Tanisaki Witness Relations

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Outline

I. Higher coinvariant algebras

II. Tanisaki witness relations

Coinvariant Algebras

Thm (Newton) $\mathbb{Q}[x_1,\ldots,x_n]^{S_n} = \mathbb{Q}[e_1,\ldots,e_n]$ where $e_i = \underbrace{\sum_{1\le i_1<\cdots<i_k\le n} x_{i_1}\cdots x_{i_k}}_{\text{elementary symmetric polynomial}}$

and $\sigma(x_i) = x_{\sigma(i)}$

Thm (Hilbert) $\langle \mathbb{Q}[x_1,\ldots,x_n]_+^{S_n} \rangle = \langle e_1,\ldots,e_n \rangle$

Def The coinvariant algebra of S_n is

$$R_n = \frac{\mathbb{Q}[x_1,\ldots,x_n]}{\langle e_1,\ldots,e_n\rangle}$$

Coinvariant Algebras

Thm (Borel) $R_n \cong \underbrace{H^*}_{\text{singular cohomology}}(\underbrace{Fl_n}_{\text{complete flag manifold}})$

Thm (Chevalley) $R_n \cong \mathbb{Q}S_n$

$\Rightarrow \dim R_n = n!$

Thm (Artin) $\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} : 0 \leq \alpha_i < i\}$ descends to a basis for R_n

$\Rightarrow \mathrm{Hilb}(R_n; q) = [n]_q!$

Thm (Lusztig–Stanley) $\mathrm{GrFrob}(R_n; q) = \sum_{T \in SYT(n)} q^{\mathrm{maj}(T)} s_{\mathrm{sh}(T)}$

Diagonal Coinvariant Algebras

Def (Garsia-Haiman '90's)

The diagonal coinvariant algebra of S_n is

$$DR_n = \frac{\mathbb{Q}[\underline{x}_n, \underline{y}_n]}{\langle \mathbb{Q}[\underline{x}_n, \underline{y}_n]_+^{S_n} \rangle}$$

where S_n acts diagonally:

$$\sigma(x_i) = x_{\sigma(i)}, \quad \sigma(y_i) = y_{\sigma(i)}$$

Thm (Haiman)

$$\mathrm{GrFrob}(DR_n; q, t) = \nabla e_n$$

- Hilbert schemes
- $n!$ conjecture
- Macdonald poly's

Super Coinvariant Algebras

Def (Zabrocki '19)

The super diagonal coinvariants are

$$SDR_n = \mathbb{Q}[\underline{x}_n, \underline{y}_n, \underline{\theta}_n] / \langle \mathbb{Q}[\underline{x}_n, \underline{y}_n, \underline{\theta}_n]^{S_n}_+ \rangle$$

where $x_i y_j = x_j y_i$, $x_i \theta_j = \theta_j x_i$, $y_i \theta_j = \theta_j y_i$, and $\underbrace{\theta_i \theta_j = -\theta_j \theta_i}_{\text{anti-commute}}$.

Conj (Zabrocki '19)

$$\mathrm{GrFrob}(SDR_n; q, t, z) = \sum_{k=0}^{n-1} z^k \Delta'_{e_{n-k-1}}(e_n)$$

Representation-theoretic model for Delta Conjecture

Super Coinvariant Algebras

- SDR_n is hard! From now on, focus on $\boxed{t=0}$ case.

Def The <u>super coinvariant algebra</u> of S_n is

$$\underline{SR_n} = \mathbb{Q}[\underline{x}_n, \underline{\theta}_n] / \langle \mathbb{Q}[\underline{x}_n, \underline{\theta}_n]_+^{S_n} \rangle.$$

- Think of θ variables as differential forms $\underline{\theta_i = dx_i}$,

$$\theta_i \theta_j = dx_i \wedge dx_j$$

Super Coinvariant Algebras

- The exterior derivative is

$$d = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} dx_i \in \mathrm{End}_{\mathbb{Q}}(\mathbb{Q}[\underline{x}_n, d\underline{x}_n])$$

$$= \frac{\partial}{\partial x_1}\theta_1 + \cdots + \frac{\partial}{\partial x_n}\theta_n$$

