

Webs, pockets, and buildings

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Based on joint work with subsets of *Christian Gaetz, Oliver Pechenik, Stephan Pfannerer, Jessica Striker, and Haihan Wu*

arXiv:2306.12501 (4-row)

arXiv:2402.13978 (2-column)

arXiv:2306.12506 (promotion permutations)

Slides: https://www.jpswanson.org/talks/2025_Connecticut_pockets.pdf

Presented at

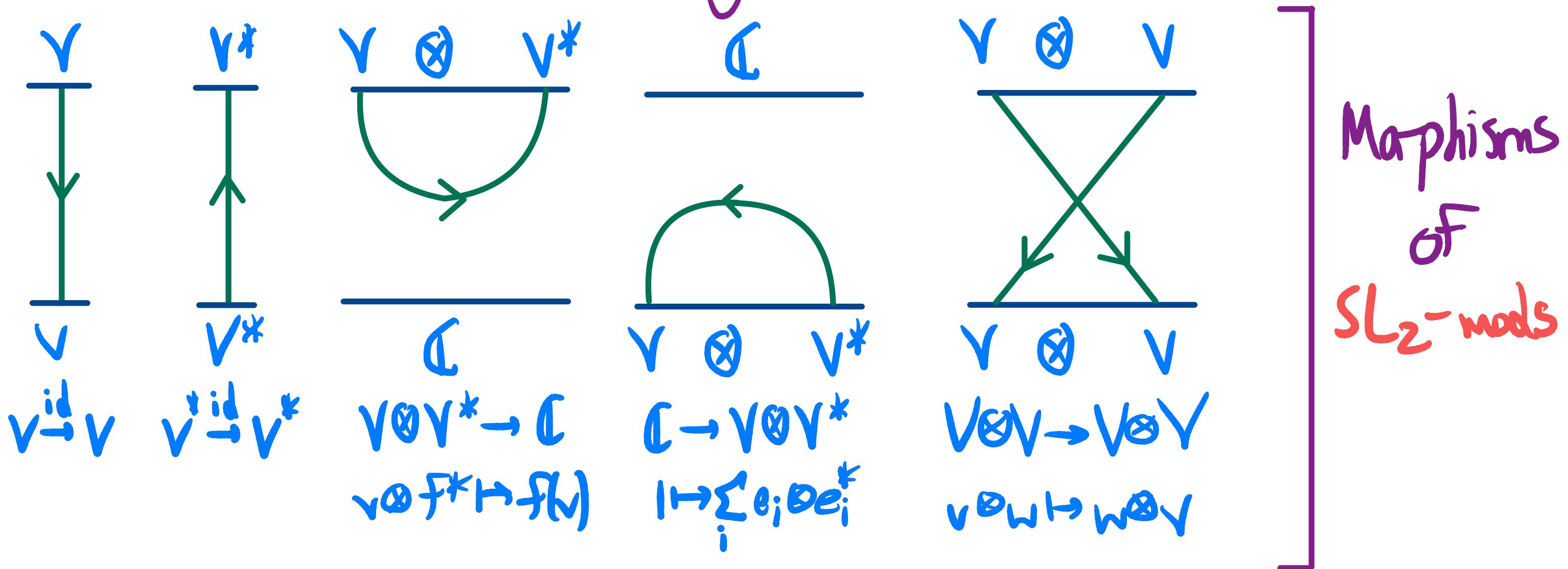
Special Session on Recent Progress on Categorification and Quantum Groups
at AMS 2025 Spring Eastern Sectional Meeting
in Hartford, CT (UConn)
February 21st, 2025

Outline

- SL_2 -webs and SL_3 -webs
 - Temperley-Lieb and non-elliptic bases
- Building embeddings
- Hourglass plabic graphs
 - (New!) SL_4 web basis
- Pockets and buildings

SL_2 -Webs

- Webs are a graphical calculus for representations.
- Let $V = \mathbb{C}^2$. Some building blocks:



SL_2 -Webs

Ex

$$\overbrace{\hspace{1cm}}^{\text{circle}} = \overbrace{\hspace{1cm}}^{\text{circle}}$$

$$\begin{array}{c}
 \downarrow \text{C} \\
 \text{V} \otimes \text{V}^* \\
 \downarrow \text{C} \\
 \downarrow \text{I} \\
 q \otimes e_i^* + e_i \otimes e_2^* \\
 \downarrow \text{J} \\
 \text{I} + \text{I} = \text{Z}
 \end{array}
 \Rightarrow \text{circle} = 2$$

Ex

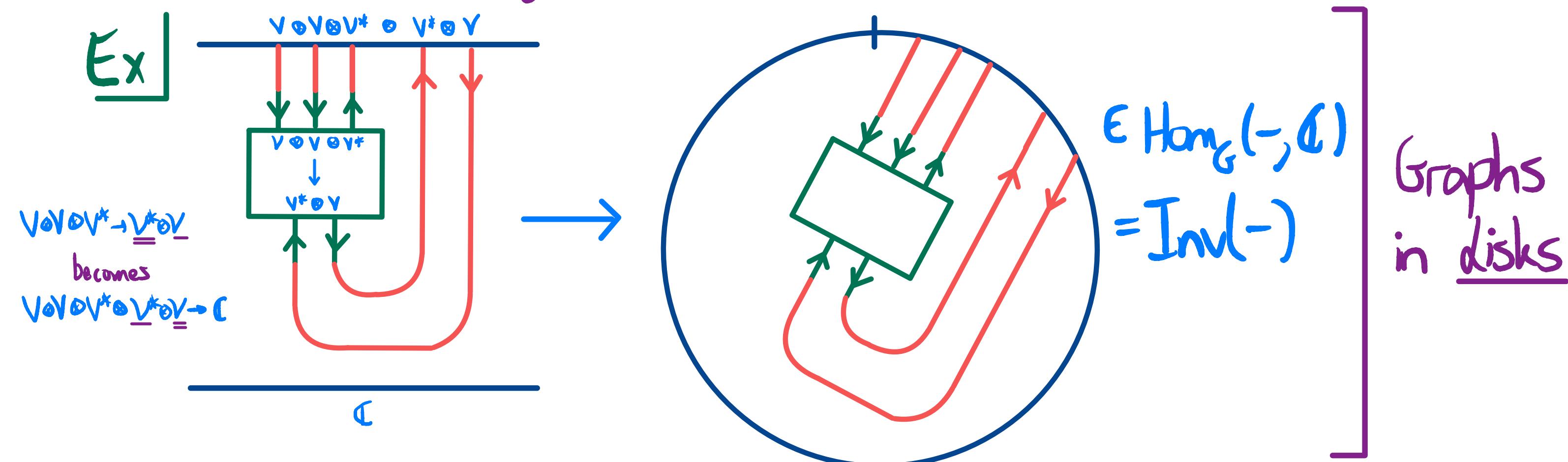
$$\begin{array}{c}
 \text{circle} \\
 = \\
 \sum (v_i \otimes e_i) - (\sum_{i=1}^n e_i^* \otimes v_i) \\
 = \\
 \sum v_i e_i
 \end{array}$$

SL_2 -Webs

- Can "rotate" factors from codomain to domain with duals:

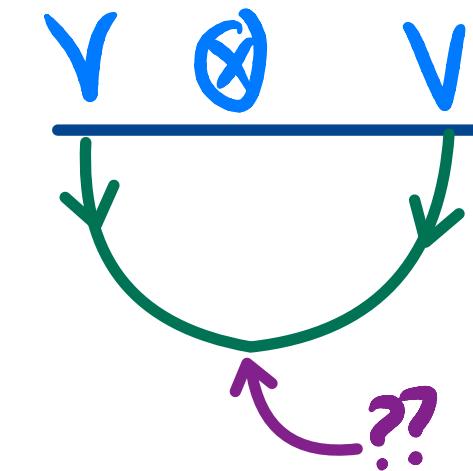
$$\text{Hom}_G(A, B \otimes \underline{C}) \cong \text{Hom}_G(A \otimes C^*, B)$$

(Tensor-hom adjunction and $\text{Hom}_\mathcal{C}(X, Y) \cong Y^* \otimes X$.)



SL_2 -Webs

- What about $\det: V \otimes V \rightarrow \mathbb{C}$? Use



maybe?

- Switch to bipartite graphs:

- Define:

The first graph shows two sets of nodes: $[x_1, x_2]$ on the left and $[y_1, y_2]$ on the right. They are connected by a single curved edge, with the expression $x_1 y_1 + x_2 y_2$ written below it.

The second graph shows two sets of nodes: $[x_{11}, x_{21}]$ on the left and $[x_{12}, x_{22}]$ on the right. They are connected by a single curved edge, with the expression $\det(x_{11} \ x_{12}, x_{21} \ x_{22})$ written below it.

The third graph shows two sets of nodes: $[k_1, k_2]$ on the left and $[x_1, x_2]$ on the right. They are connected by a single vertical edge.

SL_2 -Webs

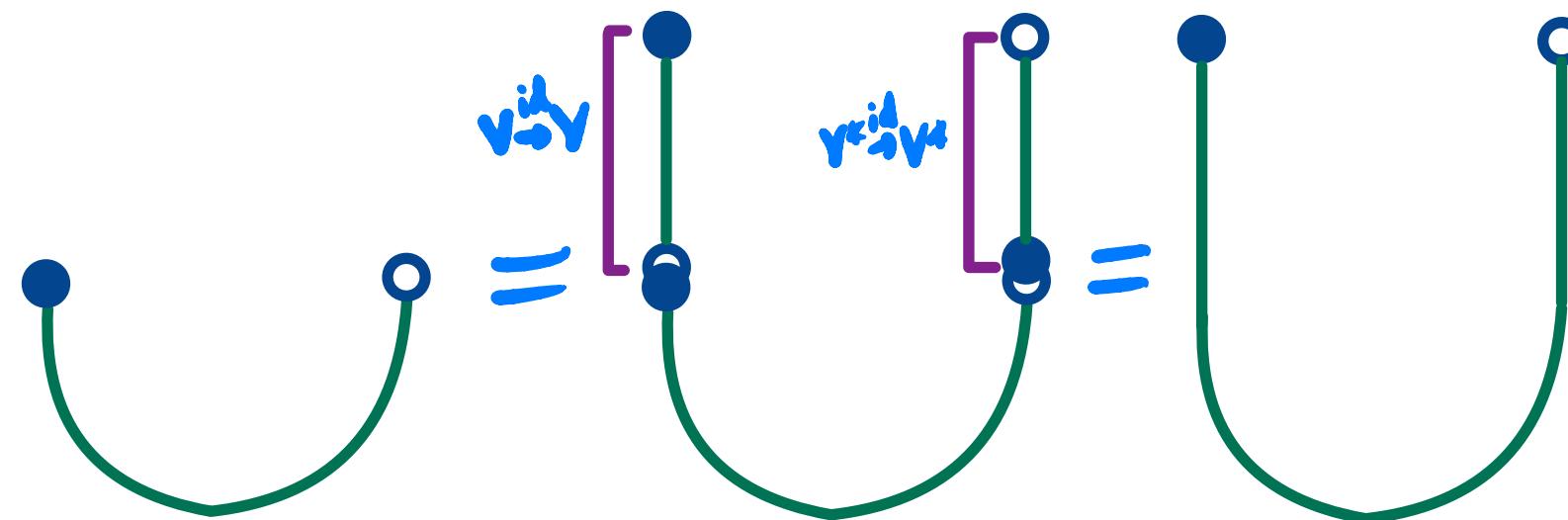
- Bipartite composition conventions

$$\bullet = V \quad \circ = V^* \quad \text{in domain}$$

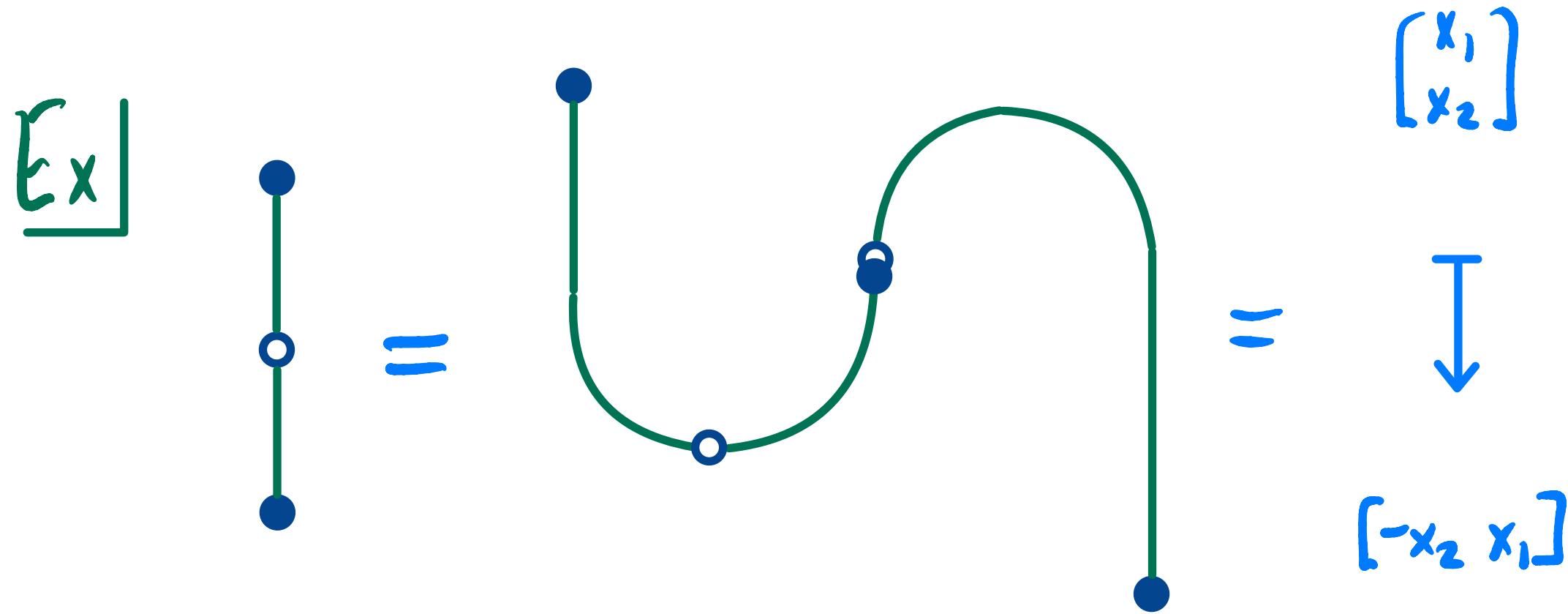
$$\circ = V \quad \bullet = V^* \quad \text{in codomain}$$

- When composing, must match and cancel \bullet pairs:

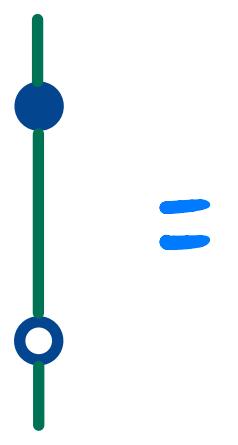
Ex



SL_2 -Webs



$\boxed{\text{Ex}}$ Have contraction relation:



SL_2 -Webs

Def An SL_2 -web is

- a bipartite graph embedded in



- with degree 2 internal vertices,
- and degree 1 boundary vertices.

SL_2 -Webs

Facts 1] Have well-defined map

$$\{SL(2) \text{ webs}\}/\text{isotopy} \longrightarrow \left\{ \begin{array}{l} \text{SL}(2) \text{ morphisms} \\ V^\pm \otimes V^\pm \otimes \dots \rightarrow V^\pm \otimes \dots \end{array} \right\}$$

2] All of $\text{Rep}(SL_2)$ is encoded in disk case

(... surjective up to linear combinations, take Karoubi envelope...)

Q] Generators and relations? Bases??

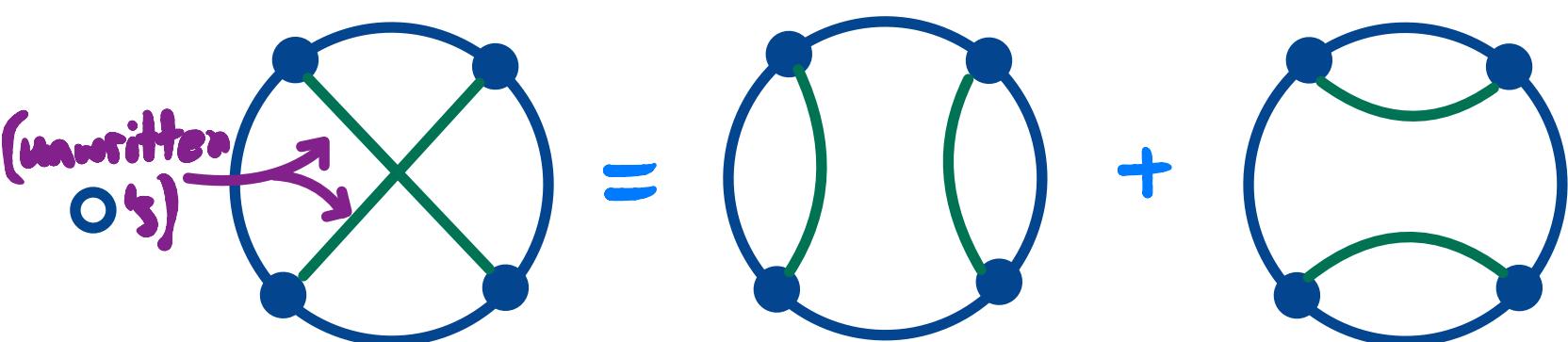
SL_2 -Webs

Ex

$$= \det \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \cdot \det \begin{pmatrix} x_{13} & x_{14} \\ x_{23} & x_{24} \end{pmatrix} \in \text{Inv}(V^{\otimes 4})$$

$$= \text{Hom}_{SL_2}(V^{\otimes 4}, \mathbb{C})$$

Ex Pliicker relations:

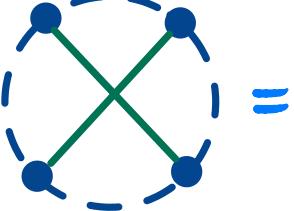
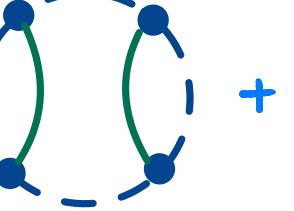
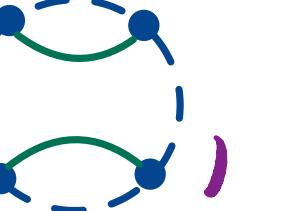


