Webs, pockets, and buildings

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Based on joint work with subsets of *Christian Gaetz, Oliver Pechenik, Stephan Pfannerer, Jessica Striker, and Haihan Wu*

arXiv:2306.12501 (4-row)
arXiv:2402.13978 (2-column)
arXiv:2306.12506 (promotion permutations)

Slides: https://www.jpswanson.org/talks/2025_Michigan_pockets.pdf

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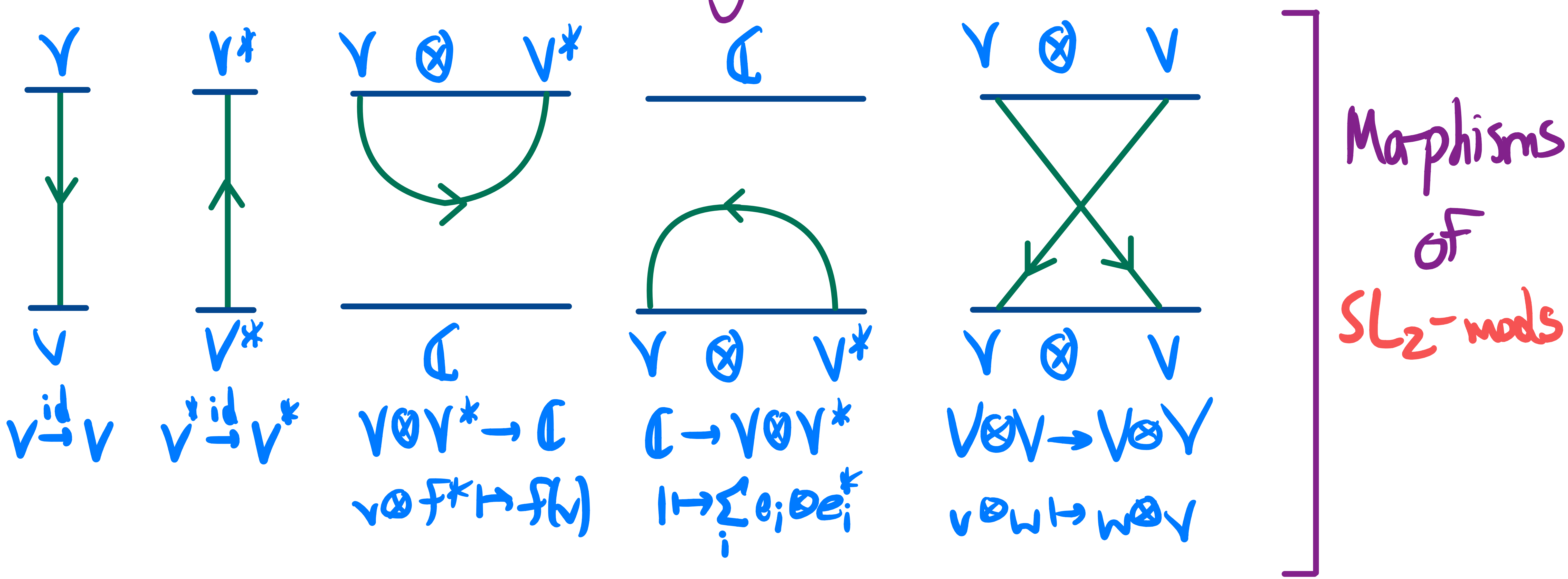
Outline

- SL_2-webs and SL_3-webs
  - Temperly-Lieb and non-elliptic bases
- Building embeddings
- Hourglass plabic graphs
  - (New!) SL_4 web basis
- Pockets and buildings

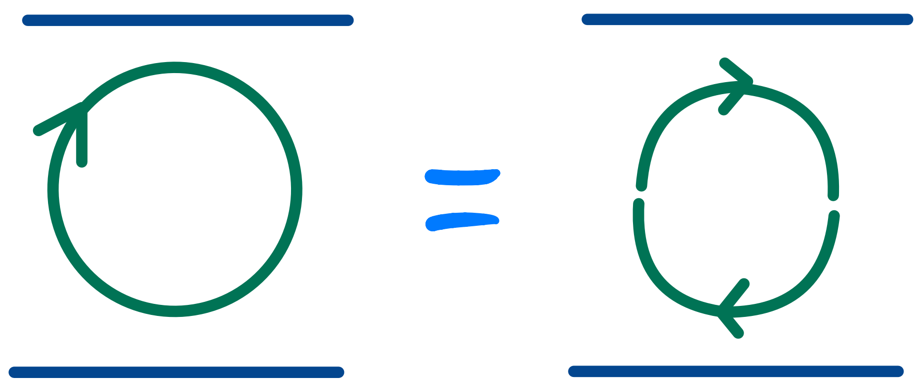
SL_2-Webs

- Webs are a graphical calculus for representations.
- Let $V = \mathbb{C}^2$. Some building blocks:



SL_2-Webs

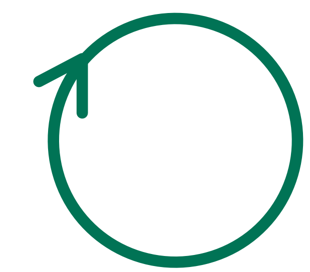
Ex



$$\mathbb{C} \to V \otimes V^* \to \mathbb{C}$$

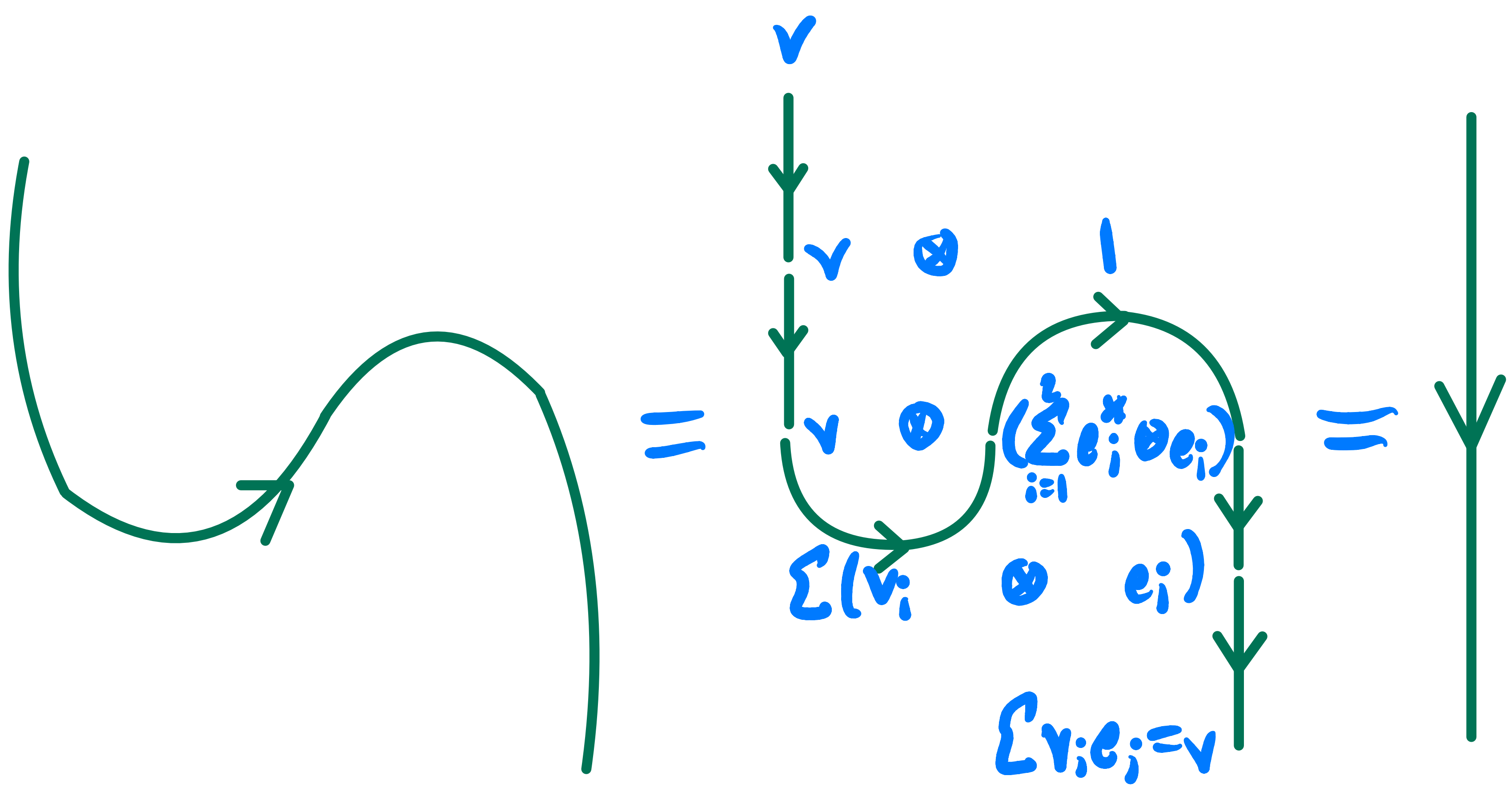
$$1 \mapsto e_1 \otimes e_1^* + e_2 \otimes e_2^* \mapsto 1+1=2$$