$\underline{\text{Thm}}$ (Solomon)

$$\langle \mathbb{Q}[\underline{x}_n, \underline{\theta}_n]^{S_n}_+ \rangle = \langle e_1, \ldots, e_n, dp_1, \ldots, dp_n \rangle$$

Super Coinvariant Algebras

Conj (Zabrocki '19; Haglund-Rhoades-Shimozono '18)

$$\mathrm{GrFrob}(SR_n; q, z) \overset{\text{still open}}{=} \sum_{\mu \vdash n} z^{n-\ell(\mu)} q^{\sum_{i=1}^{\ell(\mu)} (i-1)(\mu_i - 1)} \binom{\ell(\mu)}{m_1(\mu), \dots, m_n(\mu)}_q \omega \overbrace{Q'_\mu(\underline{x}; q)}^{\text{dual Hall-Littlewood function}}$$

$$\mathrm{Hilb}(SR_n; q, z) = \sum_{k=1}^{n} [k]! \, S[n,k] \, z^{n-k}$$

Proved by Rhoades-Wilson '24

Exactness Works!

Thm (Wallach–S. '21)

The exterior derivative complex of SR_n,

$$0 \to \mathbb{C} \to \underbrace{SR_n^0}_{=R_n} \xrightarrow{d} \underbrace{SR^1}_{\theta\text{-degree } 1} \xrightarrow{d} \cdots \xrightarrow{d} SR_n^{n-1} \to 0,$$

is exact.

Cor $\mathrm{Hilb}(SR_n; q, -q) = \sum_{k=0}^{n-1} (-q)^{n-1-k} \, \mathrm{Hilb}(SR_n^k; q) = 1$

Operator Theorem

- Top-degree component of R_n is spanned by

$$\Delta_n = \prod_{1 \le i < j \le n} (x_j - x_i) \quad \in R_n^{\mathrm{sgn}}$$

$\boxed{\text{Thm}}$ (Steinberg)

$$\mathbb{Q}\left[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right]\Delta_n \xrightarrow{\sim} R_n \text{ is a bijection!}$$

Super Operator Theorem

Thm (S.-Wallach) Alternating component of SR_n has basis

$$\{d_I \Delta_n : I \subset [n-1]\} \subseteq SR_n^{sgn}$$

where $d_I = d_{i_1} \cdots d_{i_k}$ for $\underline{d_i = \sum_{j=1}^{n} \frac{\partial^i}{(\partial x_j)^i} \theta_j}$.] generalized exterior derivatives

Thm (Rhoades-Wilson 24; conjectured by S.-Wallach)

$$\mathbb{Q}\left[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right] \{d_I \Delta_n : I \subset [n-1]\} \Delta_n \xrightarrow{\sim} SR_n$$

is a bijection! (LHS are the harmonic differential forms.)

Flip Action

- Have $\underline{2^{n-1}}$ "tent poles" $d_I\Delta_n$ which generate SR_n as an $\mathbb{Q}[x_1,\dots,x_n]$-module under the flip action

$$g\cdot w = g\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right)w = \partial_g w.$$

- Let $SR_I = \mathbb{Q}\left[\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right]d_I\Delta_n$.

By Rhoades-Wilson '24,

(not direct)

$$SR_n = \sum_{I\subseteq[n-1]} SR_I$$

Flip Action

- HRS conjecture has $\underline{2^{n-1}}$ total $\omega Q'_\mu(\underline{x};q)$'s! (Strong comp's.)

$$\mathrm{GrFrob}(SR_n;q,z)=\sum_{\mu\vdash n} z^{n-\ell(\mu)}\, q^{\sum_{i=1}^{\ell(\mu)}(i-1)(\mu_i-1)} \binom{\ell(\mu)}{m_1(\mu),\dots,m_n(\mu)}_q \omega Q'_\mu(\underline{x};q)$$

Q: Is there a filtration of SR_n adding one generator $d_I\Delta_n$ at a time whose successive quotients prove the HRS Formula?