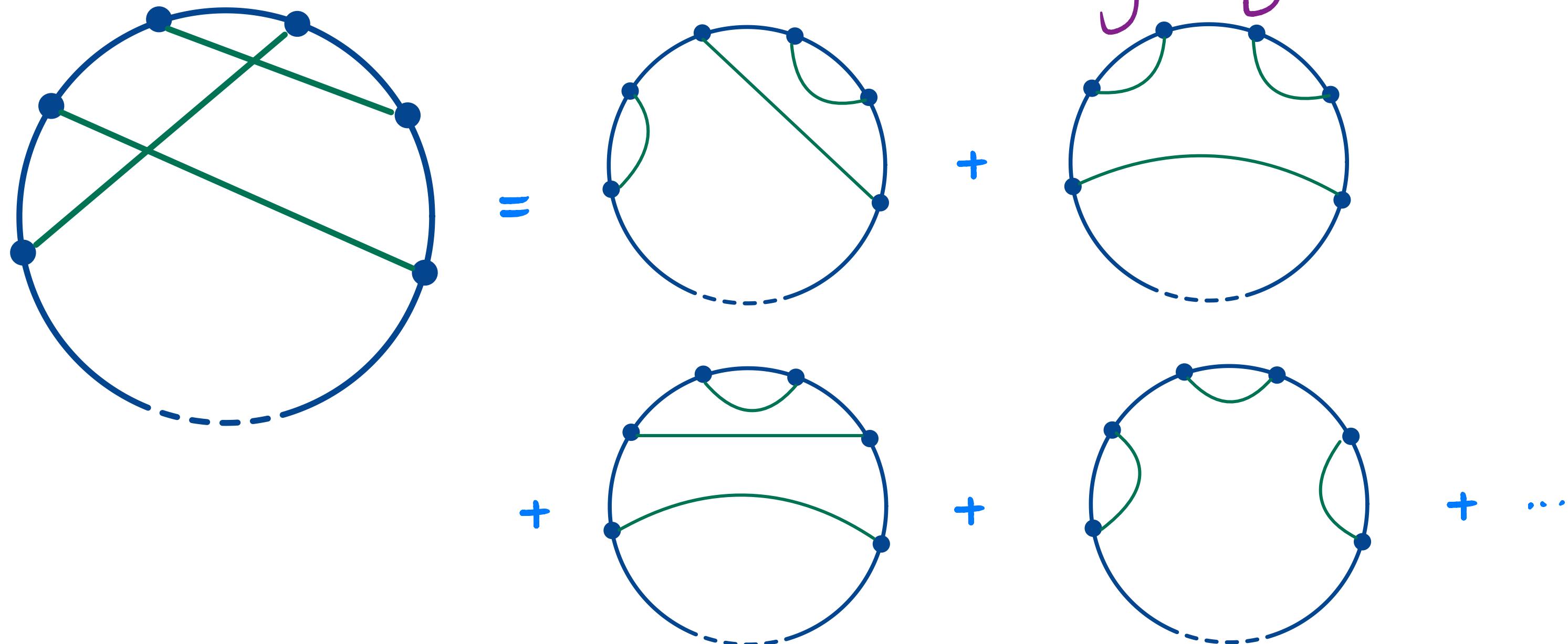
$$(x_{11}x_{23} - x_{21}x_{13})(x_{12}x_{24} - x_{22}x_{14}) =$$

$$(x_{11}x_{22} - x_{21}x_{12})(x_{13}x_{24} - x_{23}x_{14}) +$$

$$(x_{11}x_{24} - x_{21}x_{14})(x_{12}x_{23} - x_{22}x_{13})$$

Temperly-Lieb basis

- Using  =  + , can reduce any matching diagram to a linear combination of matching diagrams:



Temperly-Lieb basis

Thm] The noncrossing 2-row webs are a basis for
 $\text{Inv}_{\text{SL}_2}(V_1 \otimes \dots \otimes V_n)$ ($V_i \in \{V, V^*\}$)

called the Temperly-Lieb basis.

Pf] · Spanning: diagrams span by classical invariant theory,
noncrossing by uncrossing rule.

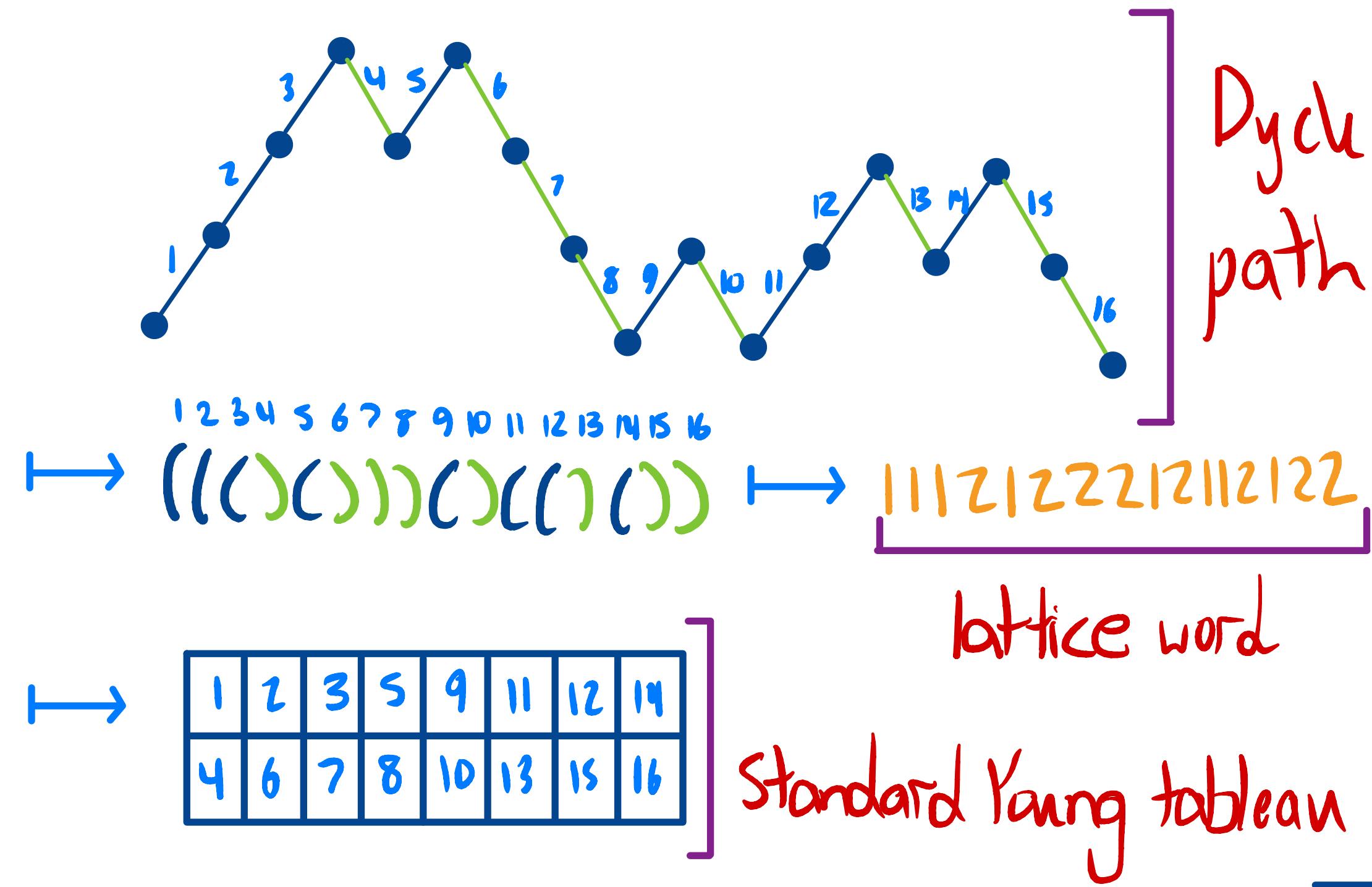
· Independence: by Pieri rule,

$$\dim \text{Inv}_{\text{SL}_2}(V^{\otimes n}) = \# \text{SYT}(2 \times \frac{n}{2}). \text{ Count!}$$

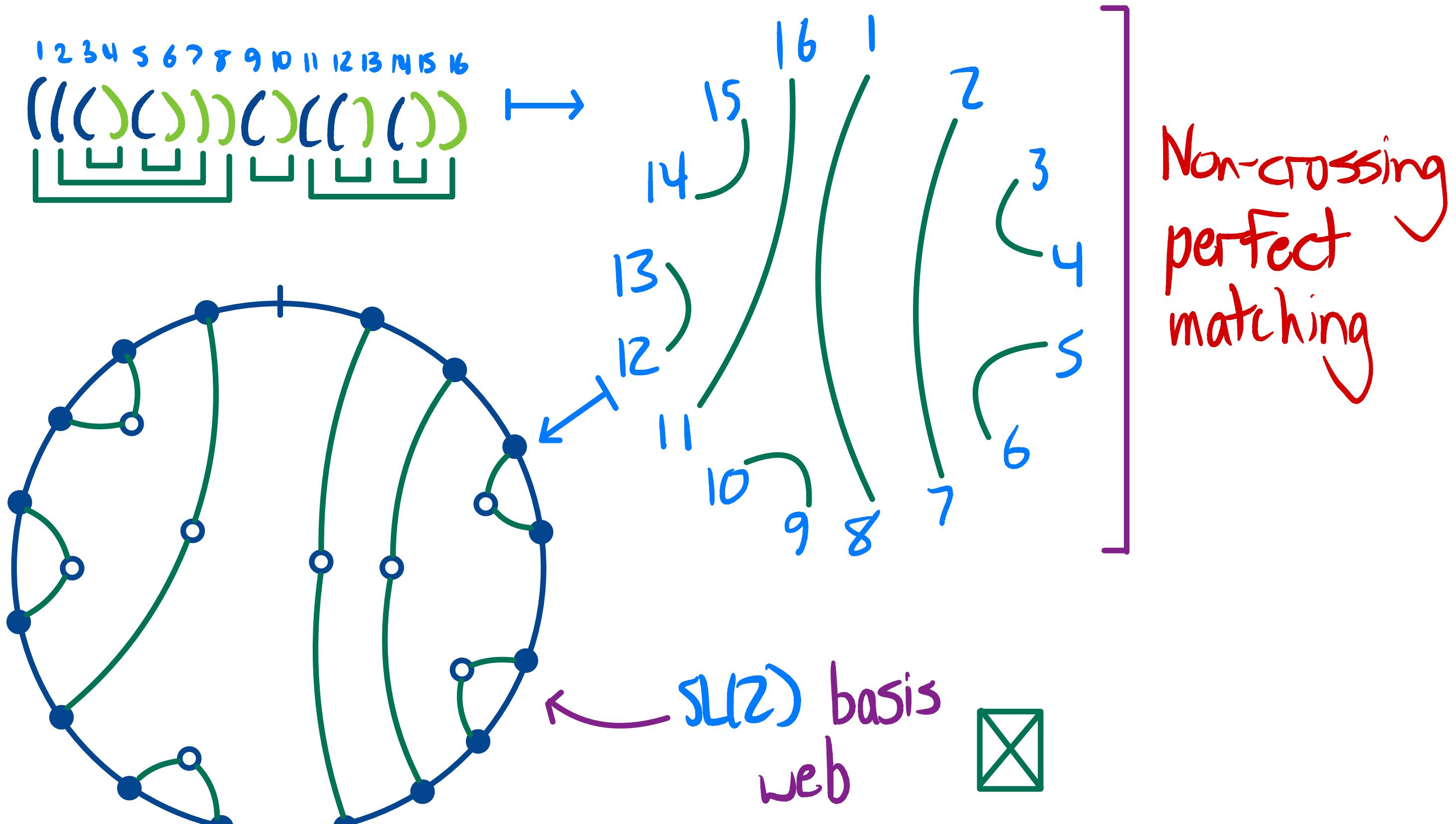


Temperly-Lieb basis

Some Catalan bijections:

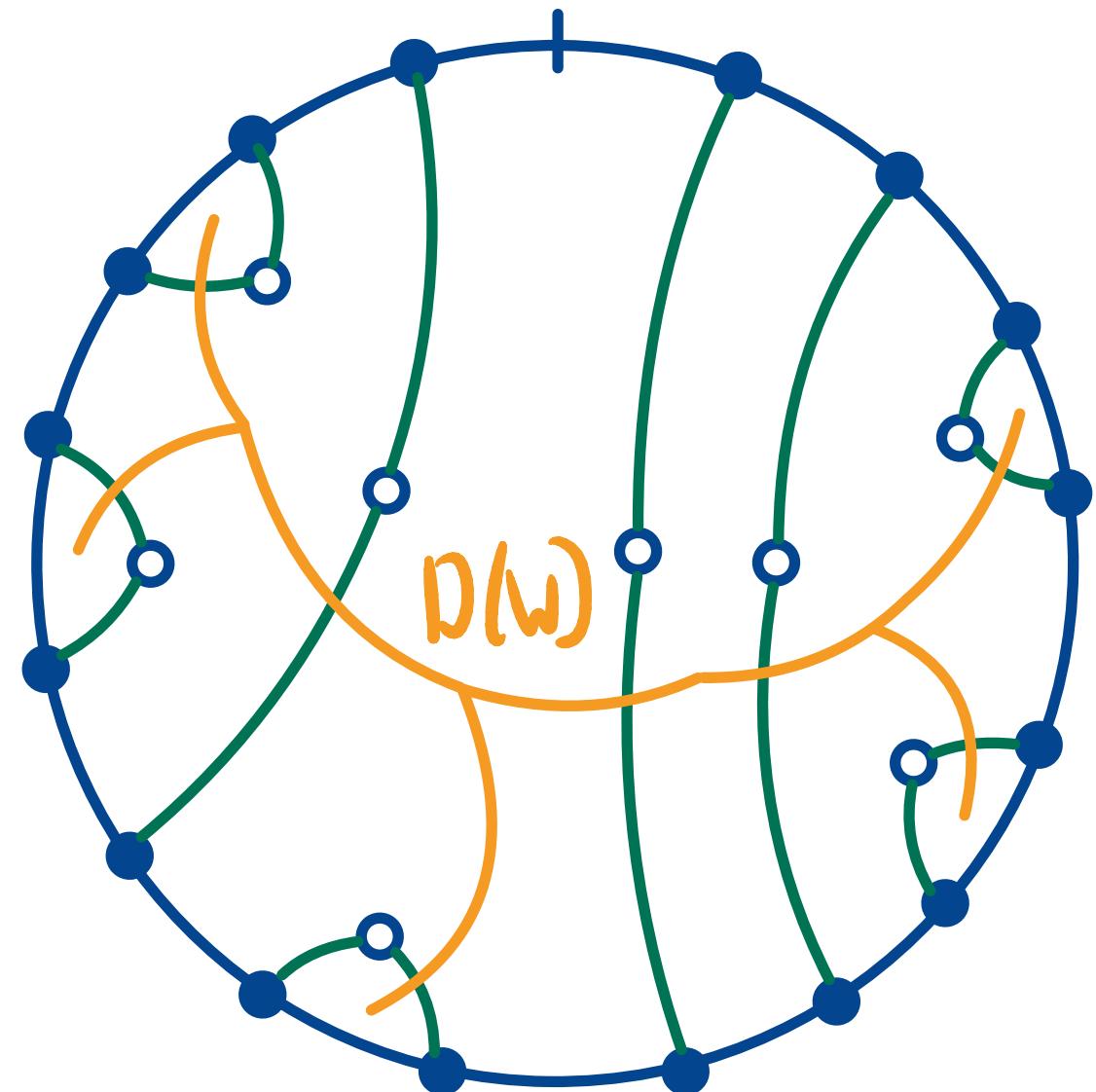


Temperly-Lieb basis



Duds and trees

Obs | The dual graph $D(W)$ of an $SL(2)$ basis web W is a tree:



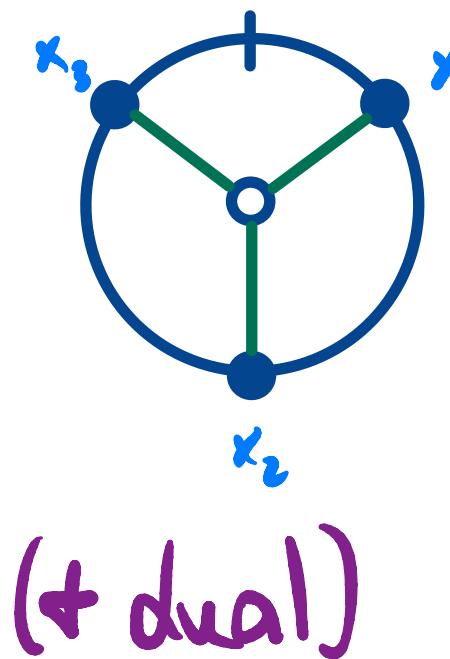
e.g. #faces of W
= #vertices of $D(W)$

SL_3 -webs

- Let $V = \mathbb{C}^3$, $V_i \in \{V, V^*\}$.

Q] Is there a nice web basis for $\text{Inv}_{SL_3}(V, 0 \dots 0, V_n)$?

- Use bipartite planar graphs in a disk built from



$$= \det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{pmatrix} \text{ (so trivalent).}$$

Have univalent boundary vertices
and connected to boundary.

SL_3 -webs

SL_3 -webs

Ex

$$= \det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & y_1 \wedge y_2 \\ 1 & 1 & 1 \end{pmatrix}$$

Ex

$$= \dots \text{a polynomial obtained by summing over } \underline{\text{proper edge}} \text{ 3-colorings...}$$

SL_3 -webs

Thm (Kuperberg '94) The generating SL_3 -web relations are

$$\begin{aligned} \text{circle} &= 3 \\ \text{double line with loop} &= 2 \cdot \text{single line} \\ \text{square web} &= \text{sum of three curved webs} \end{aligned}$$

Non-elliptic web basis

Thm (Kuperberg '94)

Call an SL_3 -web non-elliptic if it has
no 2-faces or 4-faces.

The nonelliptic webs form a basis of

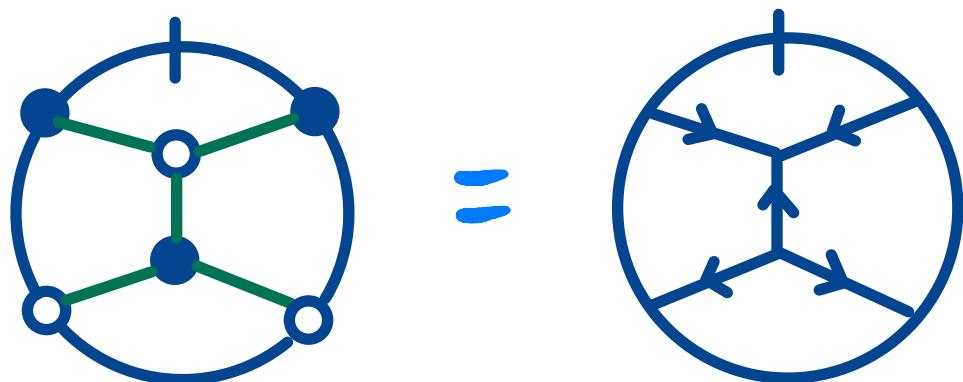
$$\text{Inv}_{SL_3}(N, \theta - \theta Y_n). \quad (V; \in \{V, V^*\})$$

Non-elliptic web basis

PF] · Spanning: similar to SL_2 case.