$\Rightarrow$



$= 2$

Ex



SL_2-Webs

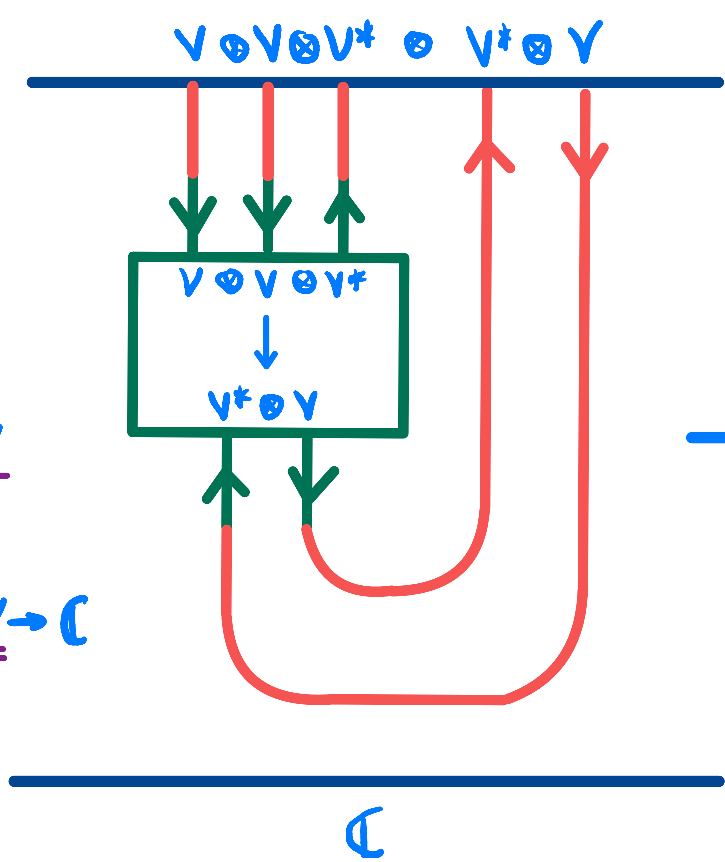
- Can "rotate" factors from codomain to domain with duals:

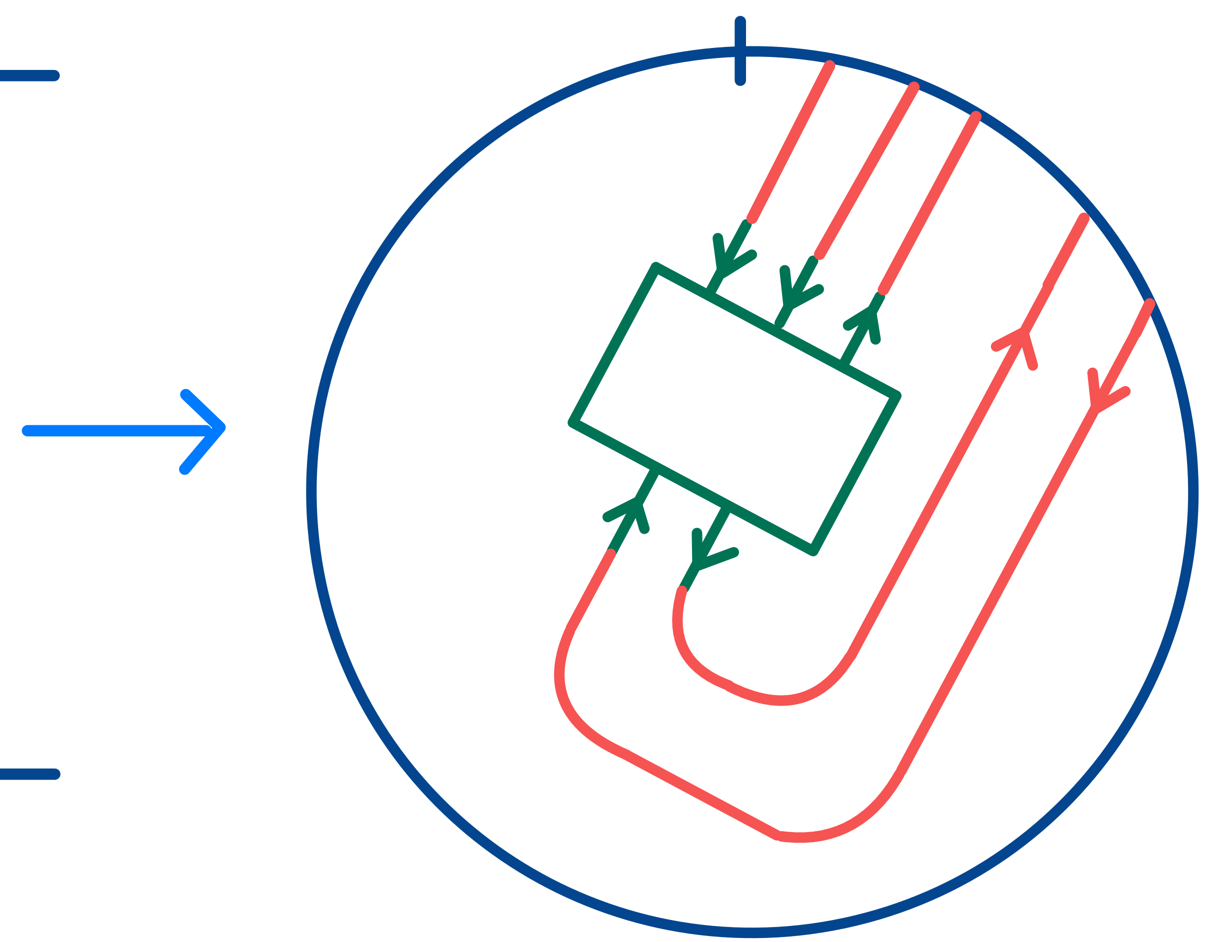
$$\mathrm{Hom}_G(A, B\otimes \underline{C}) \cong \mathrm{Hom}_G(A\otimes \underline{C^*}, B)$$

(Tensor-hom adjunction and $\mathrm{Hom}_{\mathbb{C}}(X,Y) \cong Y^*\otimes X$.)