Small Example

$\boxed{\text{Ex}}$ For $n=2$:

$$\cdot\, R_2 = \frac{\mathbb{C}[x_1,x_2]}{\langle x_1+x_2,\, x_1x_2\rangle} \cong \operatorname{span}\{x_1-x_2,\, 1\} \cong \overbrace{\text{sgn}}^{\deg 1} \oplus \overbrace{1}^{\deg 0}$$

$$\cdot\, SR_2 = \frac{\mathbb{C}[x_1,x_2,\theta_1,\theta_2]}{\langle x_1+x_2,\, x_1x_2,\, \theta_1+\theta_2,\, x_2\theta_1+x_1\theta_2\rangle} \cong \mathbb{C}\left[\tfrac{\partial}{\partial x_1}, \tfrac{\partial}{x_2}\right]\cdot \operatorname{span}\{x_1-x_2,\, \theta_1-\theta_2\}$$

$$= \operatorname{span}\{\underbrace{x_1-x_2}_{\Delta_2},\, \underbrace{\theta_1-\theta_2}_{d_1\Delta_2},\, 1\} \cong \underbrace{2\,\text{sgn}}_{\substack{\text{degs } 1,0 \\ 0,1}} \oplus 1$$

$$\cdot\, SR_{\{1\}} = \mathbb{C}\left[\tfrac{\partial}{\partial x_1}, \tfrac{\partial}{x_2}\right]\cdot(\theta_1-\theta_2) = \operatorname{span}\{\theta_1-\theta_2\}$$

$$SR_{\phi} = \mathbb{C}\left[\tfrac{\partial}{\partial x_1}, \tfrac{\partial}{x_2}\right]\cdot(x_1-x_2) = \operatorname{span}\{x_1-x_2,\, 1\}$$

Small Example

$\boxed{\text{Ex}}$ For $n=3$:

- Generators $\Delta_3 = (x_1-x_2)(x_1-x_3)(x_2-x_3)$

$$d_1 = \frac{\partial}{\partial x_1}\theta_1 + \frac{\partial}{\partial x_2}\theta_2 + \frac{\partial}{\partial x_3}\theta_3$$

$$d_2 = \frac{\partial^2}{(\partial x_1)^2}\theta_1 + \frac{\partial^2}{(\partial x_2)^2}\theta_2 + \frac{\partial^2}{(\partial x_3)^2}\theta_3$$

$$d_1\Delta_3 = (2x_1-x_2-x_3)(x_2-x_3)\theta_1 + \cdots$$

$$d_2\Delta_3 = 2(x_2-x_3)\theta_1 + \cdots$$

$$d_1 d_2 \Delta_3 = 12\theta_1\theta_2$$

- Relations: $\dfrac{\partial^2}{\partial x_1 \partial x_2} d_2\Delta_3 = 0$

$$\frac{\partial^2}{\partial x_1 \partial x_2} d_1\Delta_3 = \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right) d_2\Delta_3 \ !$$

- In Fact, $0 \subseteq SH_{\{2\}} \subseteq SH_{\{1\}} + SH_{\{2\}} = SH_3'$ has composition factors corresponding to $Q'_{(2,1)}$ ✓

Tanisaki Ideals

Thm (... Tanisaki ...)

$$\operatorname{GrFrob}(\underbrace{\mathbb{Q}[\underline{x}_n]/I_\mu}_{H^*(\overline{X}_\mu)\ (\text{Springer Fiber})}; q) = \underbrace{\operatorname{rev}_q Q'_\mu(\underline{x}; q)}_{\text{modified Hall-Littlewood functions}}$$

where

- $I_\mu = \langle e_r(S) : |S| - d_{|S|}(\mu) < r \leq |S|, S \subset [n] \rangle$ is a Tanisaki ideal
- $e_r(S) = \displaystyle\sum_{\{i_1 < \cdots < i_r\} \subset S} x_{i_1} \cdots x_{i_r}$
- $d_k(\mu) = \mu'_n + \mu'_{n-1} + \cdots + \mu'_{n-k+1}$

Twists

- Suppose for illustration that $\mathrm{Ann}\, SR_I = I_\mu$. Then

$$\mathrm{GrFrob}(SR_I;q) = \underbrace{\omega}_{d_I\Delta_n \text{ is alternating}} q^{\underbrace{\binom{n}{2}}_{\deg\Delta_n} - \underbrace{i_1 - \cdots - i_k}_{d_I}} \mathrm{GrFrob}(\mathbb{Q}[\underline{x}_n]/I_\mu; \underbrace{q^{-1}}_{\text{flip lowers deg.}})$$

$$= \omega q^{\cdots} Q'_\mu(\underline{x};q)$$

- Flip action can fully account for ω and rev_q twists!