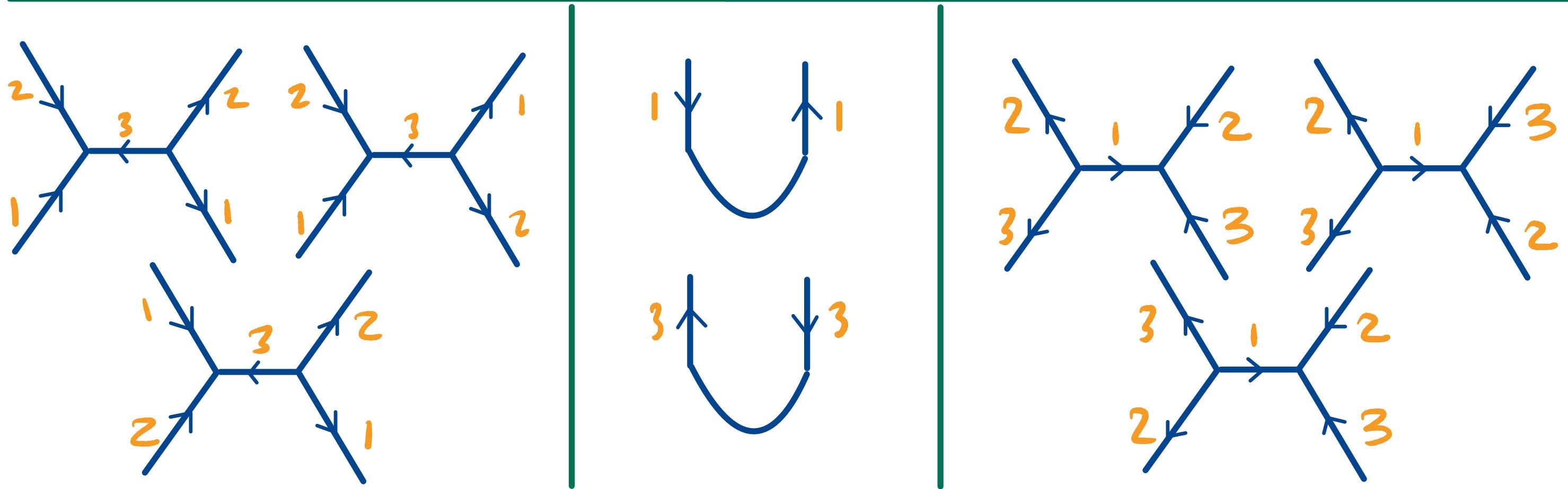
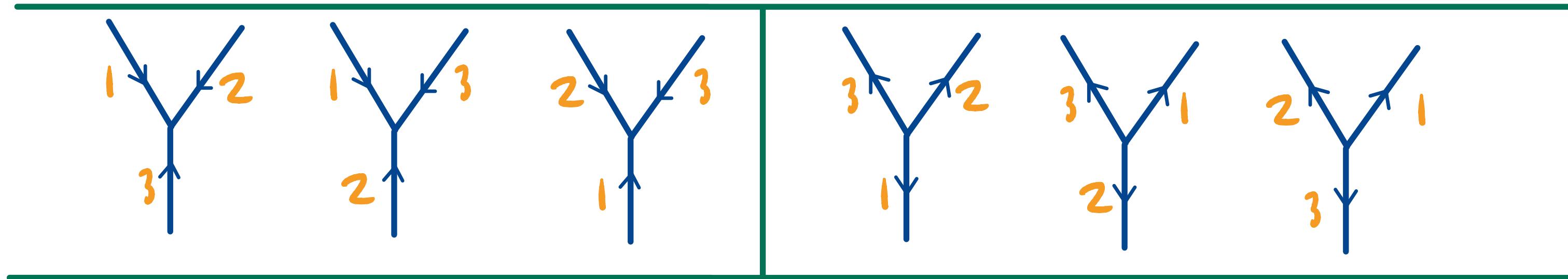
· Independence: bijection to $\text{SYT}(3 \times \frac{1}{3})$ using growth rules. (\exists other approaches)

Will use directed notation here:



SL_3 -growth rules

Kuperberg-Khorana growth rules:



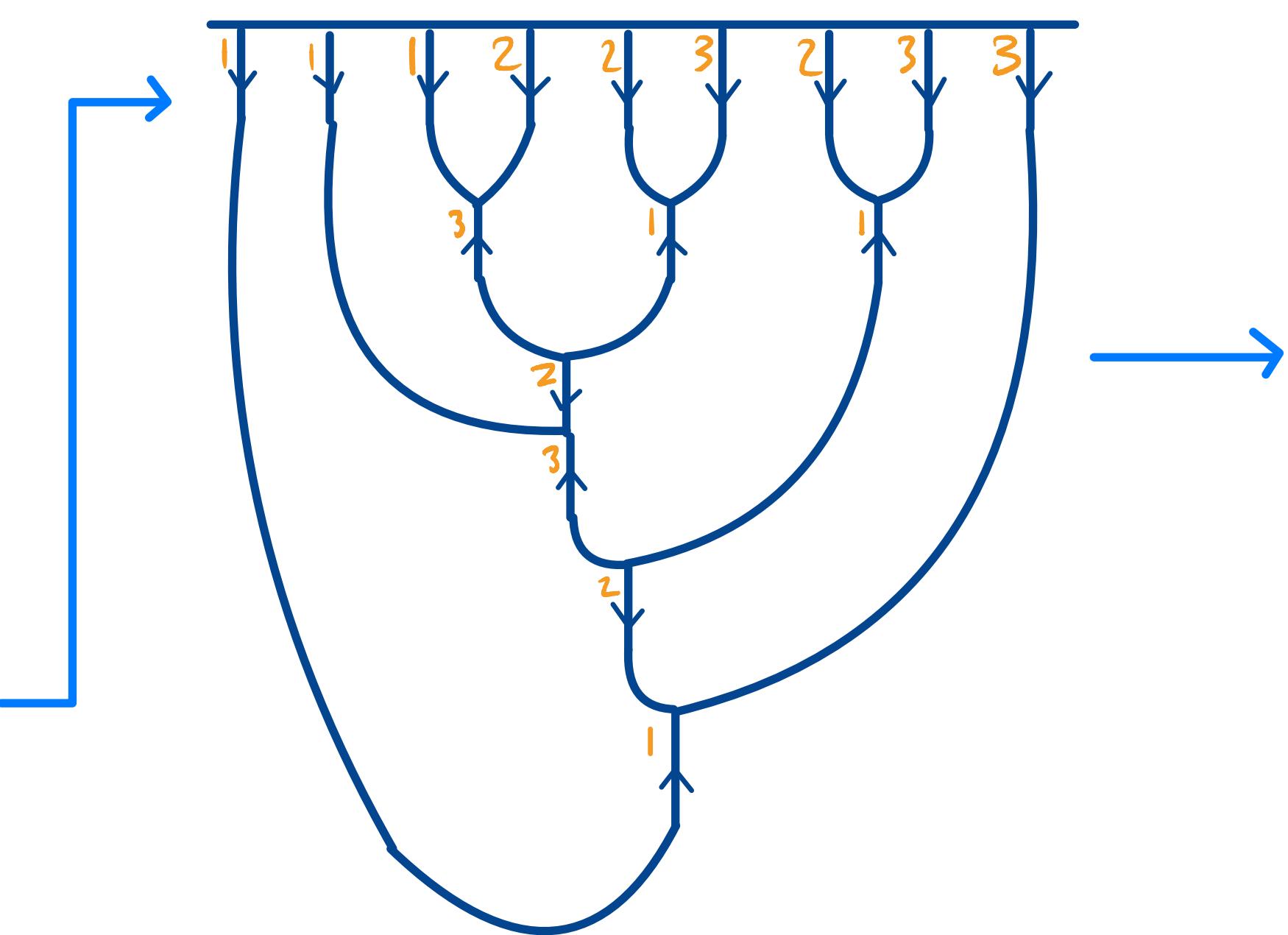
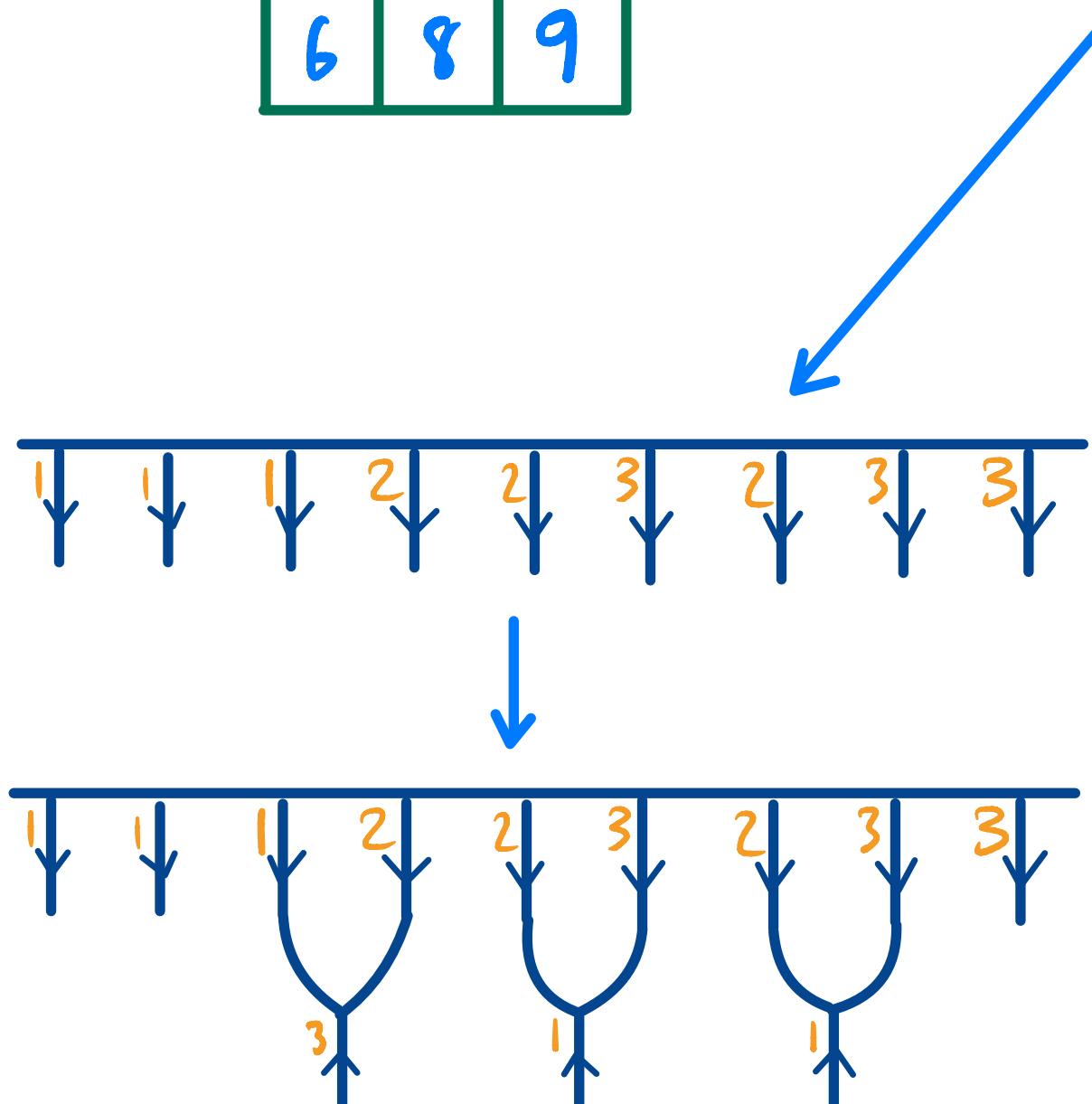
SL_3 -growth rules

Ex

$T =$

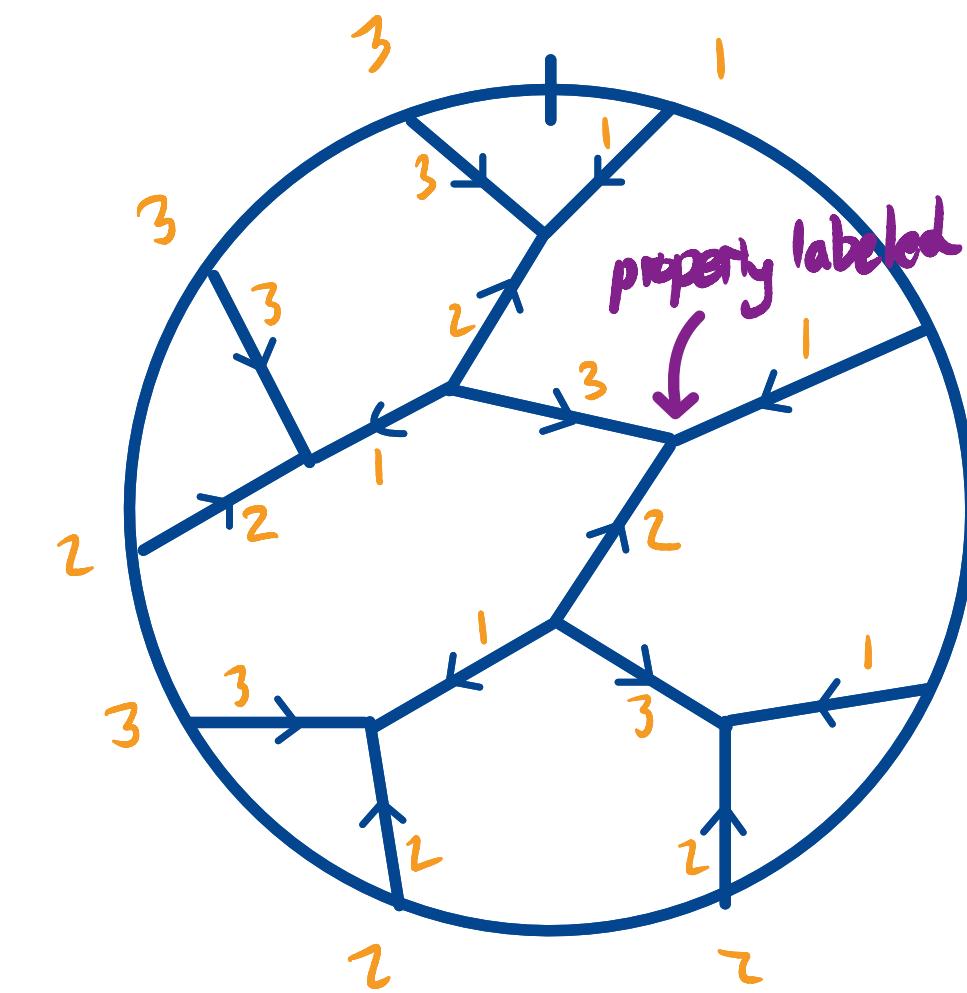
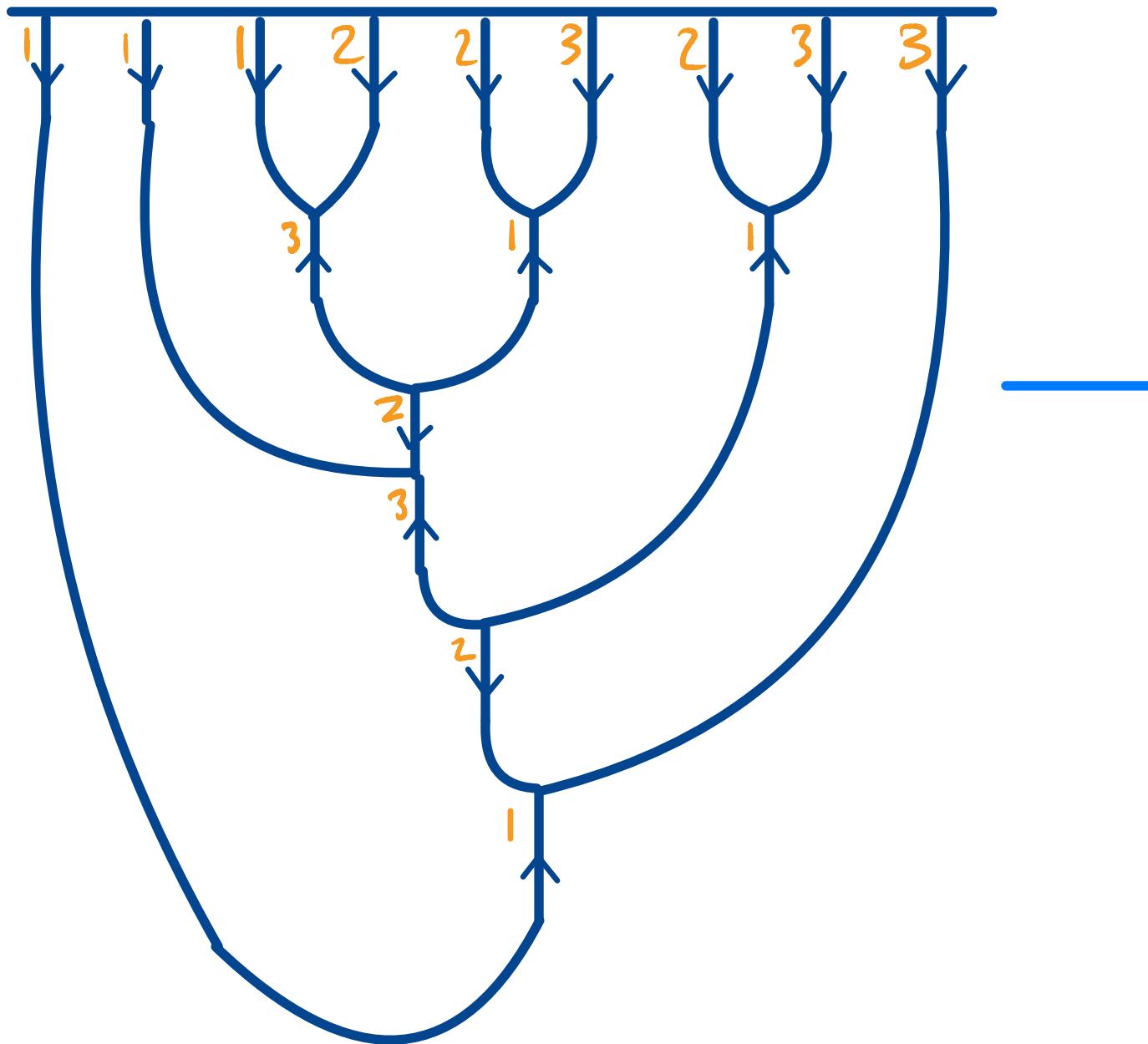
1	2	3
4	5	7
6	8	9