Ex

$V\otimes V\otimes V^* \to \underline{\underline{V^*}}\otimes \underline{V}$ becomes $V\otimes V\otimes V^*\otimes \underline{\underline{V^*}}\otimes \underline{\underline{V}} \to \mathbb{C}$



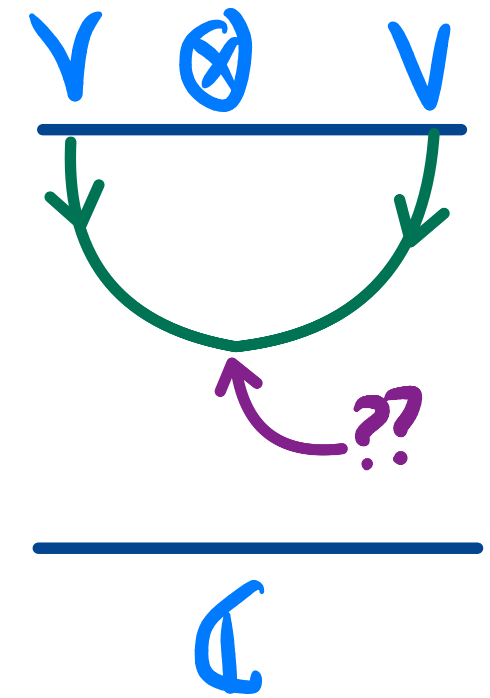


$\in \mathrm{Hom}_G(-, \mathbb{C})$
$= \mathrm{Inv}(-)$

Graphs in disks

SL_2-Webs

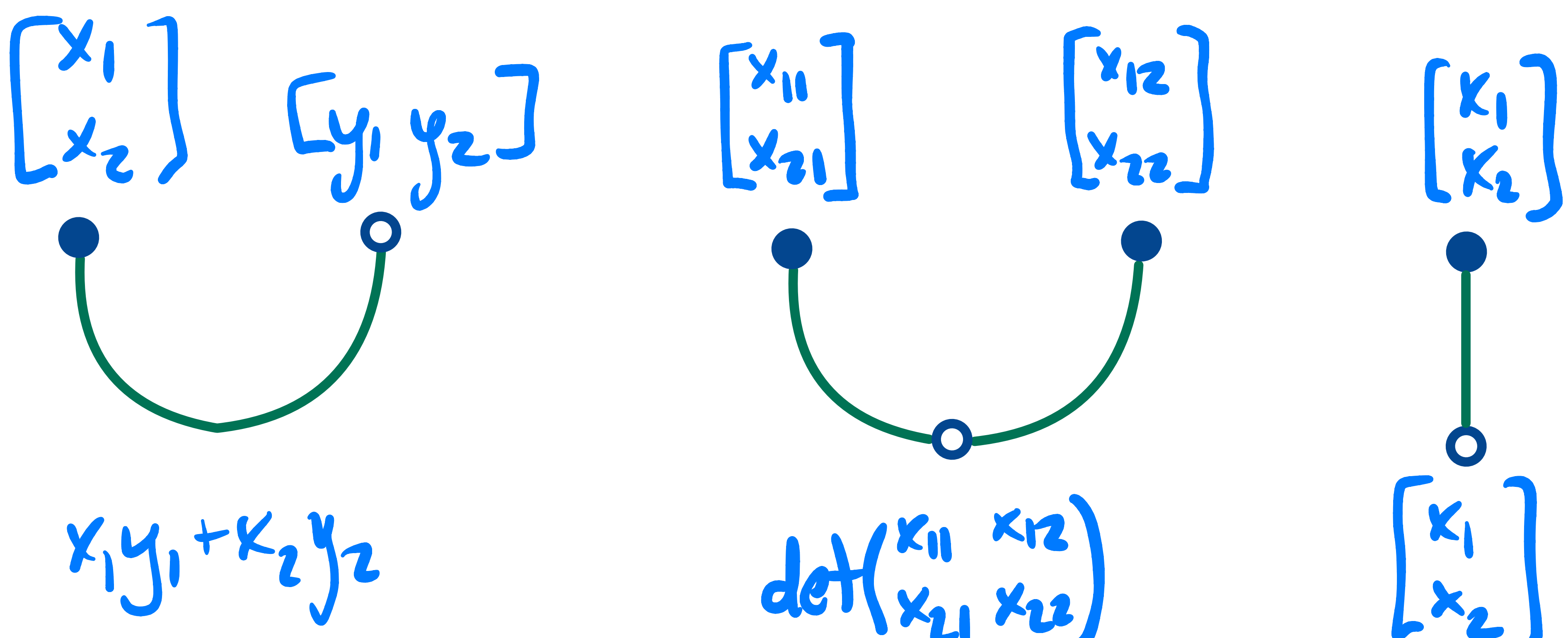
- What about $\det: V \otimes V \to \mathbb{C}$? Use



maybe?

- Switch to <u>bipartite</u> graphs:

  – Define:



SL_2-Webs

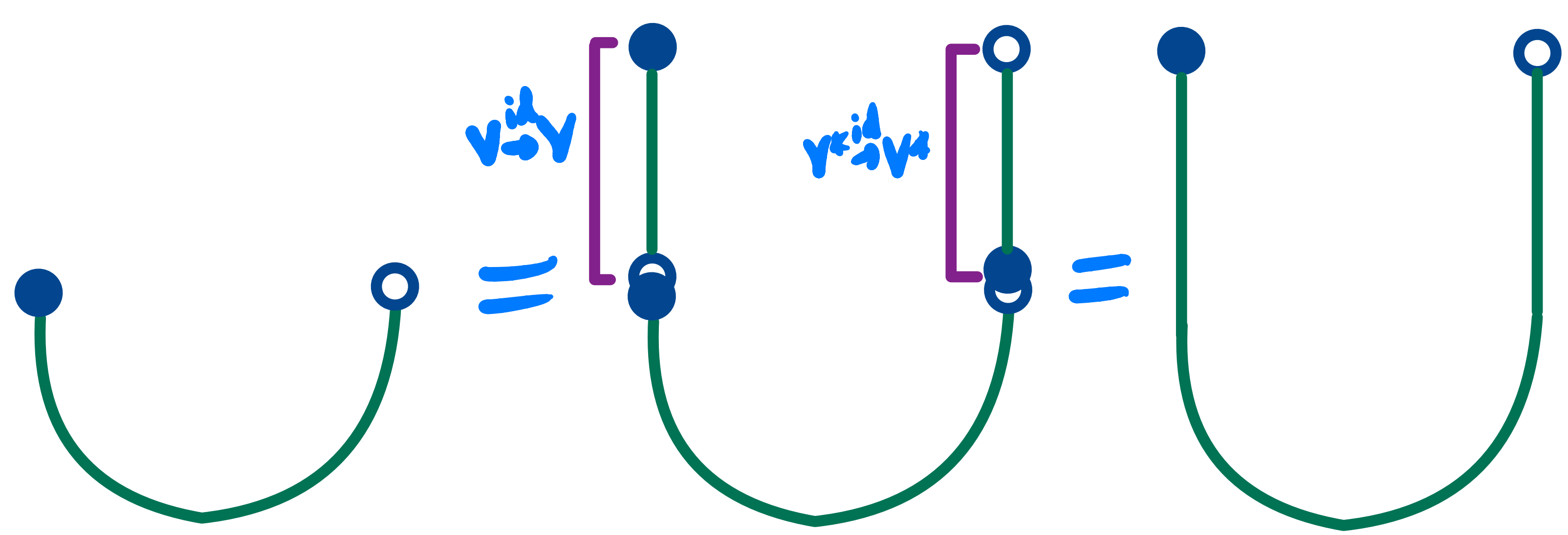
- Bipartite composition conventions

  ● $= V$ ○ $= V^*$ in domain

  ○ $= V$ ● $= V^*$ in codomain

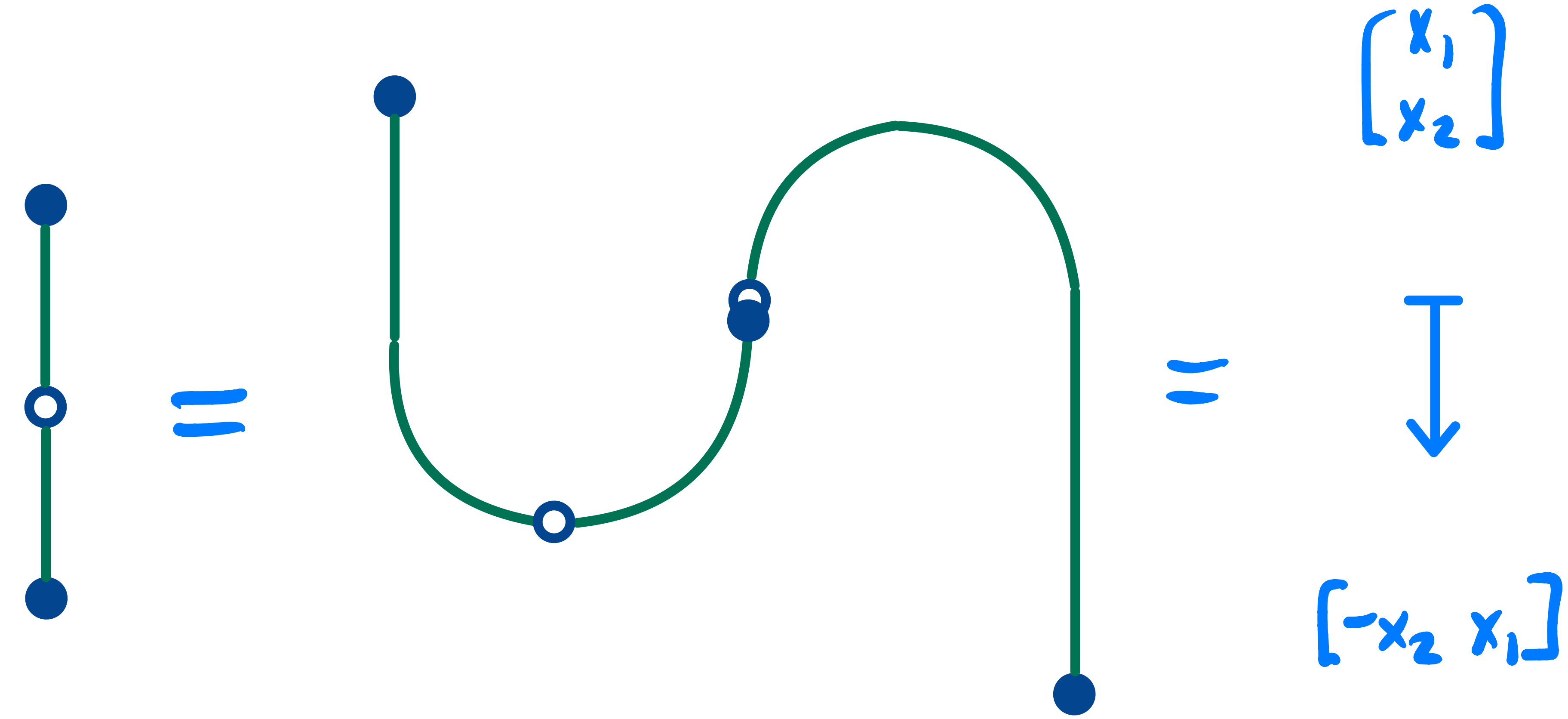
  — When composing, must match and cancel $\substack{\bullet \\ \circ}$ pairs:

Ex

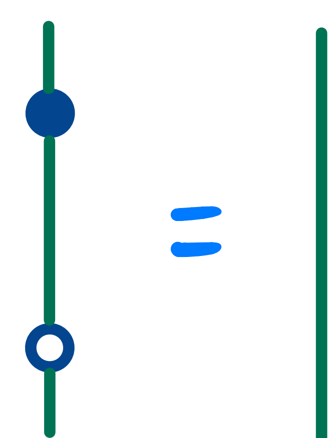


SL_2-Webs

Ex



Ex Have contraction relation:

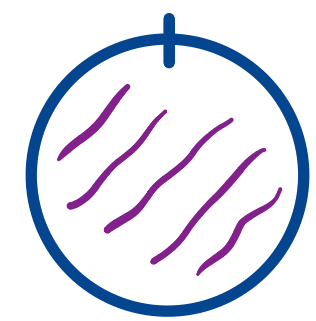


SL_2-Webs

Def An SL_2-web is

- a bipartite graph embedded in

or



- with degree 2 internal vertices,
- and degree 1 boundary vertices.

SL_2-Webs

Facts

1. Have well-defined map

$$\{SL(2)\text{ webs}\}/\text{isotopy} \longrightarrow \left\{ \begin{array}{c} SL(2)\text{ morphisms} \\ V^{\pm} \otimes V^{\pm} \otimes \cdots \to V^{\pm} \otimes \cdots \end{array} \right\}$$

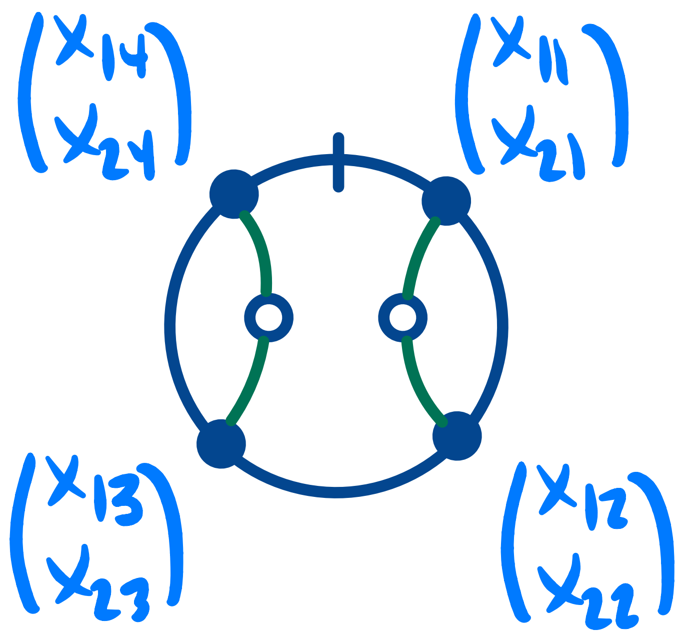
2. All of $\mathrm{Rep}(SL_2)$ is encoded in disk case

(... surjective up to linear combinations, take Karoubi envelope...)

