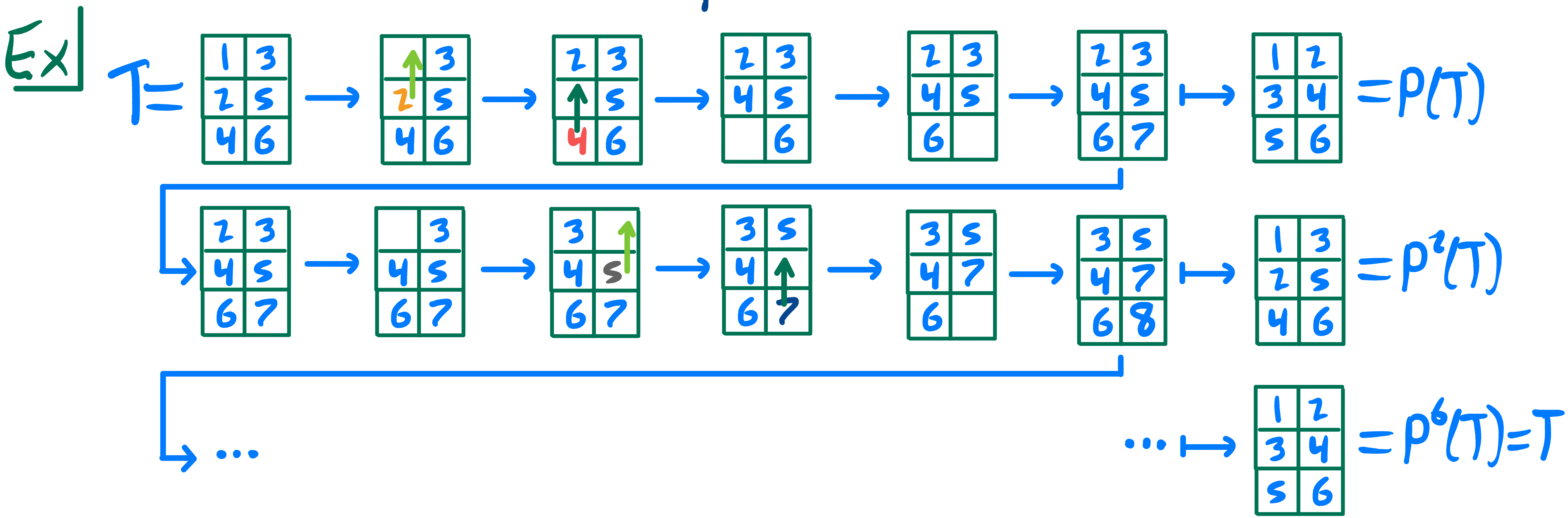
with $\text{prom}_i(T) \in S_n$ defined as follows.

Let $p^{i,j}$ be the unique entry of $P^{j-1}(T)$ which slides from row $i+1$ to row i when computing $P^j(T)$. Set

$$* \quad \boxed{\text{prom}_i(T) : j \mapsto (p^{i,j} + j - 1) \bmod n.} \quad *$$

$(n = rc)$

Promotion permutations



$$\Rightarrow \quad \begin{aligned} prom_1(T) &= 2\,5\,4\,1\,6\,3 \\ prom_2(T) &= 4\,1\,6\,3\,2\,5 \end{aligned}$$

Promotion permutations

Thm (GPPSS '23.a) Let $T \in SYT(r \times c)$. Then:

a) $prom_i(T)$ is a fixed-point free permutation

b) $prom_i(T)^{-1} = prom_{r-i}(T)$

c) $c^{-1} \circ prom_i(T) \circ c = prom_i(P(T))$ where $c = (1\ 2 \cdots n)$

d) $w_0 \circ prom_i(T) \circ w_0 = prom_i(\mathcal{E}(T))$ where $w_0 = n\ n{-}1 \cdots 2\ 1$