$\rightarrow 111223233$



SL_3 -growth rules

($T \rightarrow IIIIZZ3233$)

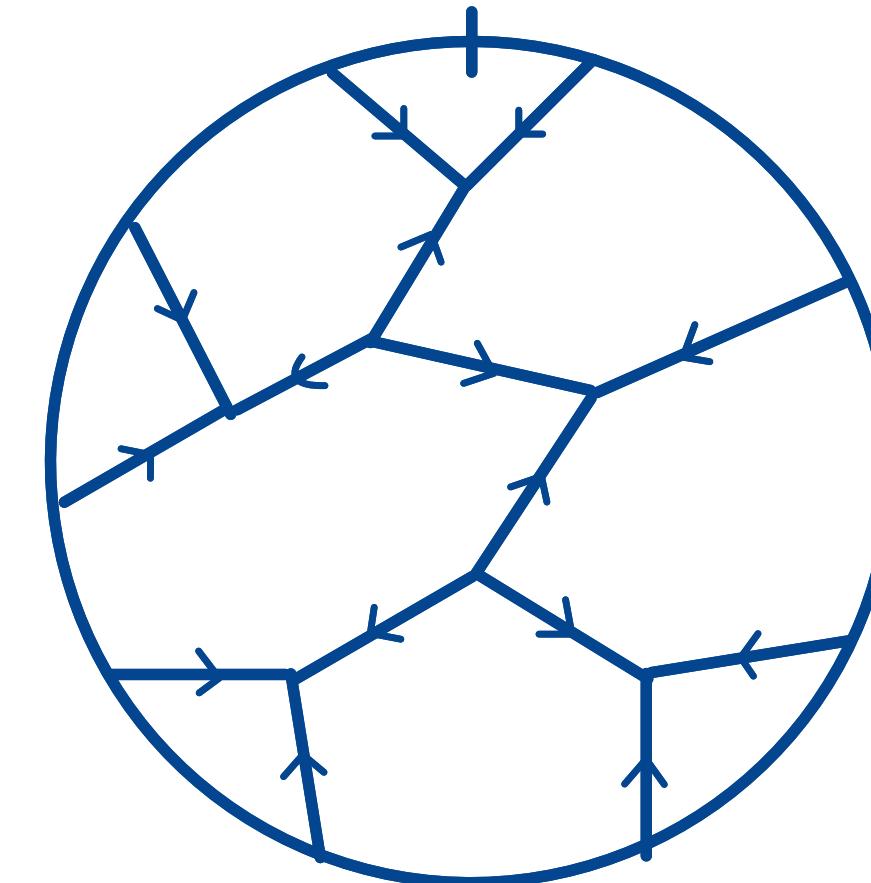


Now just erase labels!

SL_3 -growth rules

In all:

$$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 7 \\ \hline 6 & 8 & 9 \\ \hline \end{array}$$



SL_3 -growth rules

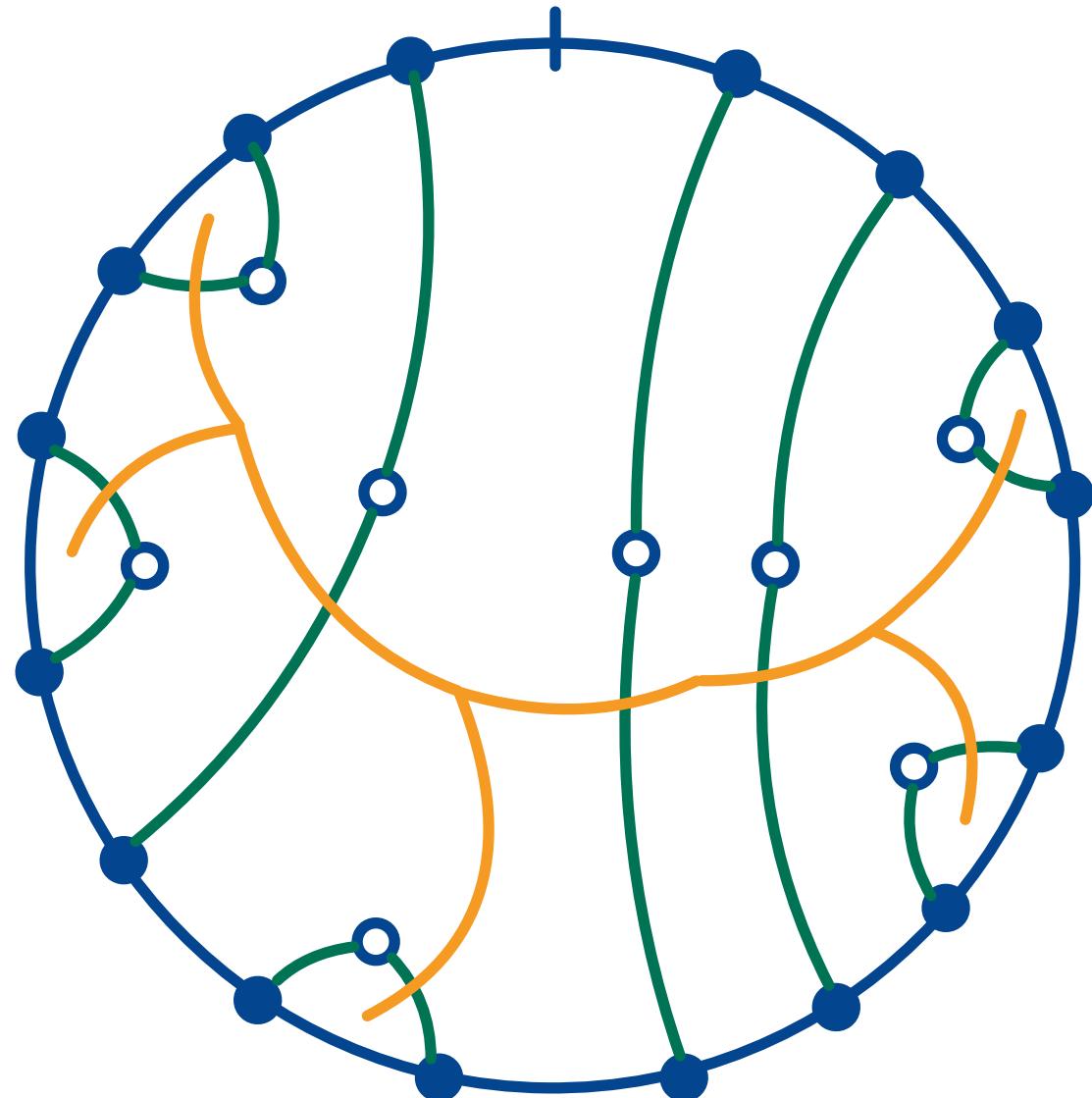
Fact

- 1 Growth algorithm is well-defined:
output is independent of choices
- 2 Growth algorithm subjects onto non-elliptic $SL(3)$ webs 

Q What's really going on here??

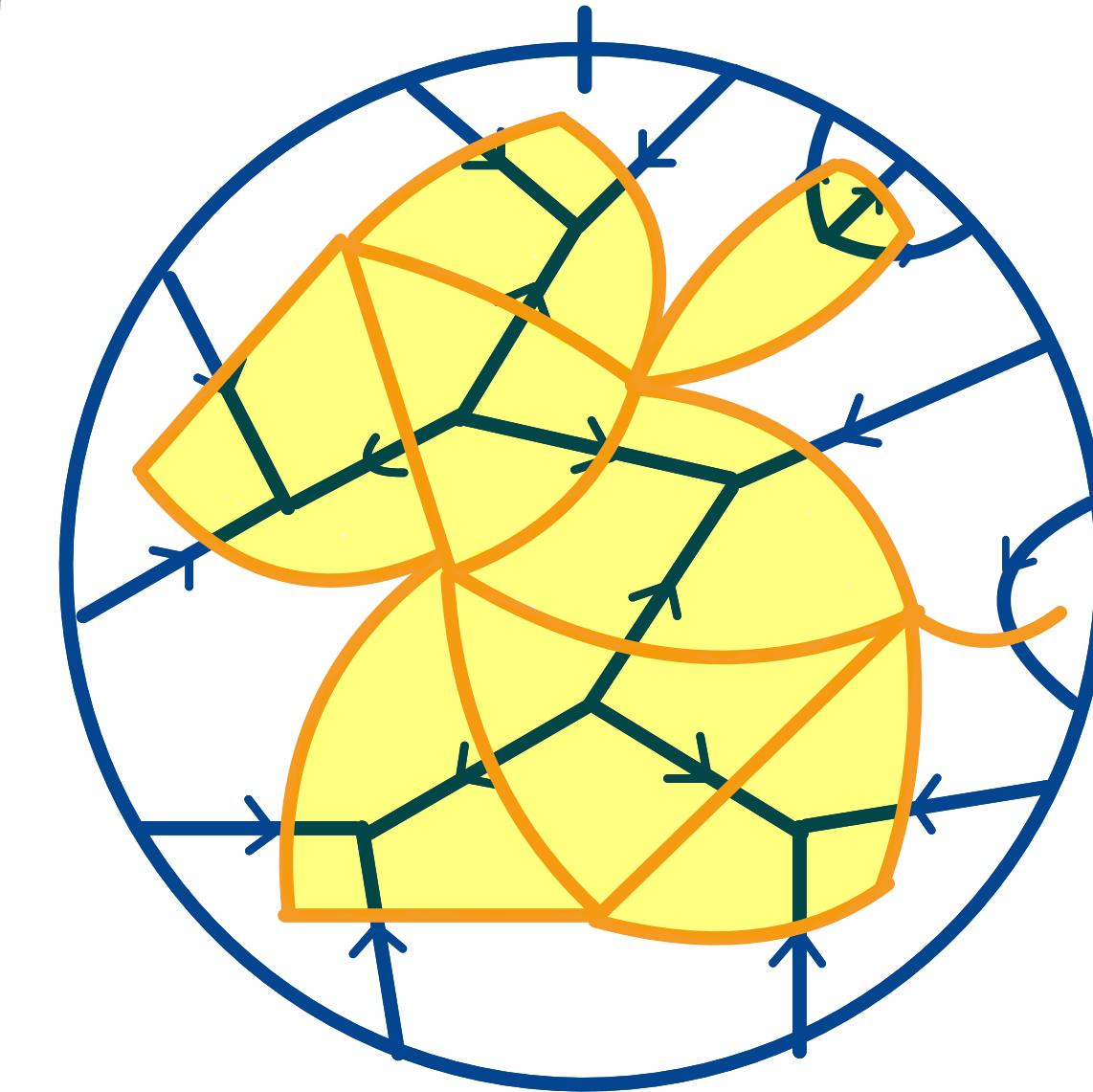
SL_3 -web duals

Obs | $D(\mathbb{W})$ for $SL(3)$ is a triangulation:



$D(\mathbb{W})$ is 1D for $SL(2)$

vs.



$D(\mathbb{W})$ is "2D" for $SL(3)$

SL_3 -web duals

- Fortinier-Kannitzer-Kuperberg showed duals of SL_3 basis webs live inside the affine building $\Delta(SL_3)$
 - growth rules build one triangle at a time
 - labels encode distance information
 - Non-elliptic condition \Leftrightarrow (AT(0))

next!

Double cosets

Obs Given groups $H \leq G$, have a "distance" on G :
 $d: G \times G \rightarrow H\backslash G / H$
 $d(p, q) = H\bar{p}^{-1}qH$

[]
a double coset

Ex For $\text{IR}_{>0} \subseteq \mathbb{C}^*$, (double) cosets are represented by polar angles. Then $d(re^{i\theta}, se^{i\phi}) = e^{i(\phi-\theta)} \text{IR}_{>0}$.
Basically $(x, y) \mapsto \arg\left(\frac{y}{x}\right) = \text{atan2}(y, x)$!

Double cosets

Ex For $\mathbb{C}[[t]]^\times \subset \mathbb{C}(t)$, coset reps are $\{t^n \mid n \in \mathbb{Z}\}$

Ex For $GL_r(\mathbb{C}[[t]]) \subset GL_r(\mathbb{C}(t))$, double coset reps are

$$t^\lambda = \begin{pmatrix} t^{\lambda_1} & & & \\ & t^{\lambda_2} & & \\ & & \ddots & \\ & & & t^{\lambda_r} \end{pmatrix}$$

where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r$
integers. $GL(r)$
dominant
weights

Ex For $PGL_r(\mathbb{C}[[t]]) \subset PGL_r(\mathbb{C}(t))$, now have

$$t^\lambda = t^{\lambda + (\Gamma)} \quad \left[\begin{array}{l} SL(r) \text{ dominant} \\ \text{weights} \end{array} \right]$$

Affine Grassmannians

Def The affine Grassmannian of $SL(r)^\vee$ is

$$AFFGr_r = PGL_r(\mathbb{C}((t))) / PGL_r(\mathbb{C}[[t]]).$$

- Have "distance"

$$d: AFFGr_r \times AFFGr_r \rightarrow \mathbb{Z}_{\text{dec}}^r / \langle 1, \dots, 1 \rangle$$

[
] sl(r) dominant
 weights

where $d(pH, qH) = d(p, q) = \lambda \Leftrightarrow p^{-1}q \in H + \lambda H$

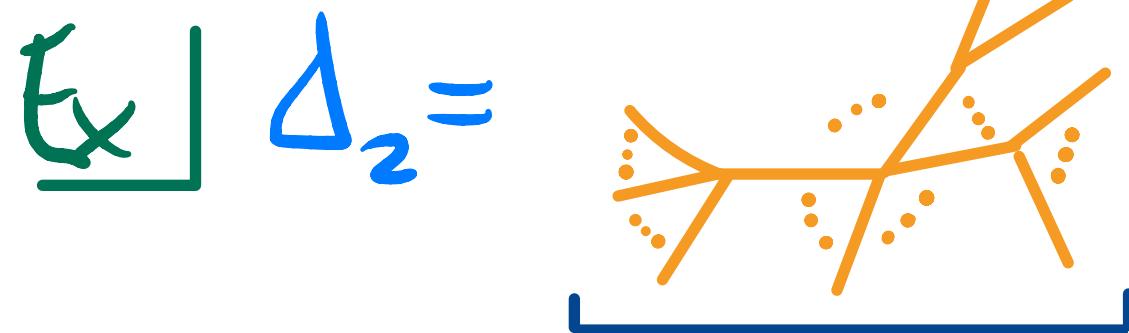
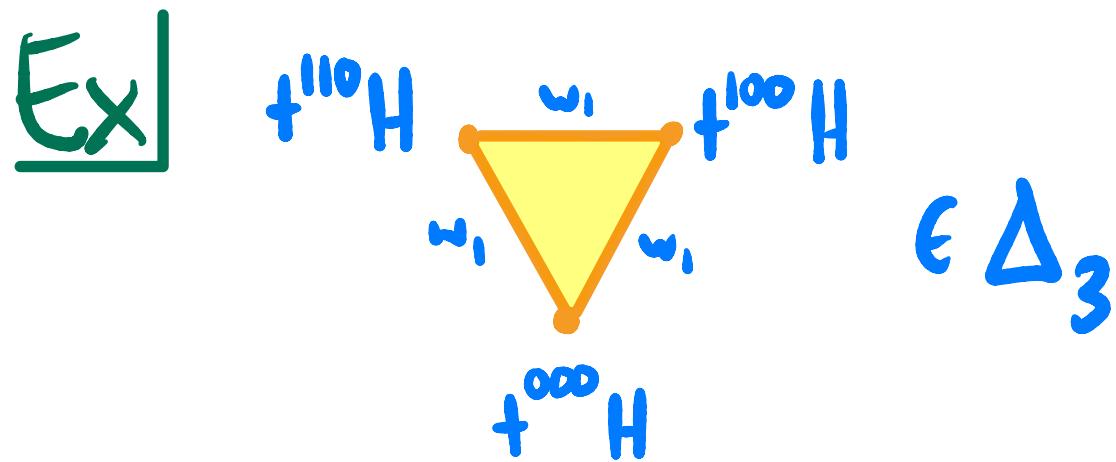
- Have $d(p, p) = 0$ $d(p, q) = d(gp, gq)$ $d(q, p) = \underline{-\text{rev}(d(p, q))}$
 for $g \in GL_r(\mathbb{C}((t)))$
 not quite (anti)symmetric

Affine Buildings

- The fundamental weights of $SL(r)$ are $\omega_i = (1^i, 0^{r-i})$
 $\omega_i^* = (0^{r-i}, -1^i) \equiv \omega_{r-i}$.
- Corresponds to $\Lambda^i V$ and $\Lambda^i V^*$.
- Essentially generates $SL(r)$ -representation theory
(e.g. Kostant envelope...)

Affine Buildings

Def The affine building on AffGr_r is the simplicial complex Δ_r whose vertices are the points of AffGr_r , and whose simplices are collections of points all of whose distances are fundamental weights.



tree where every vertex has uncountable degree

Geodesics

Def If $\lambda, \mu \in \mathbb{Z}_{\text{dec}}^r$ where $|\lambda| = |\mu| + rk$ for some $k \in \mathbb{Z}$,

say

$$\lambda \geq \mu \Leftrightarrow \forall m, \sum_{j=1}^m \lambda_j \geq \sum_{j=1}^m (\mu_j + k).$$

Def Let $\gamma = p_0 \rightarrow p_1 \rightarrow \dots \rightarrow p_n$ be a path

on the 1-skeleton of Δ^r .

Set $d(\gamma) = \sum_{i=1}^n d(p_{i-1}, p_i)$.

Geodesics

Def A ("combinatorial") geodesic from p to q

is any minimizer of $\{d(\gamma) \mid \gamma: p \rightarrow \dots \rightarrow q\}$.

Def A flag subcomplex $\Gamma \subset \Delta^r$ has coherent geodesics if

$\forall p, q \in \Gamma, \exists$ geodesic from p to q .

Affine Buildings

Thm (Fontaine-Kannitzer-Kuperberg '13)

let W be a nonelliptic basis web for $SL(3)$.