Q Generators and <u>relations</u>? <u>Bases</u>??

SL_2-Webs

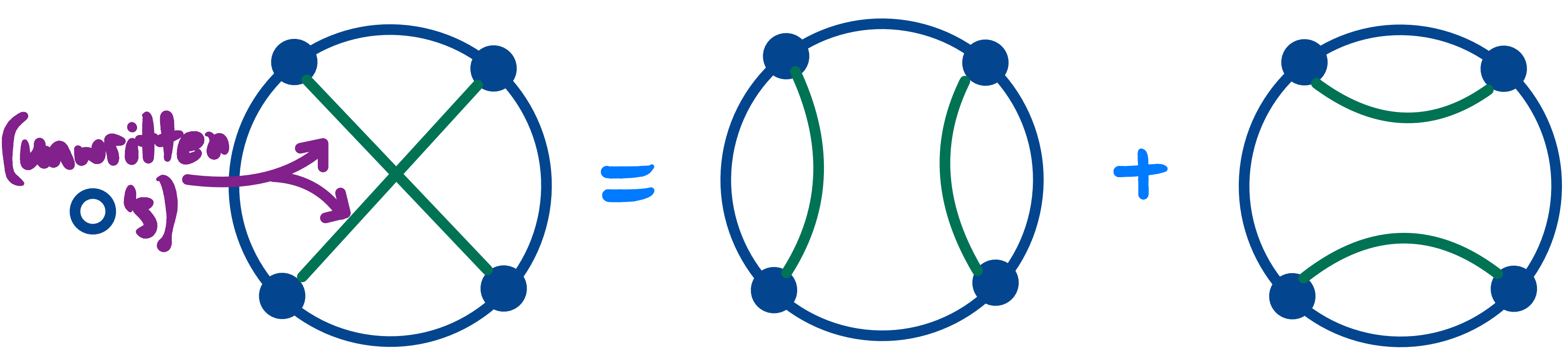
Ex



$$= \det\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \cdot \det\begin{pmatrix} x_{13} & x_{14} \\ x_{23} & x_{24} \end{pmatrix} \in \mathrm{Inv}(V^{\otimes 4})$$

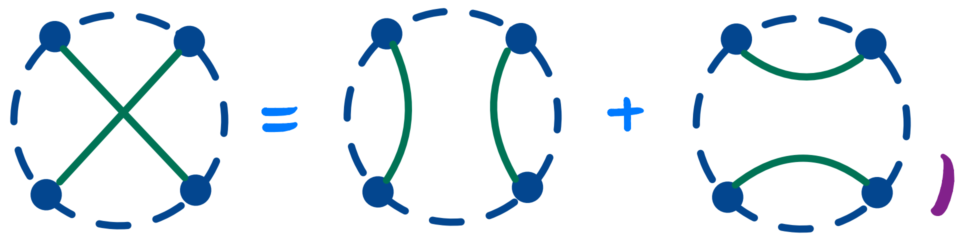
$$= \mathrm{Hom}_{SL_2}(V^{\otimes 4}, \mathbb{C})$$

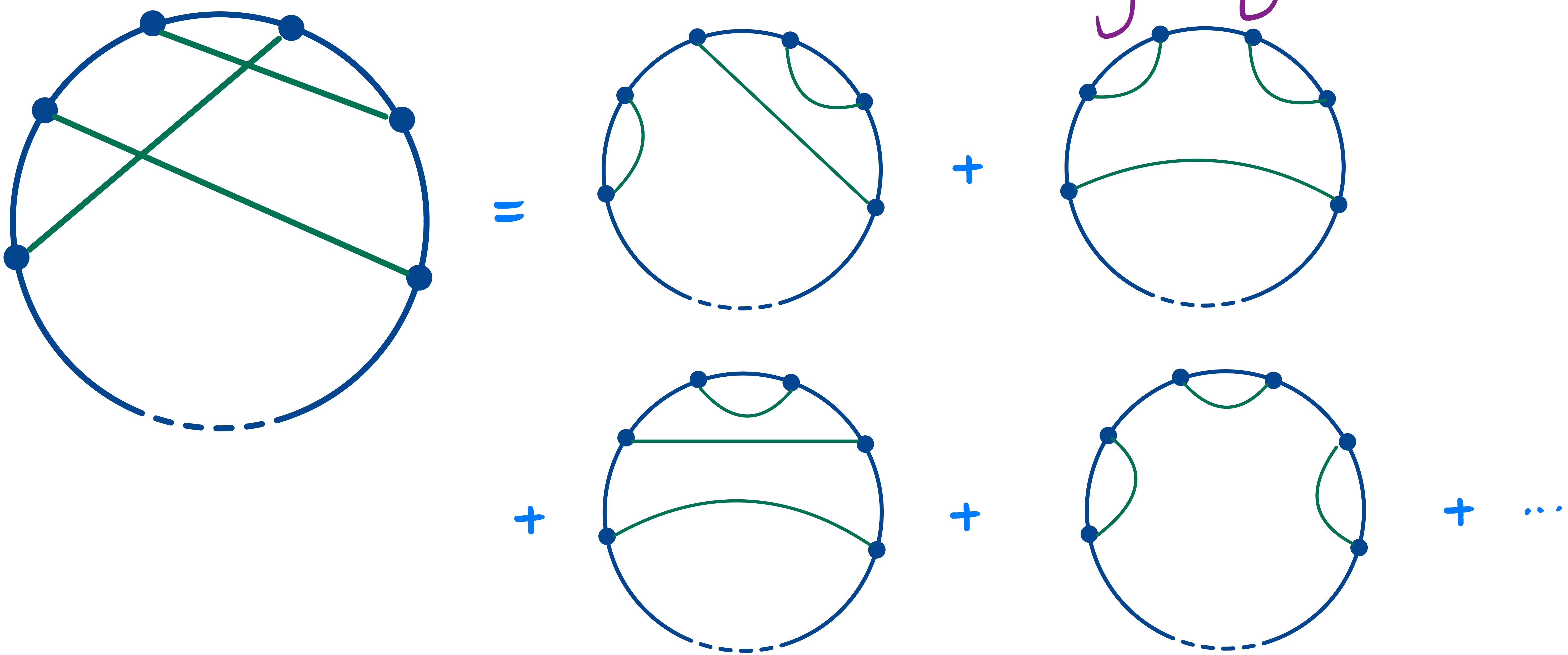
Ex Plücker relations:



$$(x_{11}x_{23} - x_{21}x_{13})(x_{12}x_{24} - x_{22}x_{14})$$
$$=$$
$$(x_{11}x_{22} - x_{21}x_{12})(x_{13}x_{24} - x_{23}x_{14})$$
$$+$$
$$(x_{11}x_{24} - x_{21}x_{14})(x_{12}x_{23} - x_{22}x_{13})$$

Temperly-Lieb basis

• Using , can reduce any matching diagram to a linear combination of matching diagrams:



Temperly-Lieb basis

Thm The <u>noncrossing</u> 2-row webs are a basis for

$$\mathrm{Inv}_{SL_2}(V_1 \otimes \cdots \otimes V_n) \qquad (V_i \in \{V, V^*\})$$

called the Temperly-Lieb basis.