Filtration Approach

$\boxed{Q}$ Is there a total order $I_1 < I_2 < \cdots$ on $2^{[n-1]}$ and a bijection $\Phi_n : 2^{[n-1]} \to \{\kappa \vDash n\}$ such that the successive filtration quotients

$$\sum_{j \leq m} SR_{I_j} \Big/ \sum_{j < m} SR_{I_j}$$

are annihilated precisely by the Tanisaki ideal $\mathcal{I}_{\Phi_n(I_m)}$?

- Would prove HRS formula, and give a basis!
- Since RW '24 proved Hilbert series formula
  $\Rightarrow$ just need "enough relations"!

Tanisaki Witness Relations

Def A **Tanisaki witness relation** is an expression of the form

$$\partial_{e_r(S)} d_I \Delta_n = \sum_j \partial_{f_j} d_{I_j} \Delta_n \quad \text{where } f_j \in \mathbb{C}[x_1, \dots, x_n].$$

Ex

$$0 = 5\partial_{e_5(\underline{S})} d_{16}\Delta_7 - 4\partial_{e_4(\underline{S})} d_{26}\Delta_7 + 3\partial_{e_3(\underline{S})} d_{36}\Delta_7$$
$$- 2\partial_{e_2(\underline{S})} d_{46}\Delta_7 + \partial_{e_1(\underline{S})} d_{56}\Delta_7$$
$$+ 3\partial_{e_5(\underline{S})} d_{25}\Delta_7 - 2\partial_{e_4(\underline{S})} d_{35}\Delta_7 + \partial_{e_3(\underline{S})} d_{45}\Delta_7$$
$$+ \partial_{e_5(\underline{S})} d_{34}\Delta_7$$

Generic Tanisaki Witness Relations

Thm ("Generic Pieri rule", S. '23)

Let $I = \{i_1 < \cdots < i_k\} \subset [n-1]$. Then

$$\sum (-1)^d \partial_{e_{n-k-d}(\underline{n-1})} d_{j_1 \cdots j_k} \Delta_n = 0$$

where the sum is over all subsets $J = \{j_1 < \cdots < j_k\} \subset [n-1]$ where

$$1 \le i_1 \le j_1 < i_2 \le j_2 < \cdots < i_k \le j_k < n$$

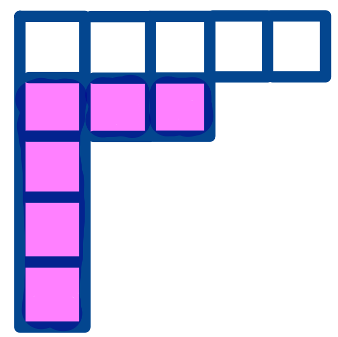
and

$$d = (j_1 - i_1) + \cdots + (j_k - i_k).$$

Generic Tanisaki Witness Relations

Rem · There are "essential" generators for I_μ, e.g.

$\underline{d}$ = 1 5 6 7 7

$\mu=(5,3,1,1,1) \Rightarrow$  w/ $n=11$

(See S. '23, Lemma 2.1)

$$\Rightarrow I_{(5,3,1,1,1)} = \mathfrak{S}_n \cdot \langle e_5(\underline{n-1}), e_6(\underline{n-2}), e_7(\underline{n-3}), e_7(\underline{n-4}), e_1(\underline{n}), \ldots, e_n(\underline{n}) \rangle$$

· If $\mu \vdash n$ w/ $\ell(\mu)=n-k$ and $\mu \neq (1^n)$, then $\underbrace{e_{n-k}(\underline{n-1})}_{\text{generic Pieri gives a relation!}} \in I_\mu$

Generic Tanisaki Witness Relations

- Using a (non-trivial) bijection from [S.'23], solves 1-form case.