There exists an embedding $\phi: D(W) \rightarrow \Delta^3$ (of flag simplicial complexes)
s.t. $\phi(D(W))$ has coherent geodesics

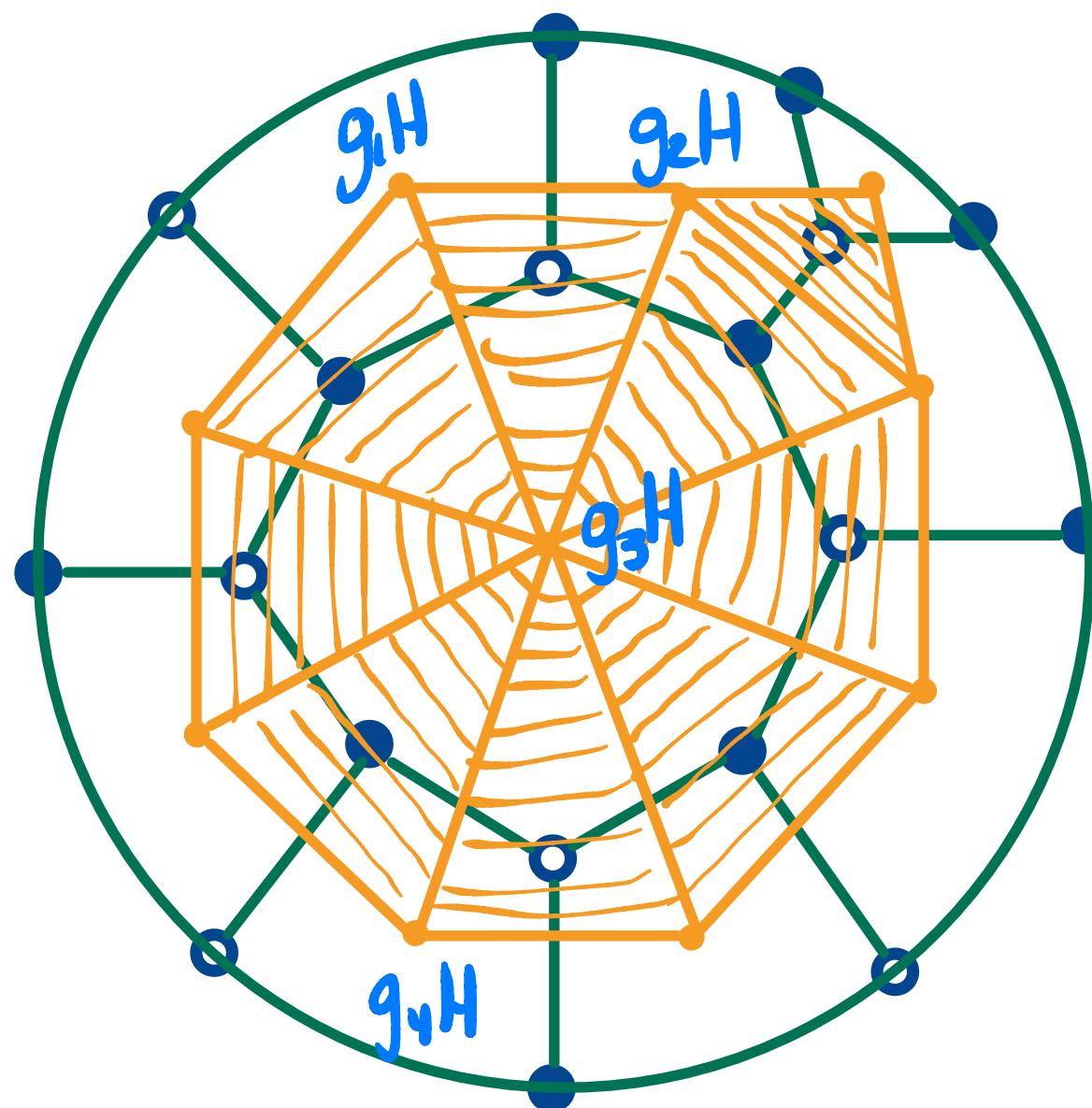
and

in $D(W)$

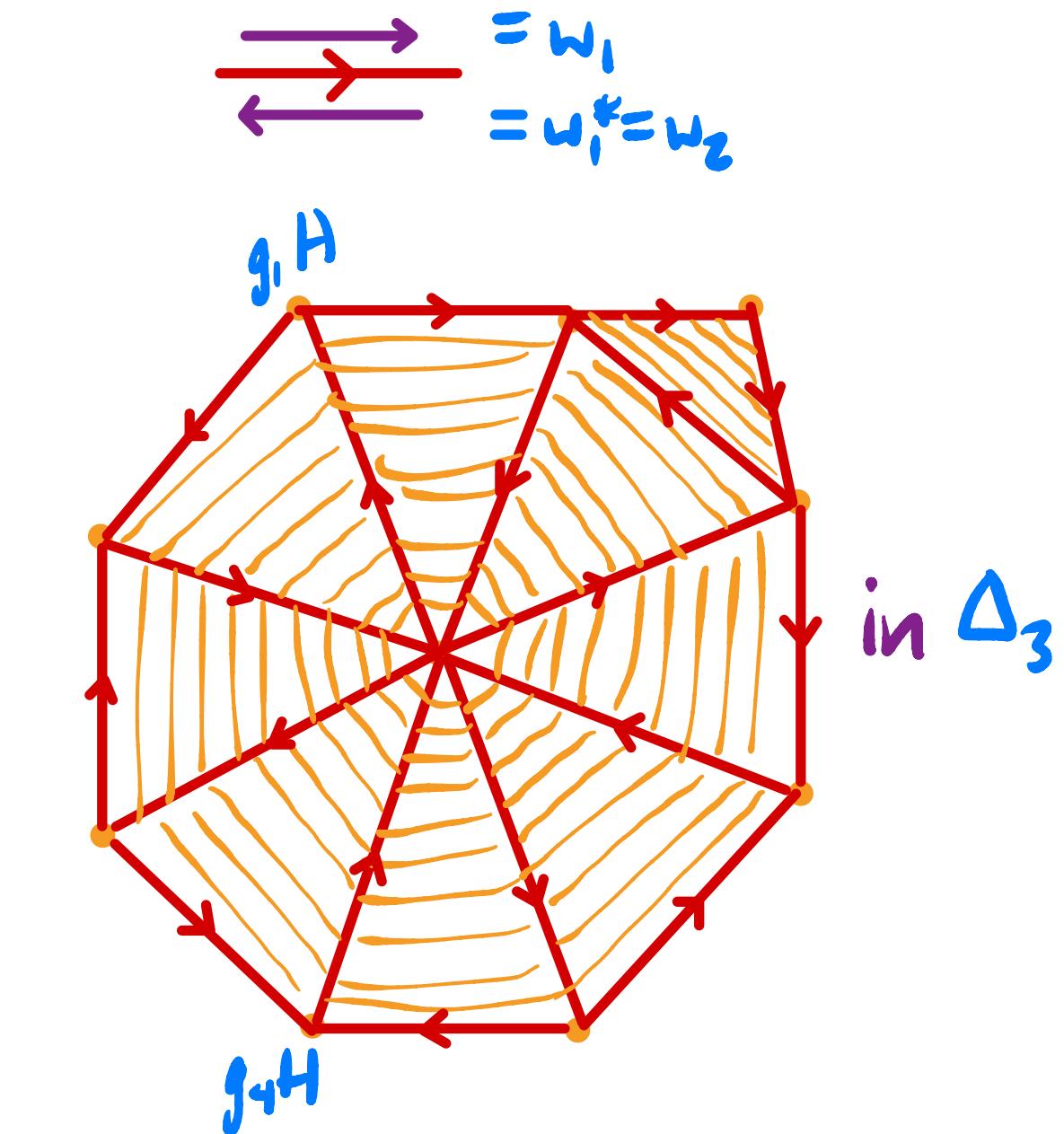
$$F_0 \text{---} F_1 \quad \Rightarrow \quad d(\phi(F_0), \phi(F_1)) = w_1$$

Affine Buildings

Ex There exist $g_1, g_2, g_3, g_4, \dots$ s.t.



$$\begin{aligned} d(g_1, g_2) &= w_1 \\ d(g_2, g_3) &= w_1 \\ &\vdots \\ d(g_4, g_1) &= \underline{\sum w_i} \\ &\vdots \\ &\text{geodesic distance} \end{aligned}$$

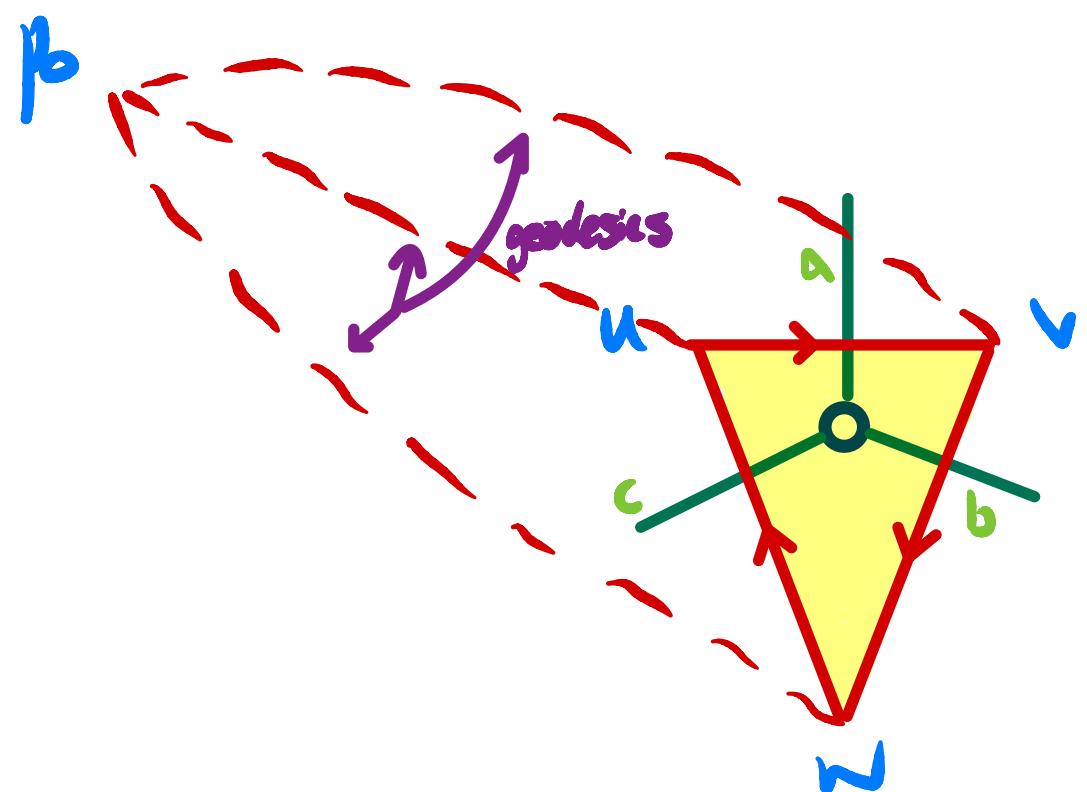


Affine Buildings

Q Growth rule label meaning?

A Let p_0 be base face.

Then



Fact $d(p_0, w) - d(p_0, v) \in S_r \cdot n_i$

Hence $d(p_0, w) - d(p_0, v) = e_a$

$$d(p_0, v) - d(p_0, w) = e_b$$

$$\underline{d(p_0, w) - d(p_0, u) = e_c}$$

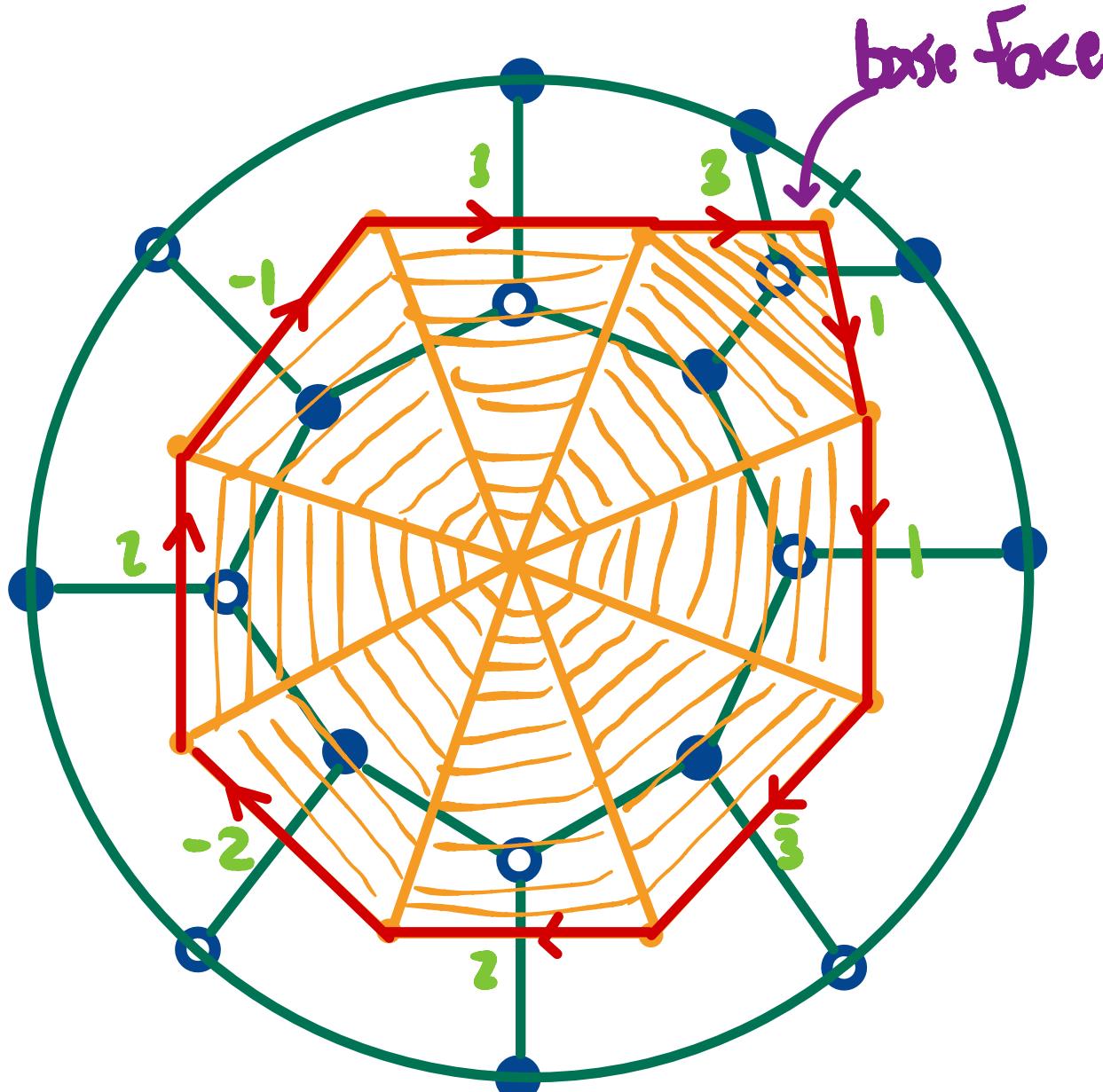
$$0 = e_a + e_b + e_c$$

$$\Leftrightarrow \{a, b, c\} = \{1, 2, 3\}$$

\Leftrightarrow proper labeling!

Affine Buildings

Ex



Boundary word:

$$1\bar{1}3\bar{2}\bar{2}2\bar{1}3\bar{3} \cong 111221322333$$

$$= \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 6 \\ \hline 4 & 5 & 8 & 9 \\ \hline 7 & 10 & 11 & 12 \\ \hline \end{array}$$

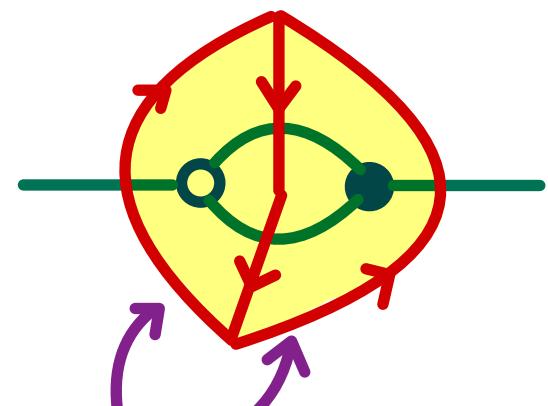
Connects to promotion
evacuation
Satake fibers

:

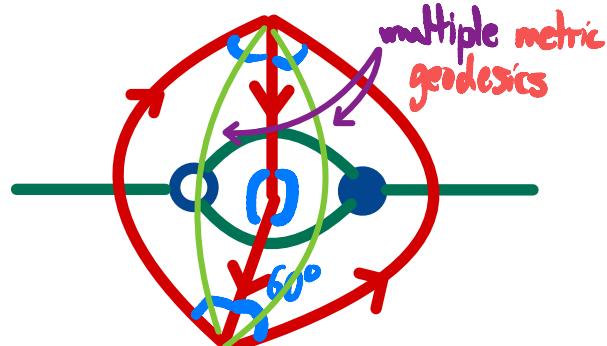
Affine Buildings

Q Non-elliptic condition meaning?

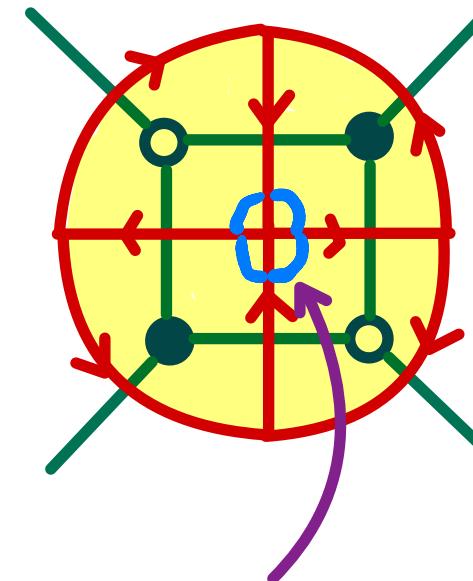
A



must be equilateral



Not CAT(0)!



Angle $4 \cdot 60^\circ = 240^\circ < 360^\circ$!

Also not CAT(0)!

(Also not coherent.)

The web basis problem

Problem (Khavinson-Kuperberg '96)

Give a web basis* for $\text{Inn}_{SL_r}(N \otimes \cdots \otimes V_n)$ for $r \geq 4$.