Pf • <u>Spanning</u>: diagrams span by classical invariant theory, noncrossing by uncrossing rule.

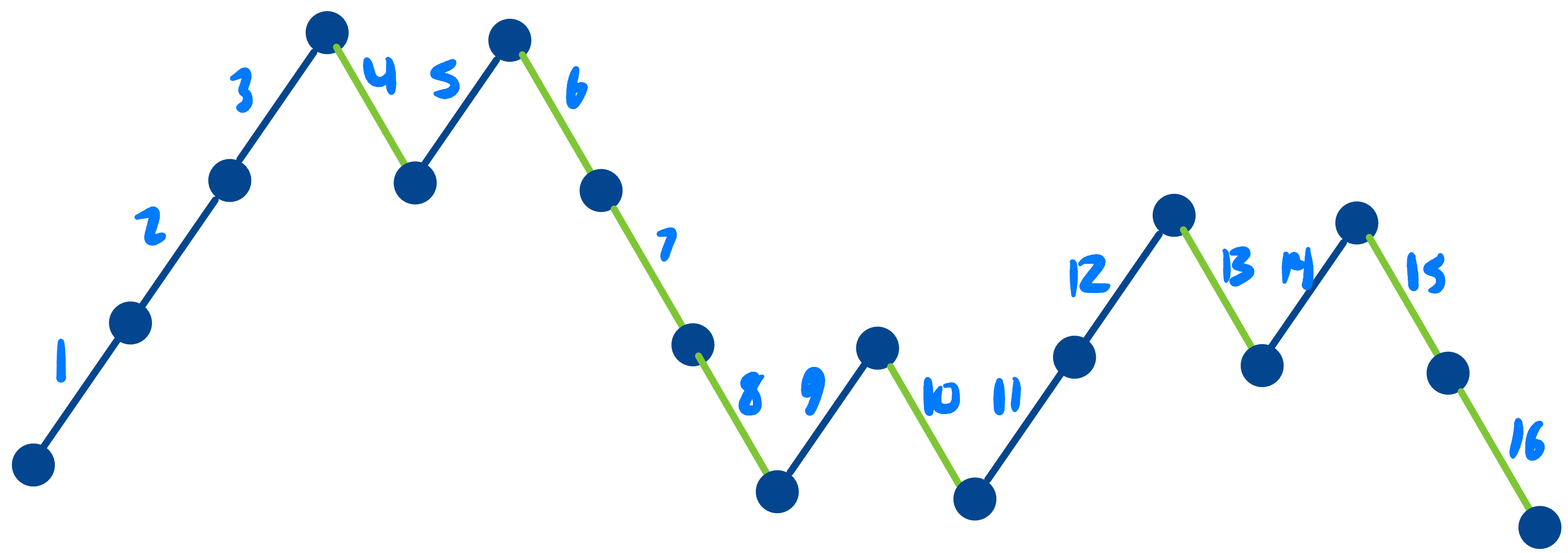
• <u>Independence</u>: by Pieri rule,

$$\dim \mathrm{Inv}_{SL_2}(V^{\otimes n}) = \#\mathrm{SYT}(2 \times \tfrac{n}{2}).$$ Count!

→

Temperly-Lieb basis

Some Catalan bijections:



Dyck path

$\mapsto$ ((()()))()((()()))