Ex $n=3$, $k=1$ has $\alpha \in \{(2,1),(1,2)\}$ with $I \in \{\{1\},\{2\}\}$.

Ideal $\mathcal{I}_{(2,1)}$ generated by $e_1(\underline{3}), e_2(\underline{3}), e_3(\underline{3})$ and $\mathfrak{S}_3 \cdot \underbrace{e_2(\underline{2})}_{x_1x_2}$.

Have

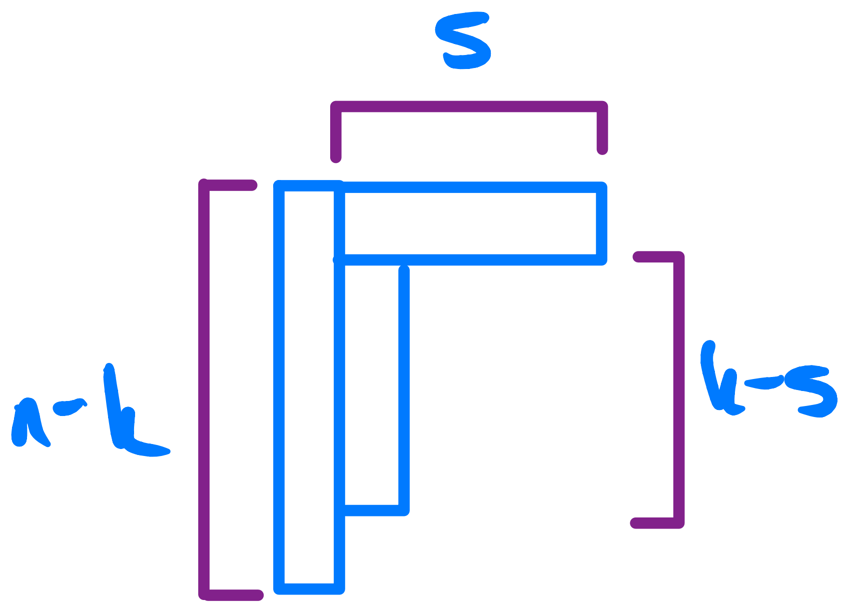
$$\partial_{e_2(\underline{2})} d_{\{2\}} \Delta_3 = 0$$

$$\partial_{e_2(\underline{2})} d_{\{1\}} \Delta_3 = \partial_{e_1(\underline{2})} d_{\{2\}} \Delta_3.$$

In fact, annihilators of $SR_{\{2\}}$ and $(SR_{\{1\}}+SR_{\{2\}})/SR_{\{2\}}$ are precisely $\mathcal{I}_{(2,1)}$.

Extreme Tanisaki Witness Relations

Ex Let

$\mu =$ 

$\Rightarrow$ essential generators

$$e_{n-k}(\underline{n-1}),\ \underbrace{e_{n-s}(\underline{n-2}), \dots, e_{n-s}(\underline{n-s})}_{\text{relations?}}$$

Extreme Tanisaki Witness Relations

Thm ("Extreme hook relations," S. '23)

Let $I=\{i_1<\cdots<i_k\}\subset[n-1]$ have $1\le s\le k$ s.t.

$i_1,\dots,i_{k-s+1}\le n-k$ and $i_{k-s+1+j}\le n-s+j$ for $1\le j<s$.

Pick $0\le u\le s$. Then

$$\sum(-1)^d \Delta_s(j_{k-s+1},\dots,j_k)\binom{d+u}{u}\partial_{e_{n-s-d}(\underline{n-s+u})} d_J \Delta_n = 0$$

where the sum is over $J=\{j_1<\cdots<j_k\}\subset[n-1]$ s.t.