*with desirable properties for use in applications:

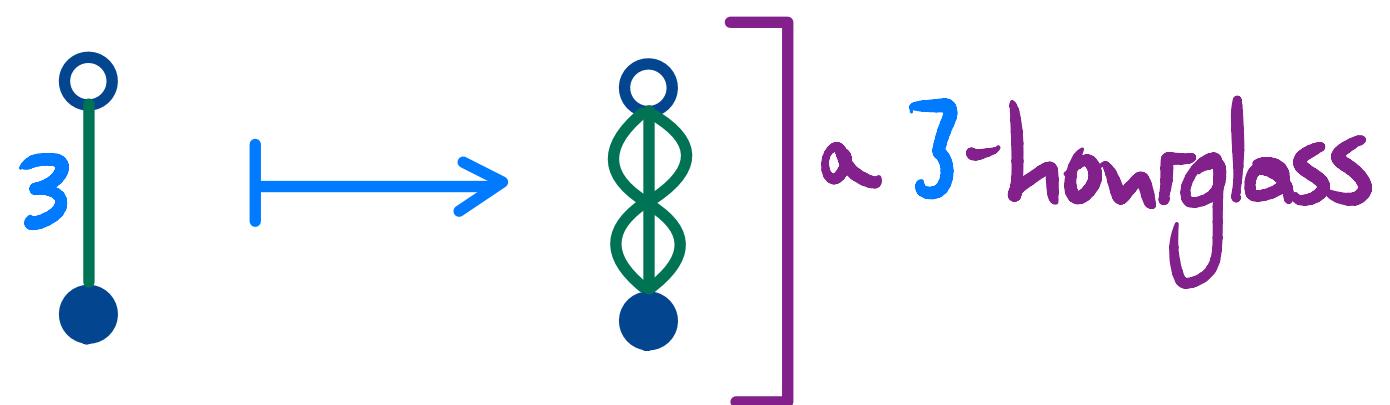
- testability
 - reduction rules
 - rotation invariance
-

Hourglass plabic graphs

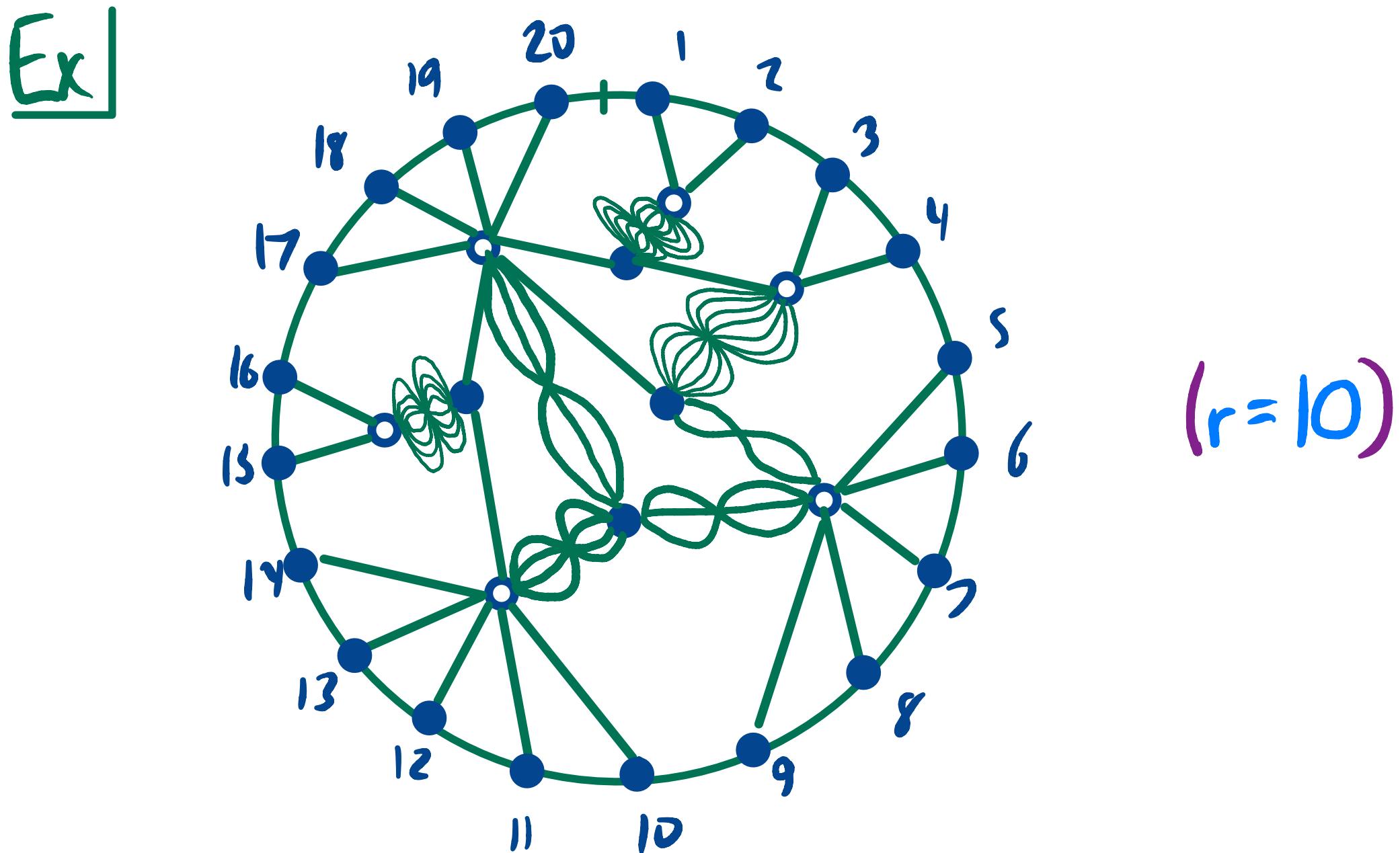
DfF ([Gaetz-Pechenik-Pfannerer-Striker-S. '23])

An r -hourglass plabic graph (r -HPG) is a planar bipartite graph embedded in a disk with edge weights in $[r]$ which sum to r around internal vertices, and boundary vertices have degree 1.

An edge with weight m is drawn as an m -hourglass:

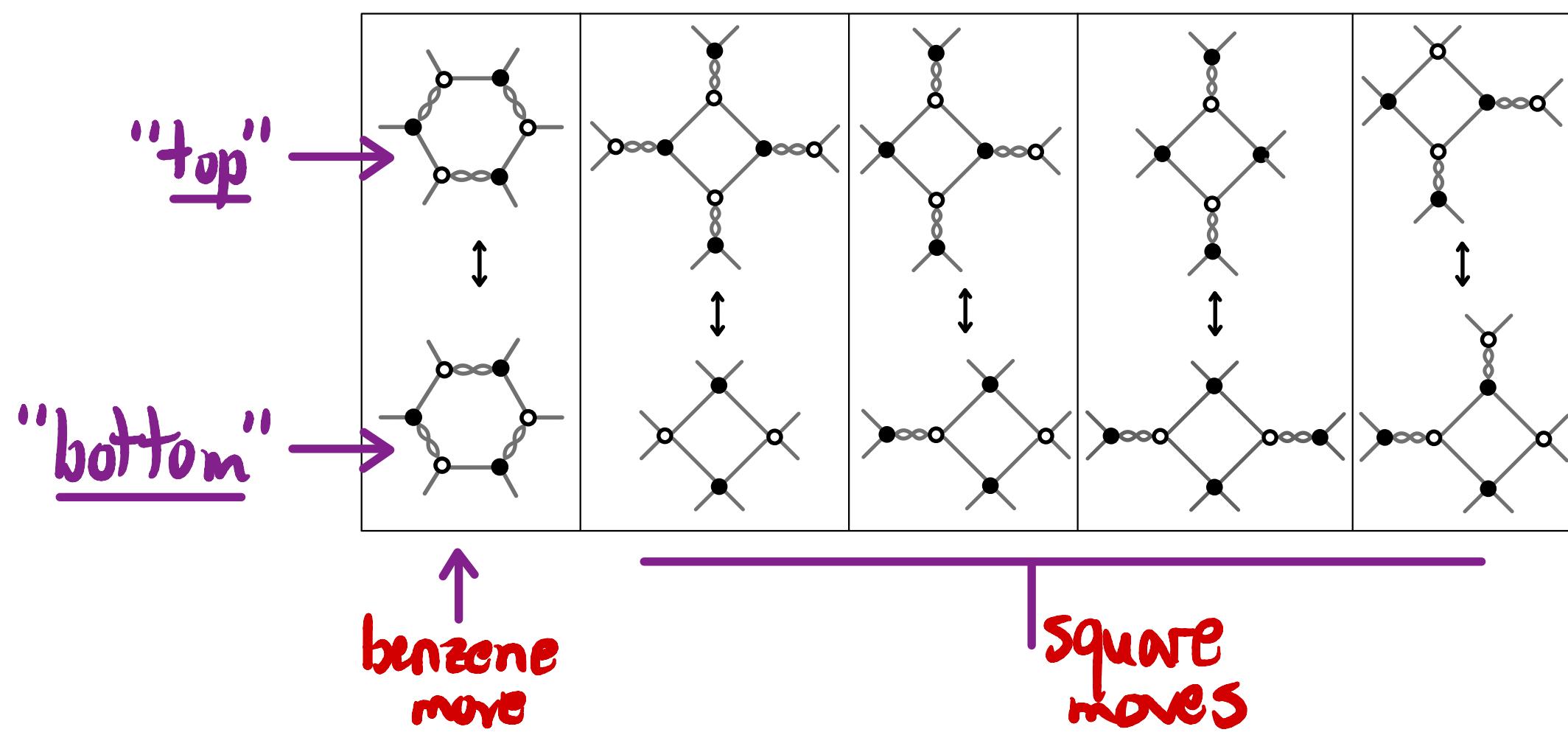


Hourglass plabic graphs



$r=4$ moves

Thm (GPPSS '23) Two contracted, fully reduced 4-HPG's have the same trip permutations

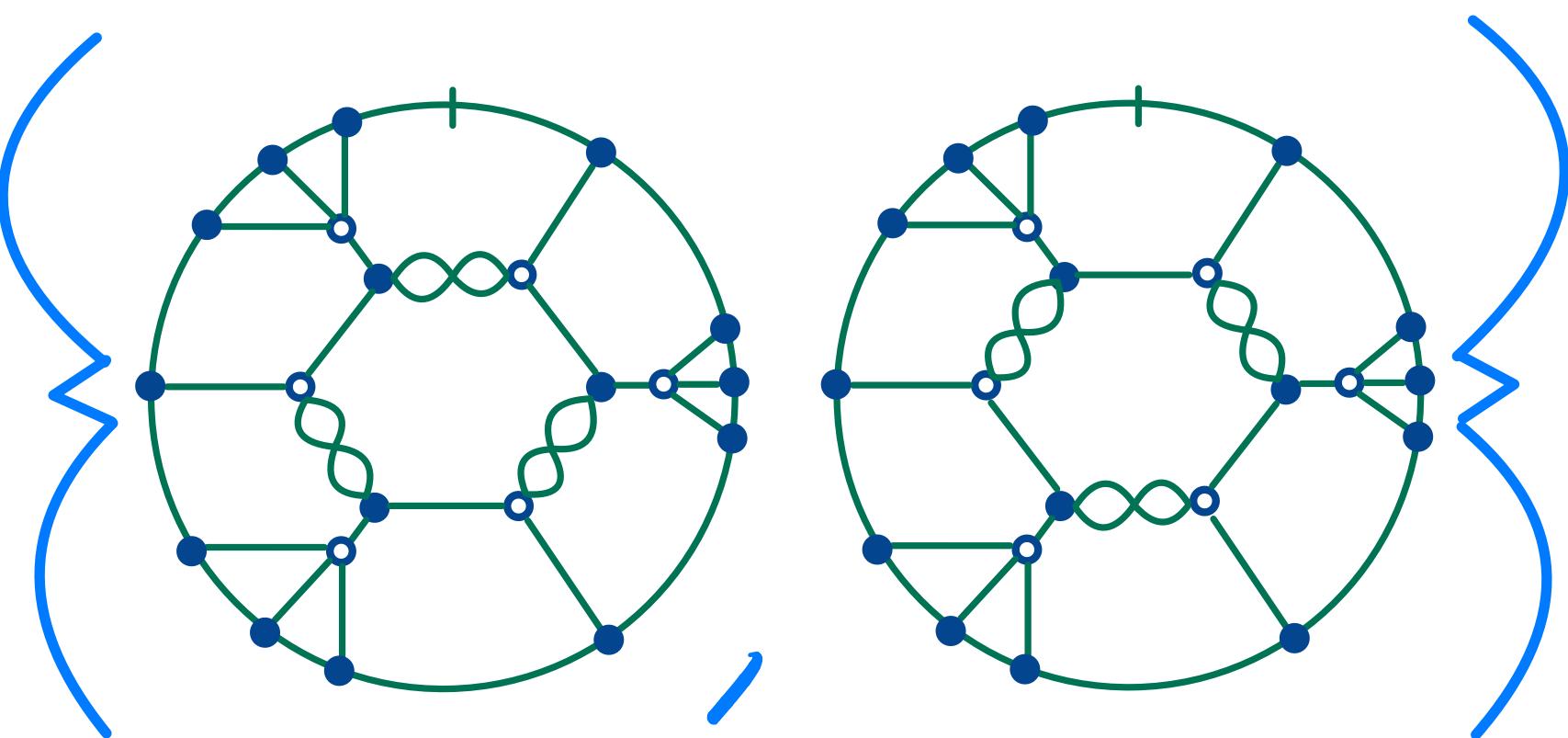


≈ 4 moves

Thm (GPPSS '23) There is a bijection
between $\text{SYT}(4 \times \square)$ and such move-classes.
It sends $\text{prom};(T)$ to $\text{trip};(G)$.

Ex

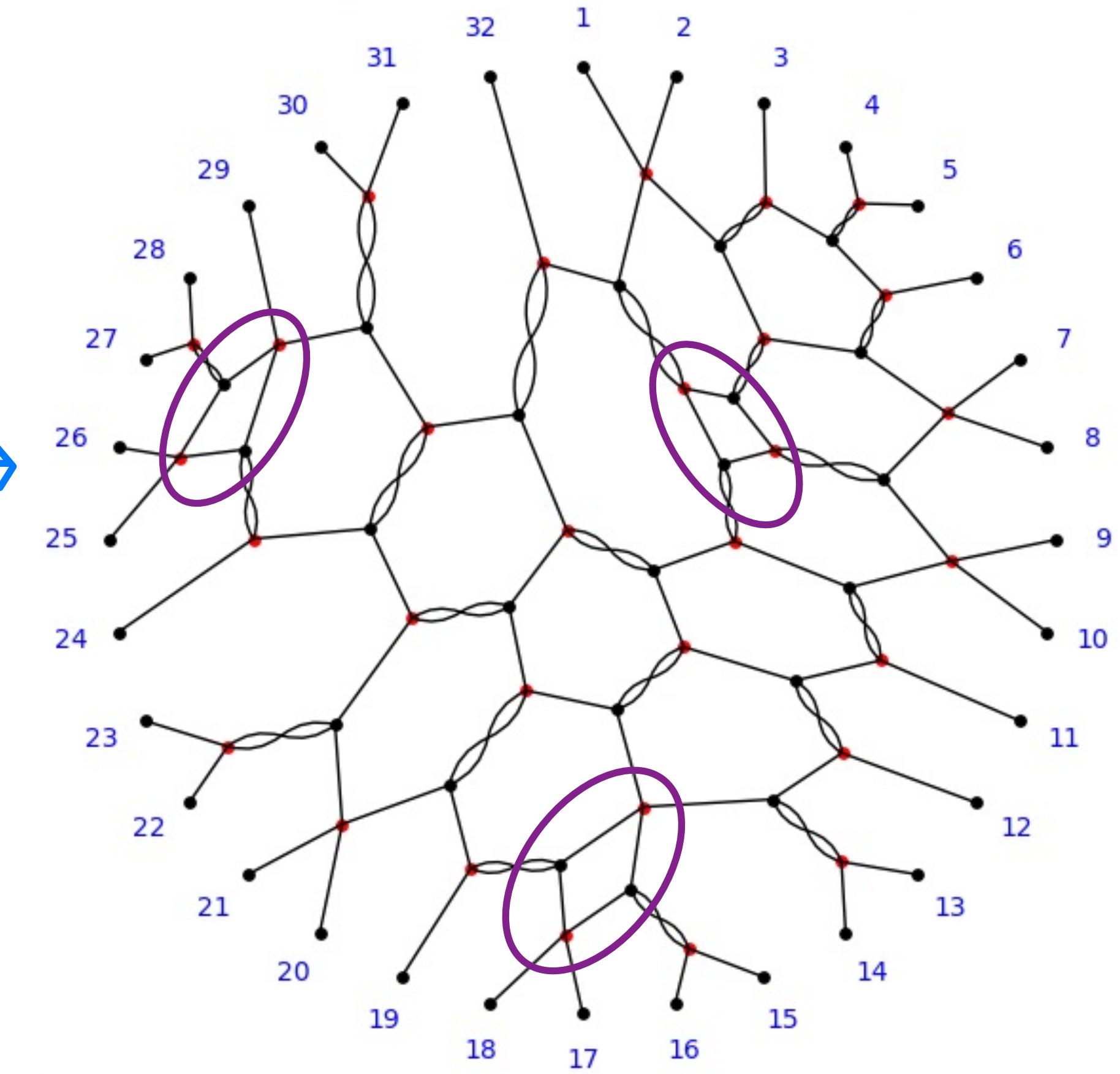
$$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 6 \\ \hline 3 & 5 & 10 \\ \hline 4 & 7 & 11 \\ \hline 8 & 9 & 12 \\ \hline \end{array}$$



≈ 4 moves

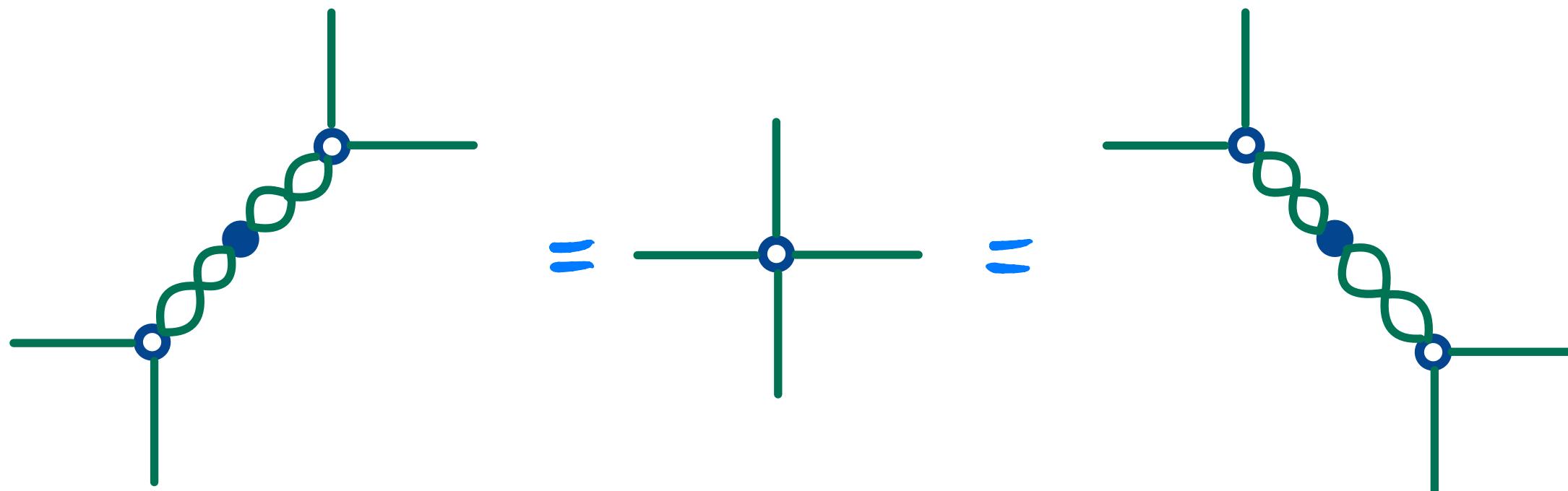
Ex

1	3	4	7	8	17	19	23
2	5	6	9	14	18	21	24
10	12	13	15	16	25	26	28
11	20	22	27	29	30	31	32