$\mapsto$ 1112122212112122

lattice word

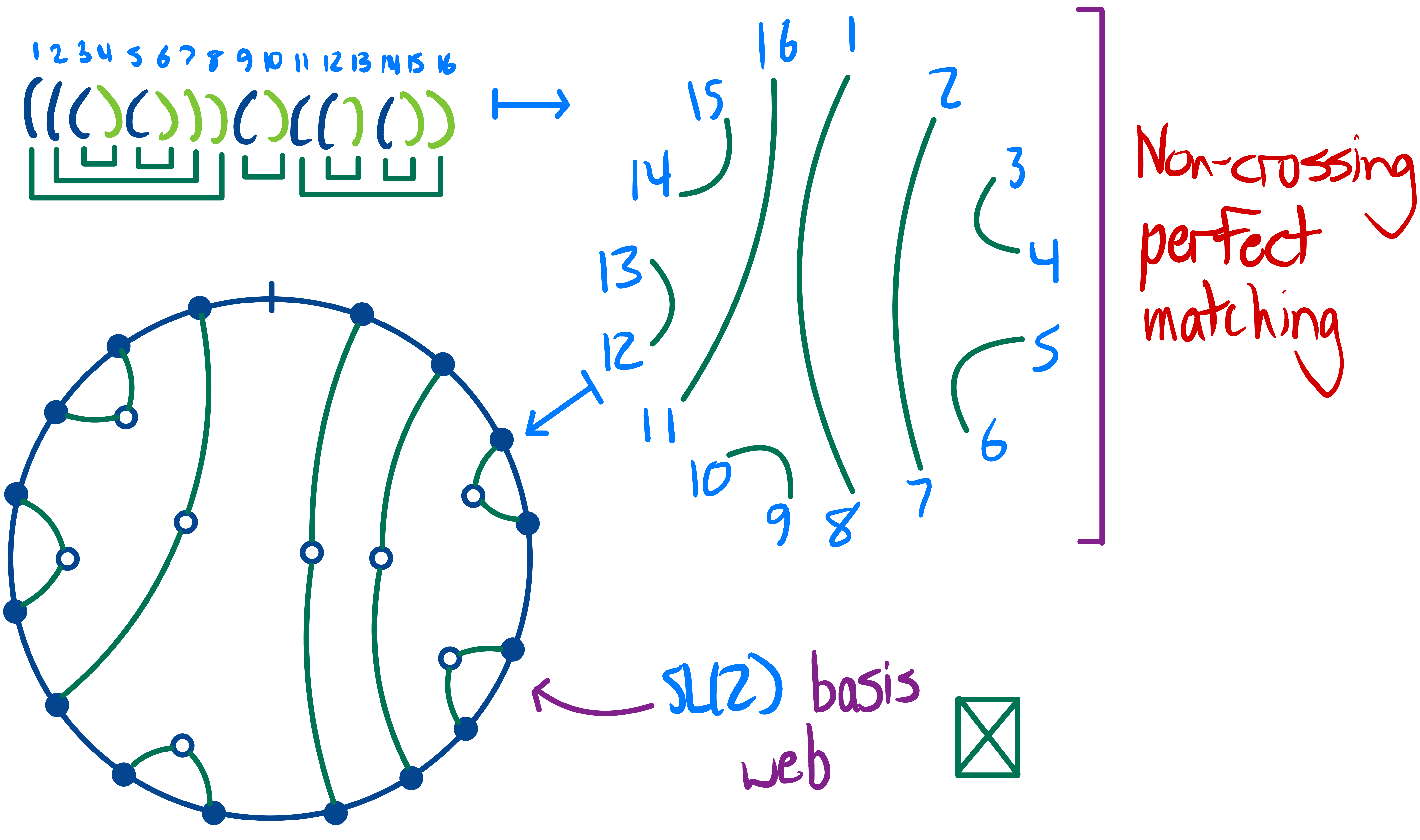
$\mapsto$

1	2	3	5	9	11	12	14
4	6	7	8	10	13	15	16

Standard Young tableau

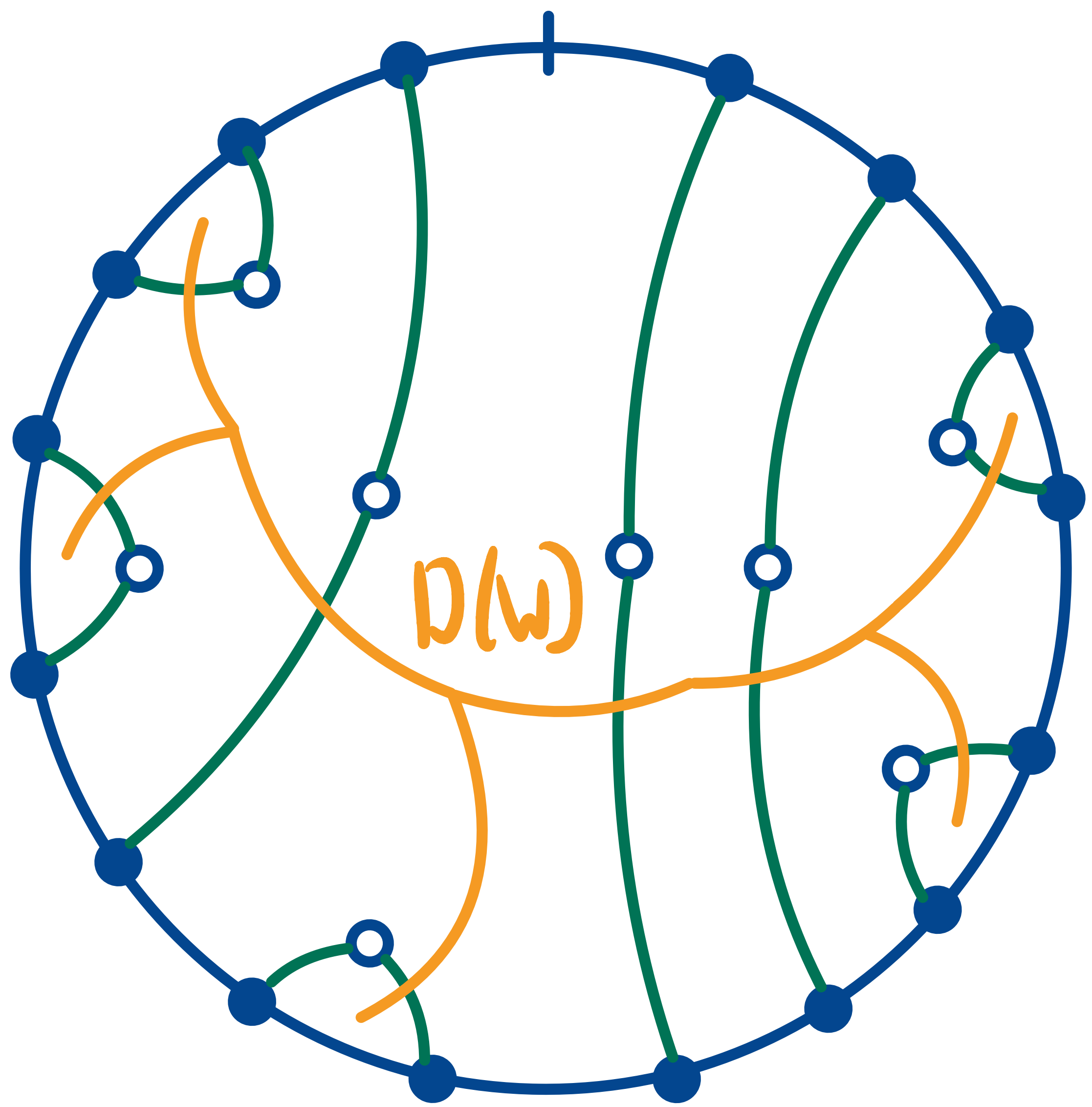
$\longrightarrow$

Temperly-Lieb basis



Duals and trees

Obs The dual graph D(W) of an SL(2) basis web W is a tree:



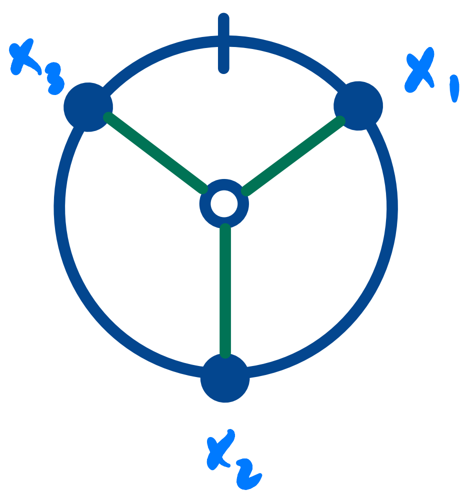
e.g. # faces of W = # vertices of D(W)

SL_3-webs

- Let $V = \mathbb{C}^3$, $V_i \in \{V, V^*\}$.

  Q: Is there a nice web basis for $\mathrm{Inv}_{SL_3}(V_1 \otimes \cdots \otimes V_n)$?

- Use bipartite planar graphs in a disk built from



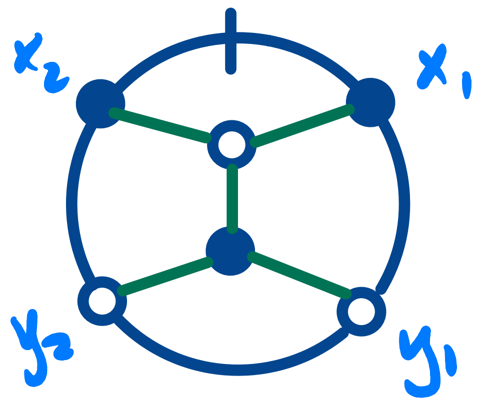
$= \det\begin{pmatrix} | & | & | \\ x_1 & x_2 & x_3 \\ | & | & | \end{pmatrix}$ (so trivalent).

(+ dual)

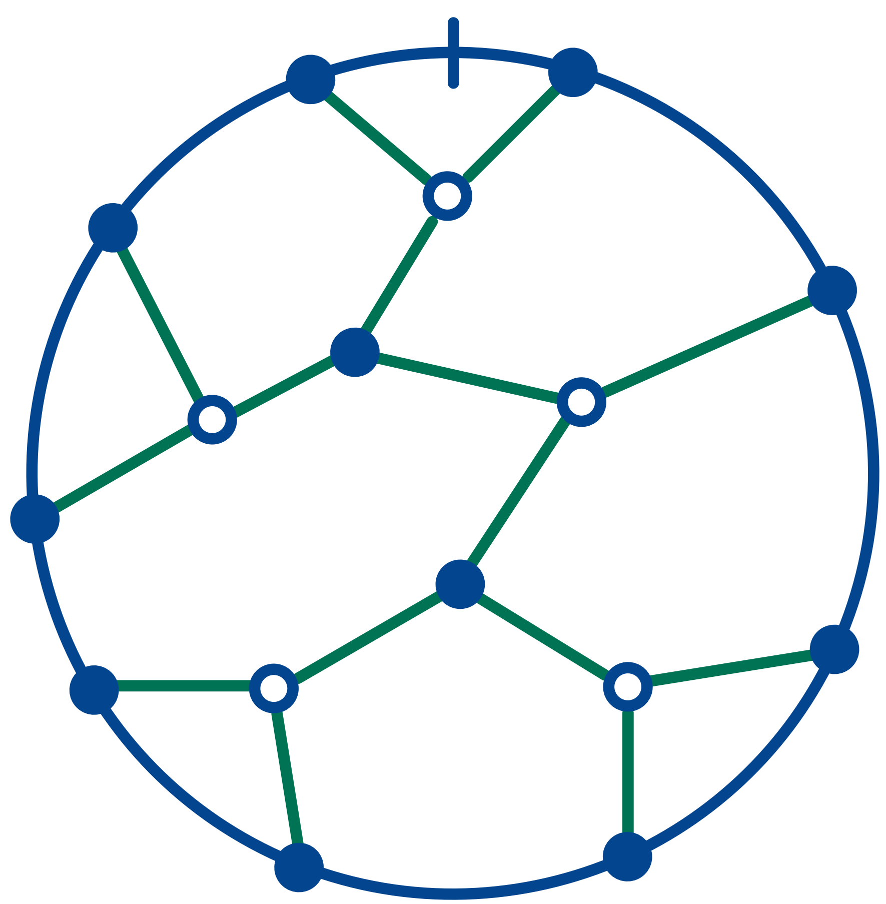
Have univalent boundary vertices and connected to boundary.