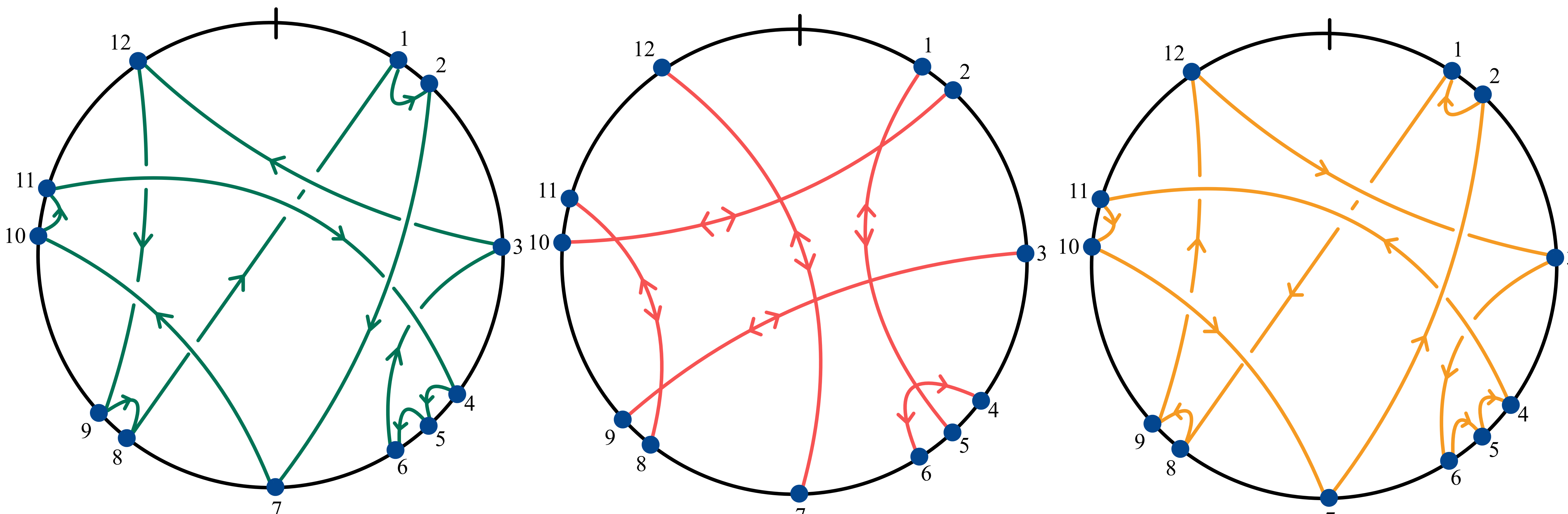
e) $Aexc(prom_i(T)) = \{e \mid e \text{ is in the first } i \text{ rows of } T\}$

where $Aexc(\pi) = \{i : \pi^{-1}(i) > i\}$

Promotion permutations

Cor $\mathrm{prom}_\bullet(T)$ is a combinatorial model manifesting the dihedral structure on $SYT(r \times c)$!

Ex



prom_1 $\quad$ prom_2 $\quad$ prom_3

(Here $T=\{1,2\}\bar{4}\{1,3,4\}2\{\bar{3},\bar{2}\}\{3,4\}T$ is a more general fluctuating tableaux.)

Promotion permutations

Cor When r is even, $\mathrm{prom}_{\frac{r}{2}}(T)$ is a perfect matching.

Note When $r=2$, we recover the Catalan bijection!
When $r=4$, we show in GPPSS '23.b that $\mathrm{prom}_2(T)$ has no 4-crossings.

Open problems

Open Problem 1 ("Higher rank Catalan objects")

Characterize the codomain of $\text{prom}_\bullet$.

Open Problem 2 ("Combinatorial web duality")

Describe $\text{prom}_\bullet(\text{transpose}(T))$ in terms of $\text{prom}_\bullet(T)$, without going through tableaux.

Alternate definitions

See GPPSS '23.a for six different characterizations of prom.:

- Bender-Knuth involution swap positions
- Decorated local rule diagrams
- Row slides
- Antiexcedance sets
- First balance point conditions
- Kashiwara crystal raising algorithm

THANKS!