Pockets

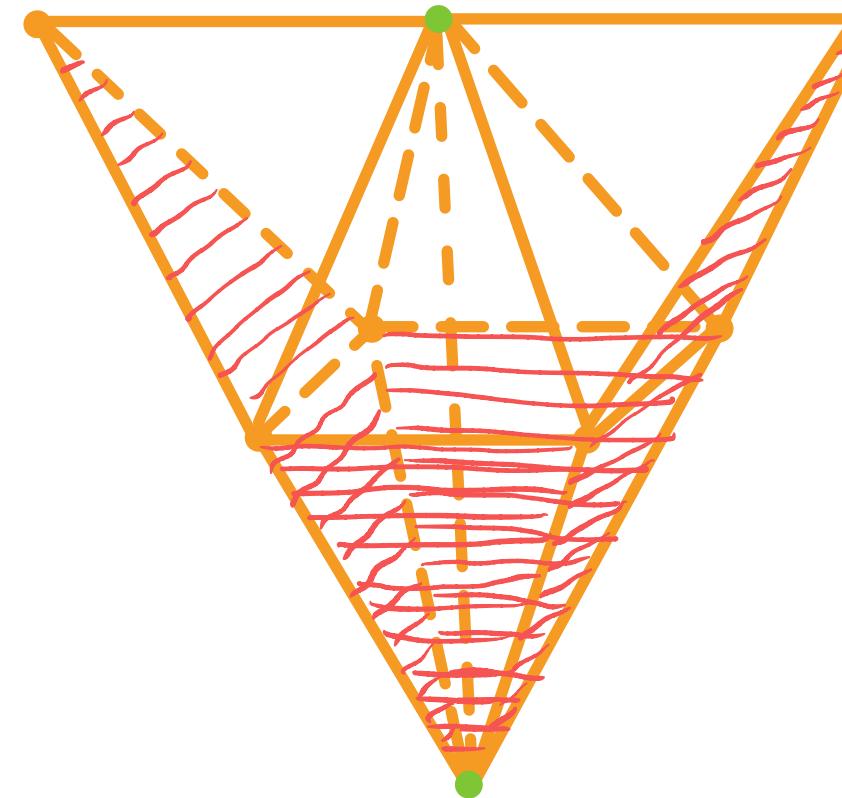
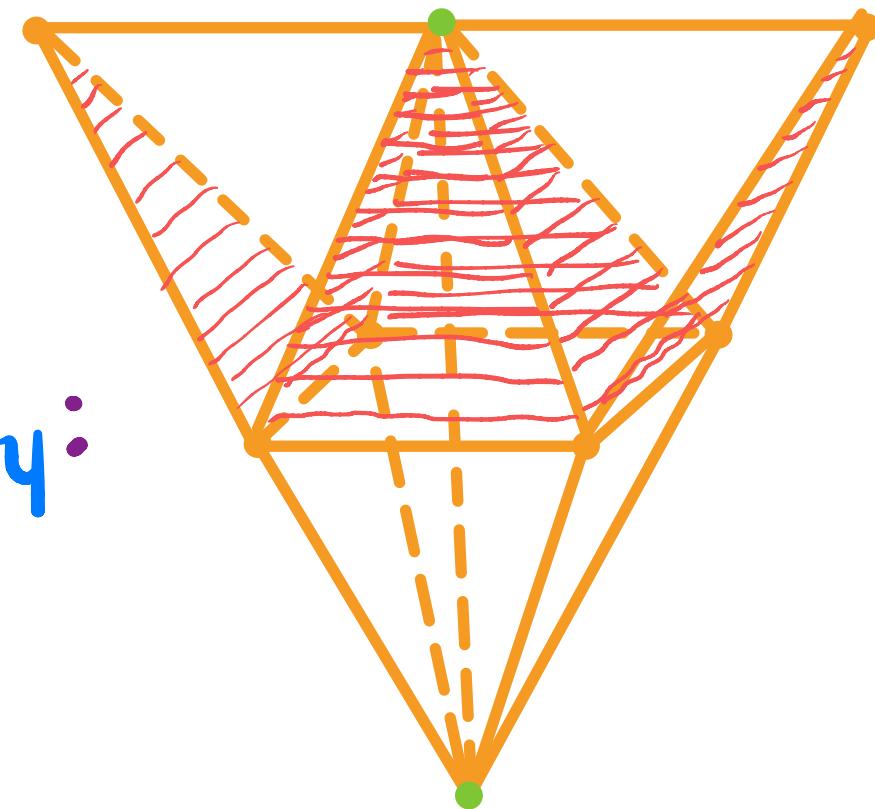
- We (GSSW) show there is a 3D analogue for the $\text{SL}(4)$ web equivalence classes: pockets in Δ_4 .
- Requires expanding sources/sinks via $I=H$ moves:



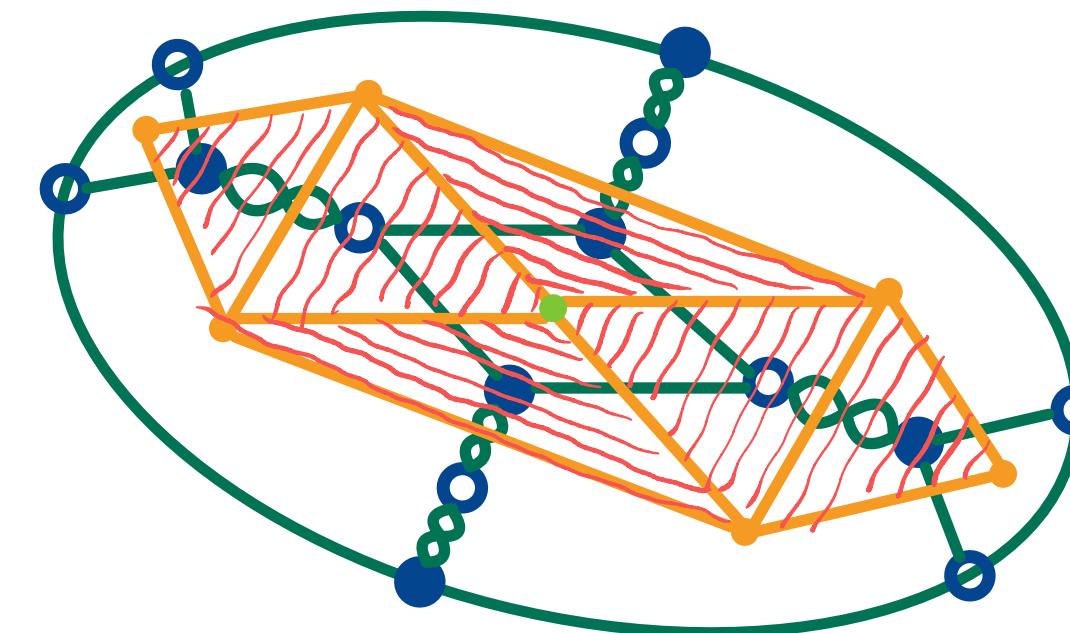
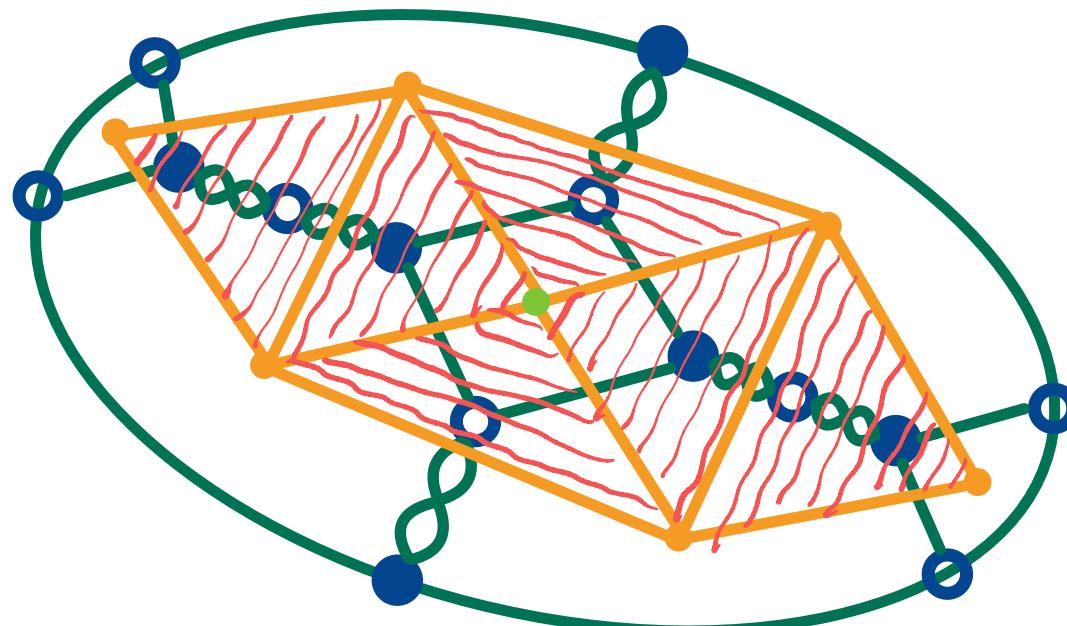
Pockets

Ex

In Δ_4 :



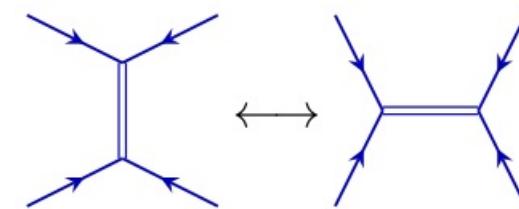
Pocket of
square+benzene+IH
class of
 $\bar{4}\bar{3}1,2\bar{3}\bar{3}\bar{2}\bar{4}\bar{3}3,4\bar{3}\bar{1}$



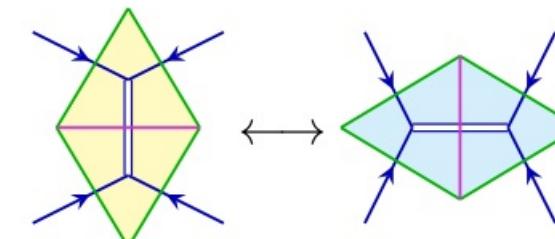
+6 more

Pockets

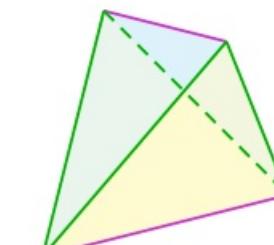
- Build pockets from 4-HPG basis web move classes:



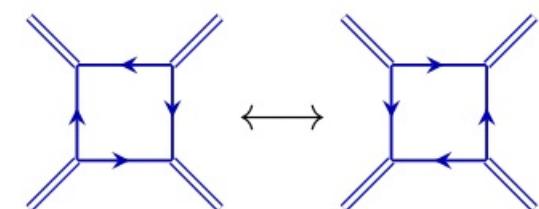
IH move between webs



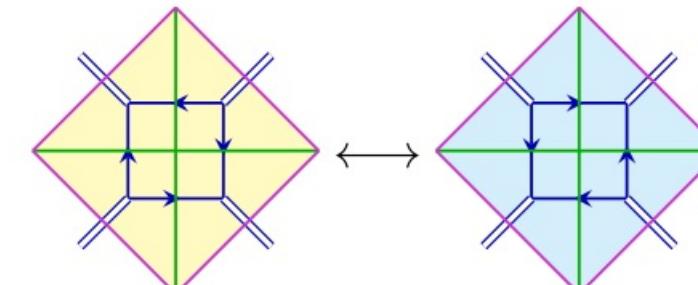
Dual diskoids



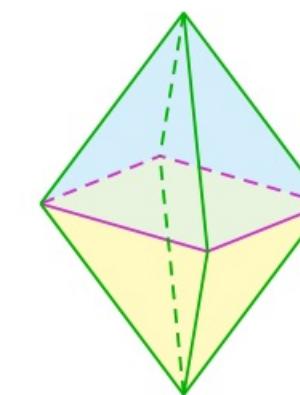
tetrahedron



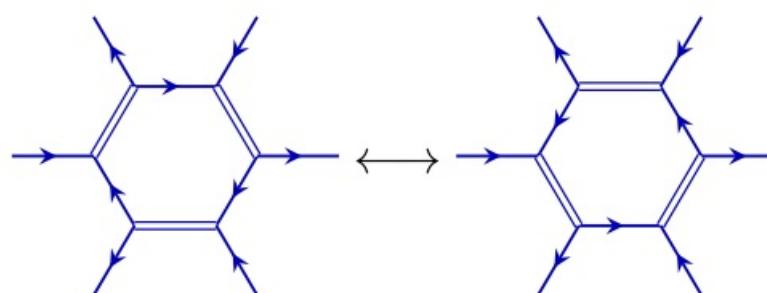
Square move between webs



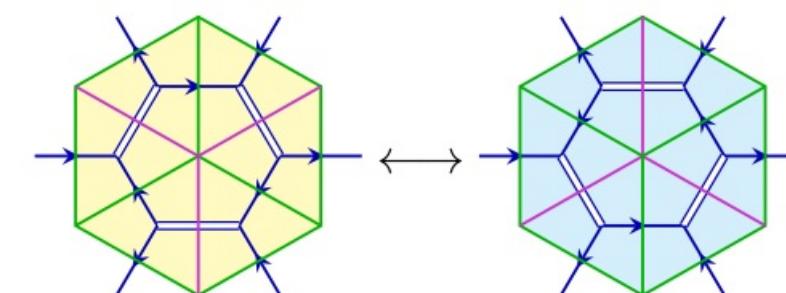
Dual diskoids



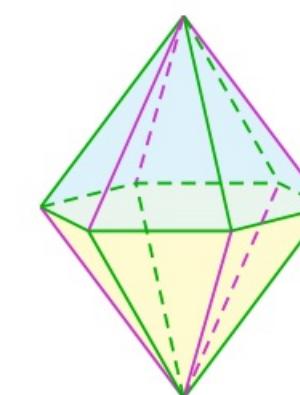
octahedron



Benzene move between webs



Dual diskoids



dodecahedron

Pockets

Thm (GSSW '24+) The product $P = P(T)$ is coherent, (AT(0)).

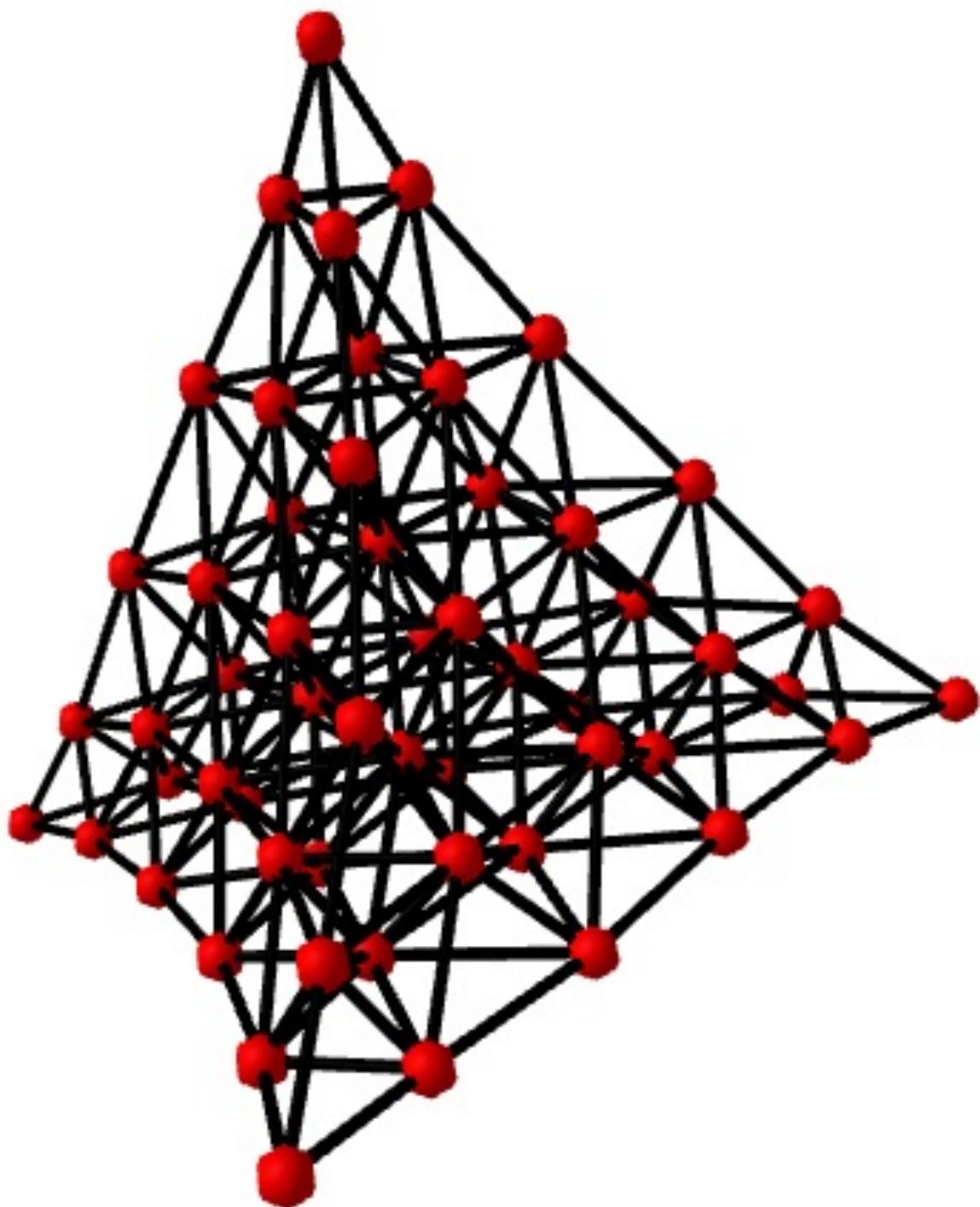
The webs are "sections".