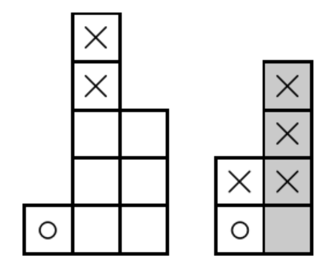
$$j_1=i_1,\dots,j_{k-s}=i_{k-s}$$

$$d=(j_{k-s+1}-i_{k-s+1})+\cdots+(j_k-i_k)\ge 0.$$

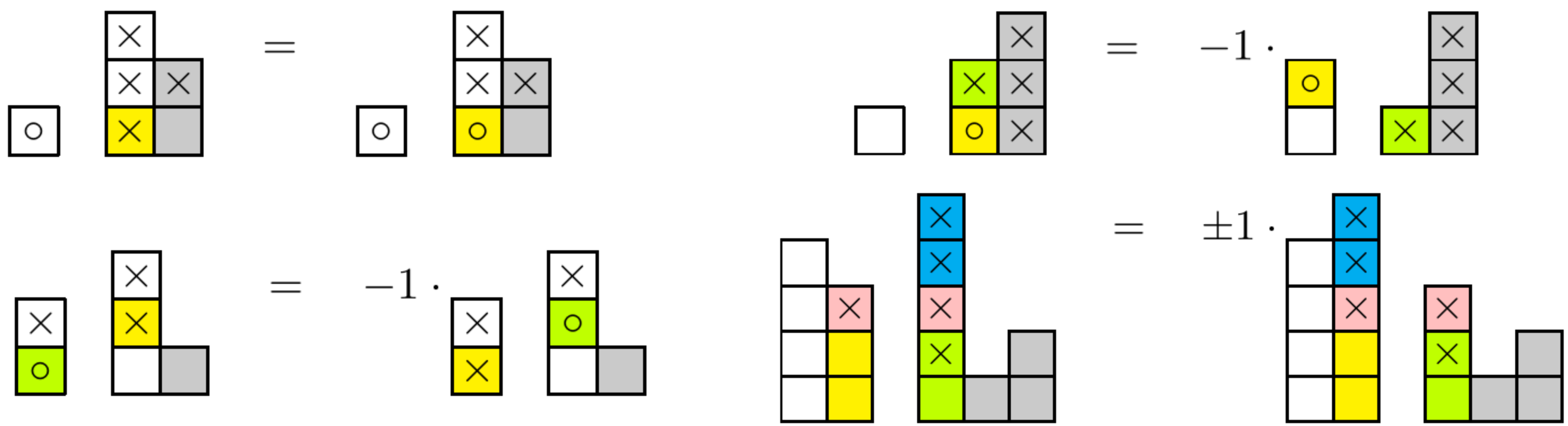
Proofs

- Combinatorial model for terms in $\partial_{e_r(\underline{n-m})} d_I \Delta_n$ using marked staircases and relations:

Example 4.4. The marked 6-staircase



has monomial weight $x_2^3 x_3^3 x_6 \theta_2 \theta_5 \theta_6$, sign $(-1)^7 \operatorname{sgn} \Delta_3(2,1,3) = 1$, and order $1 \cdot (5 \cdot 4) \cdot (2 \cdot 1) \cdot (4 \cdot 3 \cdot 2) = 960$. The weight is thus $960 x_2^3 x_3^3 x_6 \theta_2 \theta_5 \theta_6$, which represents a term in $\partial_{e_2(\underline{5})} \mathrm{d}_{123} \Delta_6$.



Proofs

- For generic Pieri rule, build intricate sign-reversing involution inductively.
- For extreme hook relations, similar but harder.
  - Involves $\mathfrak{S}_s$-actions on marked staircases reminiscent of Lie theory, e.g.

$$\sigma \cdot^{\alpha} \Gamma = \sigma \cdot (\Gamma + \alpha) - \alpha \quad \text{for } \alpha \in \mathbb{Z}^s \text{ fixed, } \Gamma \in \mathbb{Z}^s, \sigma \in \mathfrak{S}_s$$

$$\sigma \cdot (\gamma_1, \ldots, \gamma_s) \stackrel{\text{def}}{=} (\gamma_{\sigma^{-1}(1)}, \ldots, \gamma_{\sigma^{-1}(s)})$$

  - Also new Vandermonde evaluation identities

Further Directions

- **Main problem:** find enough relations to complete the program! Figure out order!

Q What is a combinatorial description for their coefficients? Is there a geometric/algebraic/topological interpretation?

graded-positivity gives hope!

THANKS!