Thm (GSSW '24+) Given an irreducible component of a Satake fiber of $\Delta = \Delta(\mathrm{SL}_4^\vee)$ indexed by T , there is a dense open set U s.t. every point of U extends uniquely to a configuration $P(T) \hookrightarrow \Delta$ which preserves distances.

Pockets

Ex | Product of 5×5 ASM:

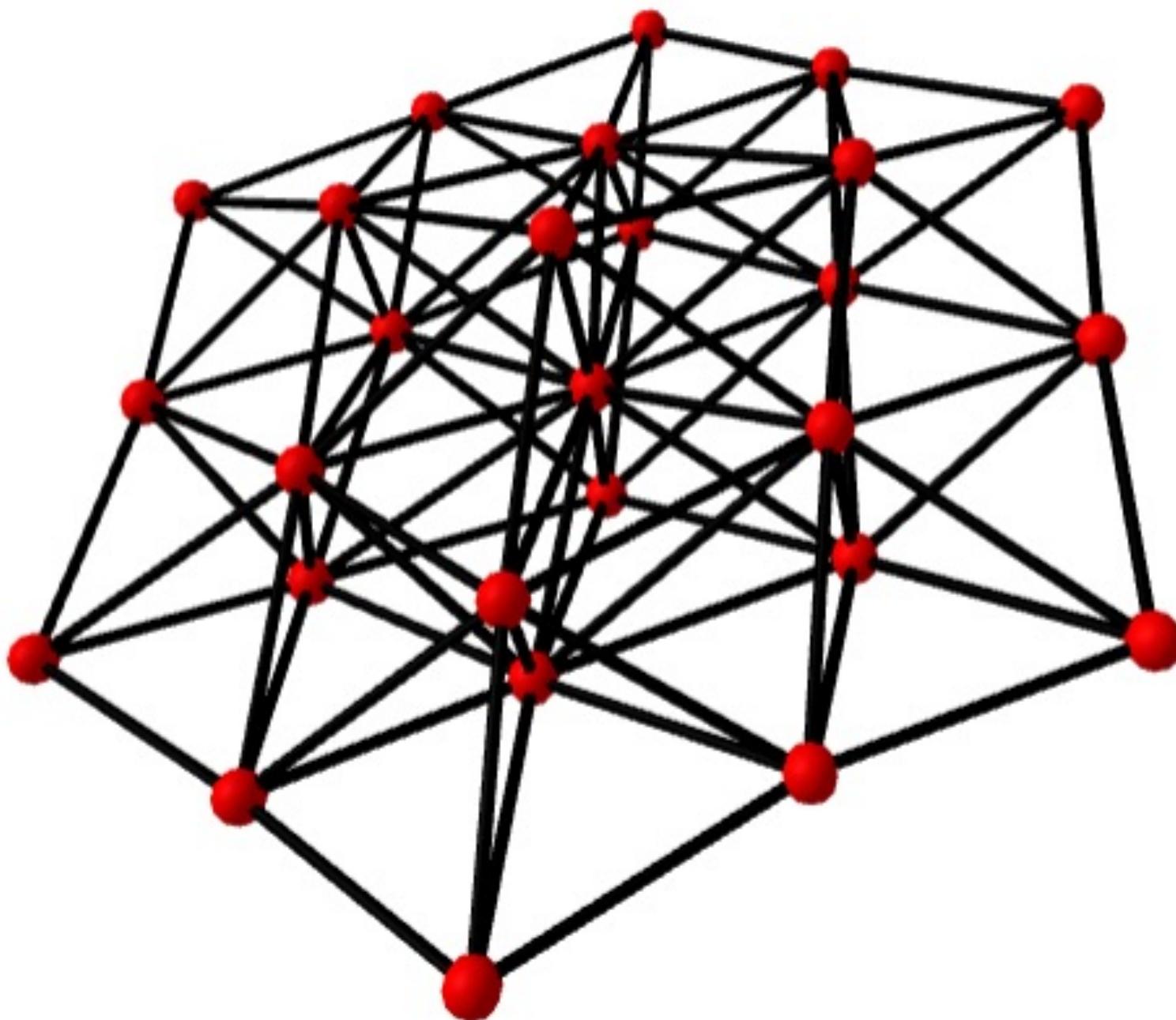
$$L = 1^s 2^s 3^s 4^s$$



Note | Related to height functions, octahedral recurrence, tilings of the Aztec diamond, distributive lattices

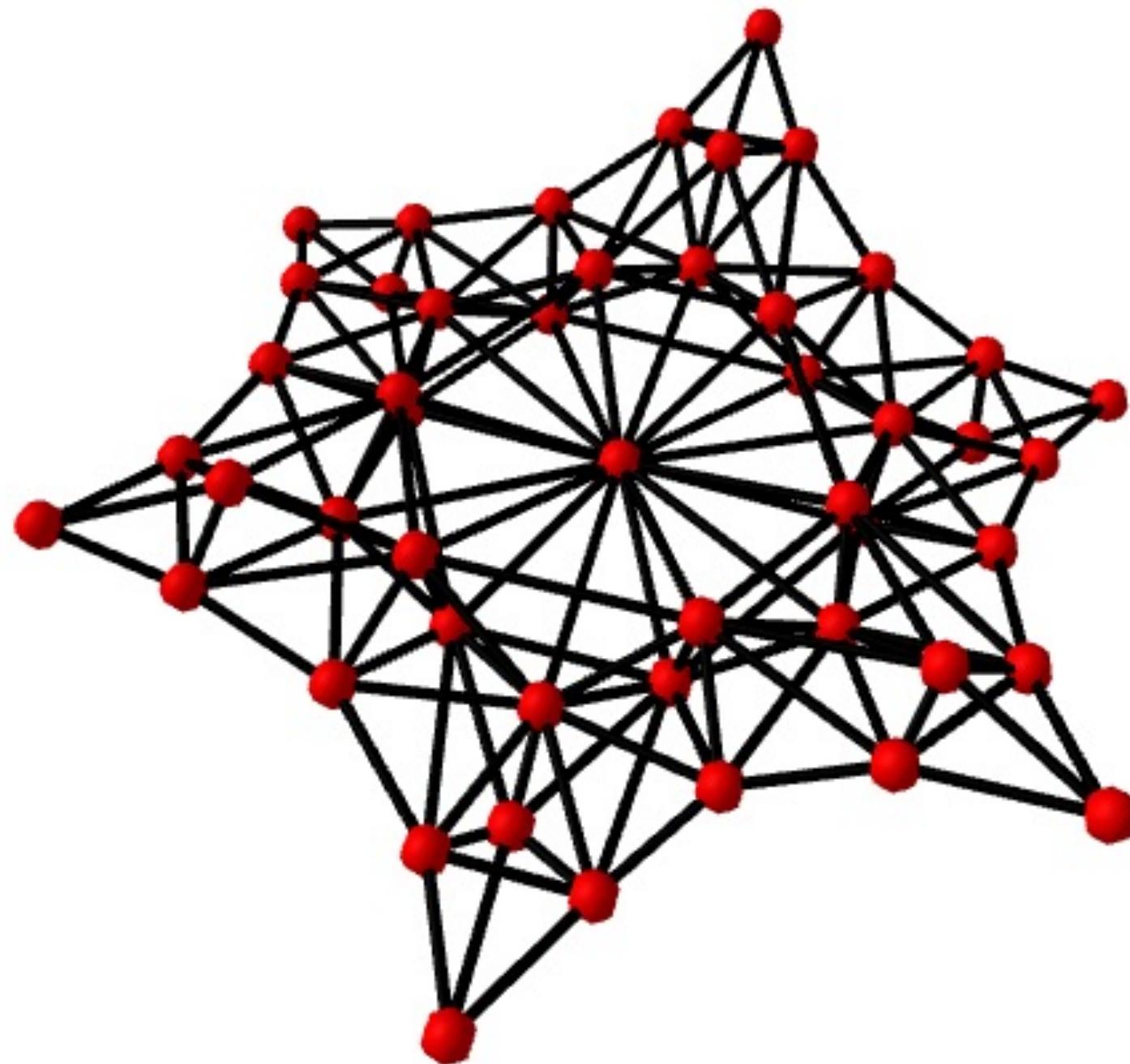
Pockets

Ex | Product of $2 \times 2 \times 2$ PP:



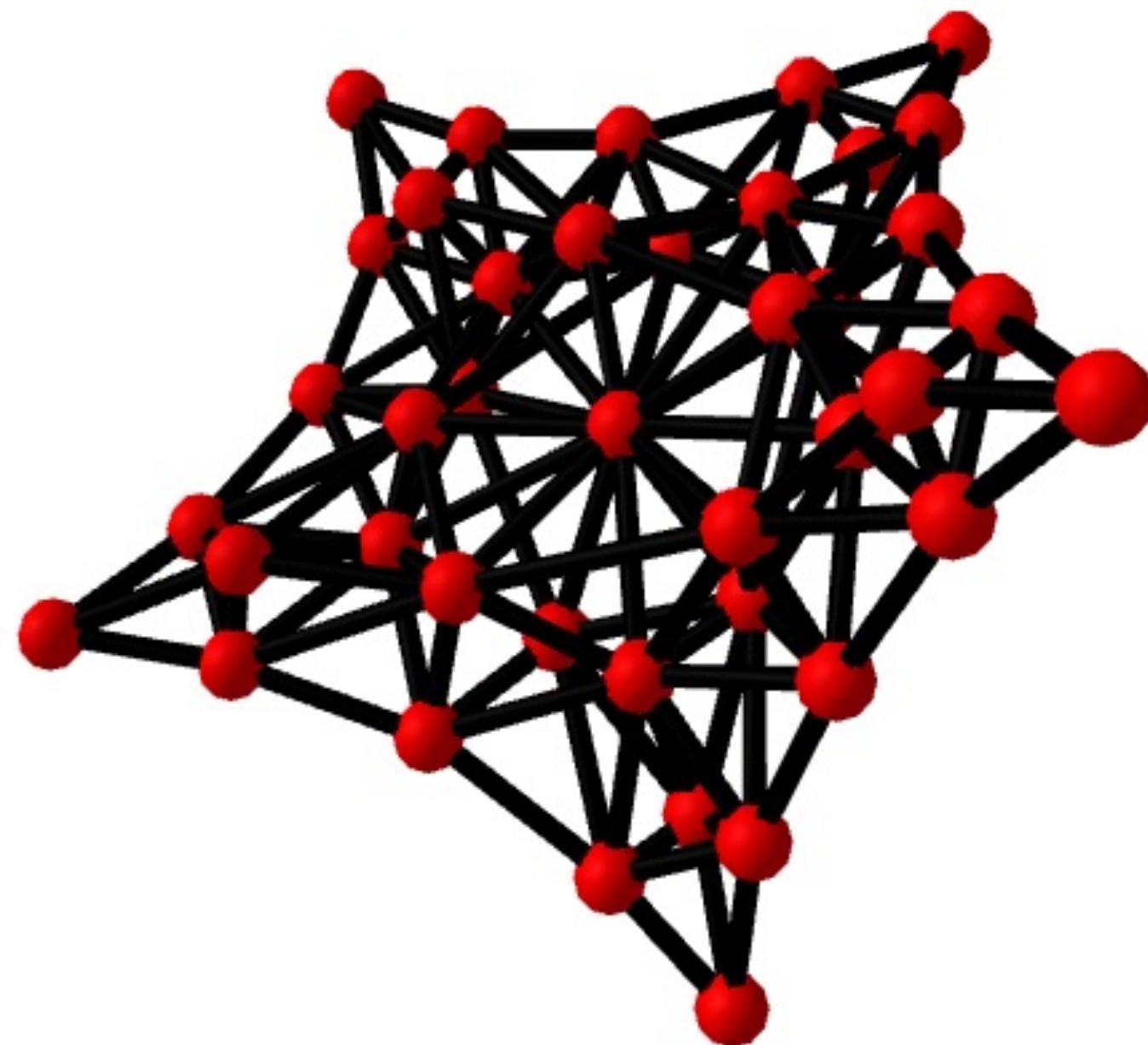
Pockets

Ex] Pocket of a "chained hexagon":



Pockets

Ex] Product of a "chained pentagon":

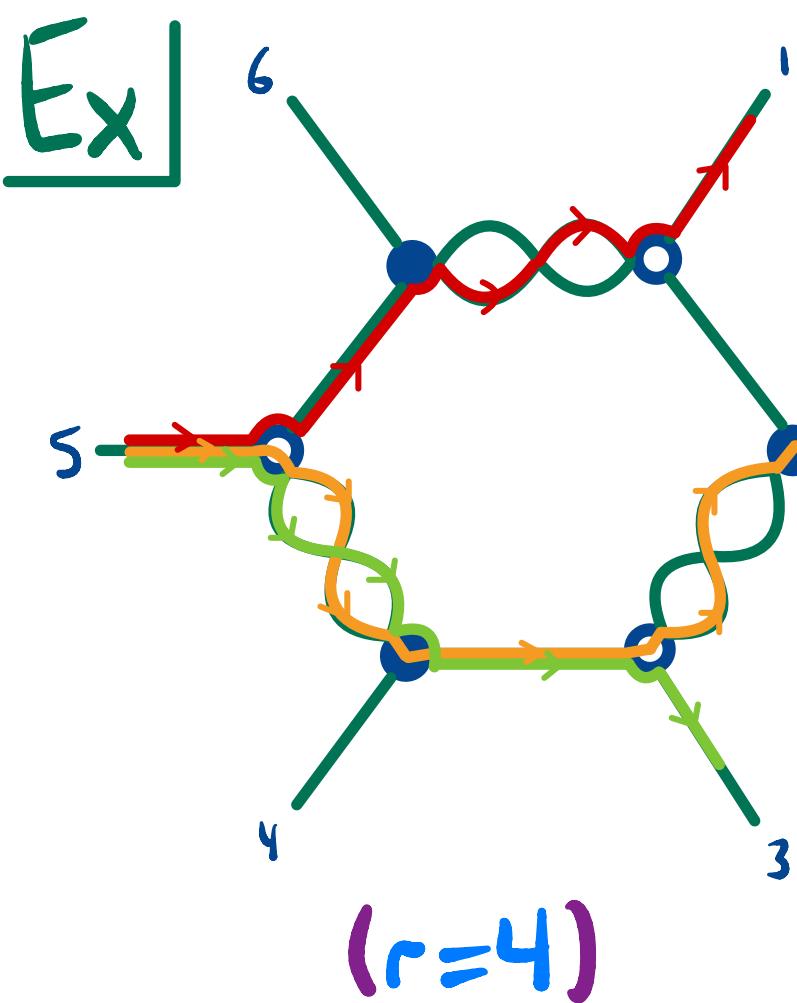


Note] Not realizable
in \mathbb{R}^3

THANKS!

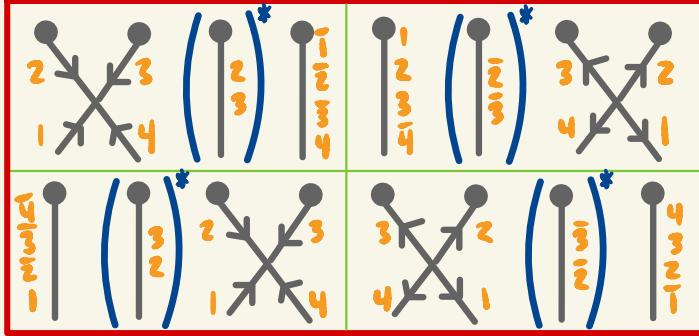
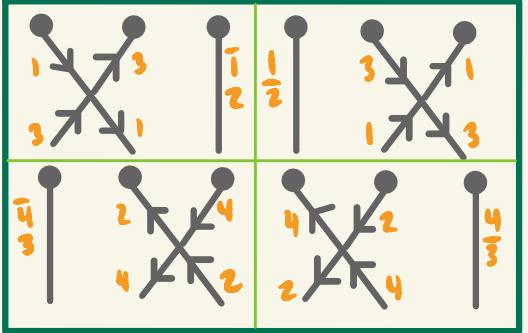
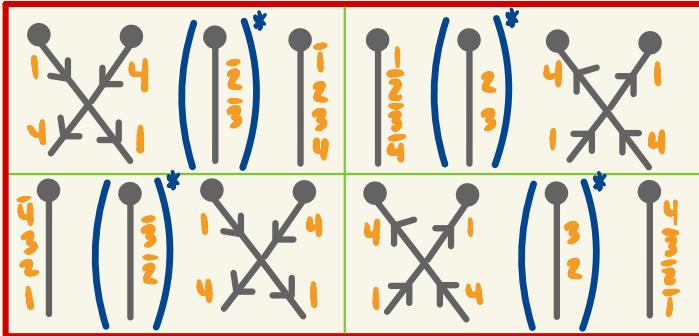
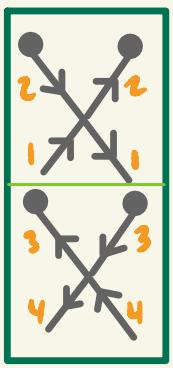
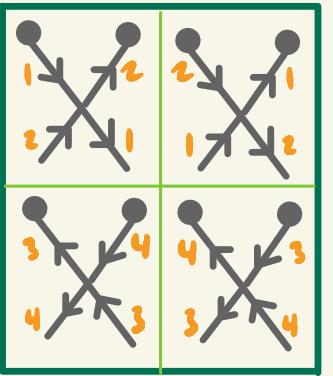
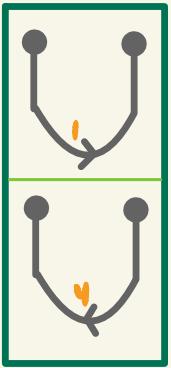
Trip permutations

Def (GPPSS '23) An r -hourglass plabic graph has trip permutations $\text{trip}_1, \dots, \text{trip}_{r-1}$ where trip_i takes the i th left at white and i th right at black:



$$\begin{aligned} &\rightarrow = \text{trip}_1 = (135)(642) \\ &\rightarrow = \text{trip}_2 = (14)(25)(36) \\ &\rightarrow = \text{trip}_3 = (531)(246) \end{aligned}$$

Note
 $\text{trip}_i = \text{trip}_{r-i}^{-1}$!



SL(4) growth rules