SL_3-webs

SL_3-webs

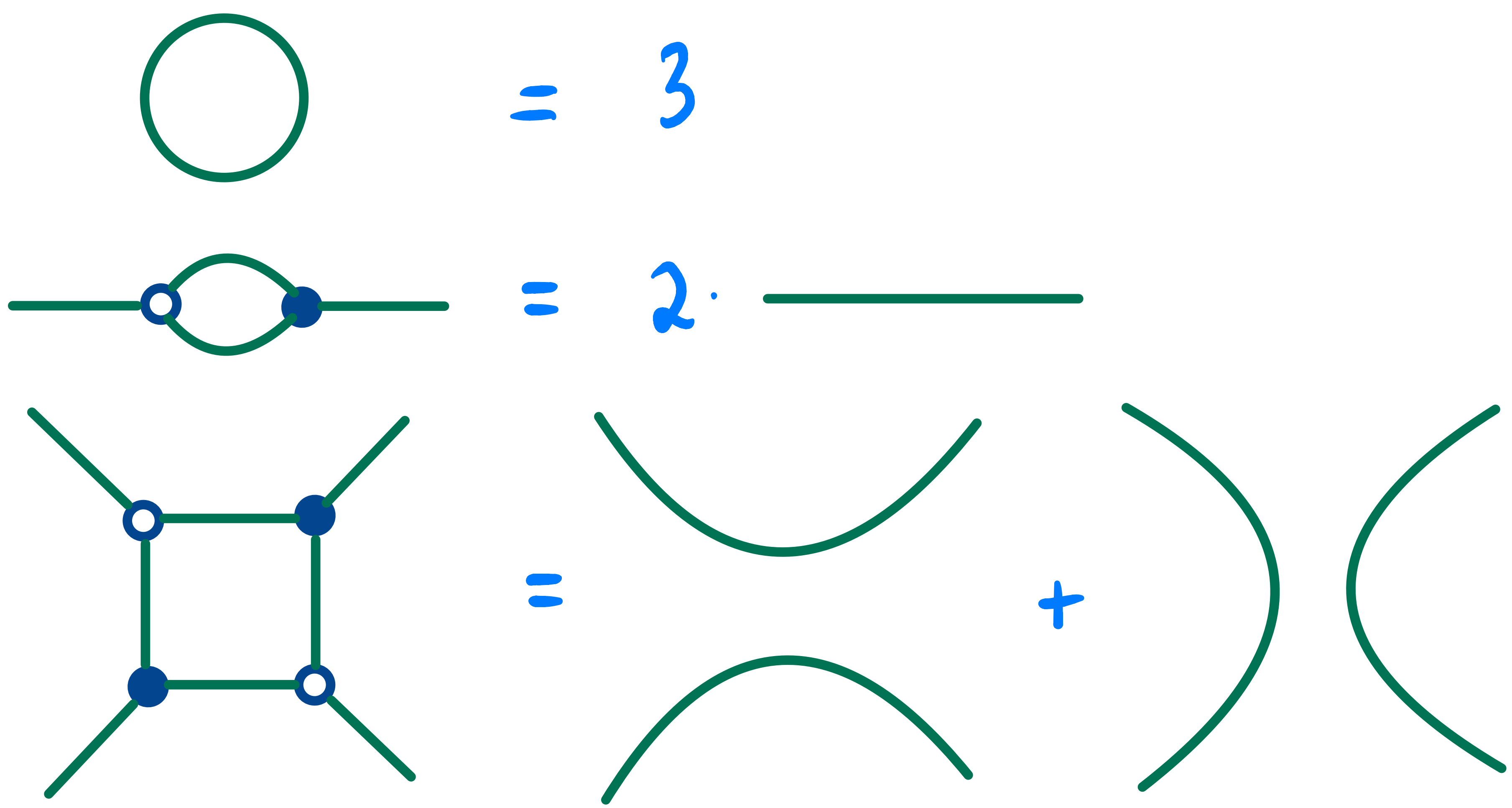
Ex 

$= \det\begin{pmatrix} | & | & | \\ x_1 & x_2 & y_1 \wedge y_2 \\ | & | & | \end{pmatrix}$

Ex 
$=$ … a polynomial obtained by summing over <u>proper edge 3-colorings</u> …

SL_3-webs

Thm (Kuperberg '94) The generating SL_3-web relations are



Non-elliptic web basis

Thm (Kuperberg '94)

Call an SL_3-web **non-elliptic** if it has no 2-faces or 4-faces.

The non-elliptic webs form a basis of

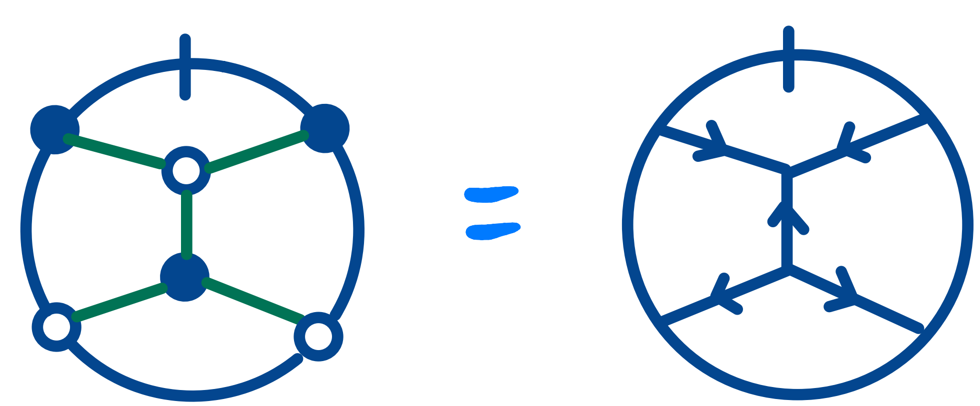
$$\mathrm{Inv}_{SL_3}(V_1 \otimes \cdots \otimes V_n). \qquad (V_i \in \{V, V^*\})$$

Non-elliptic web basis

Pf · <u>Spanning</u>: similar to SL_2 case.

· <u>Independence</u>: bijection to $SYT(3 \times \frac{n}{3})$ using growth rules. ($\exists$ other approaches)

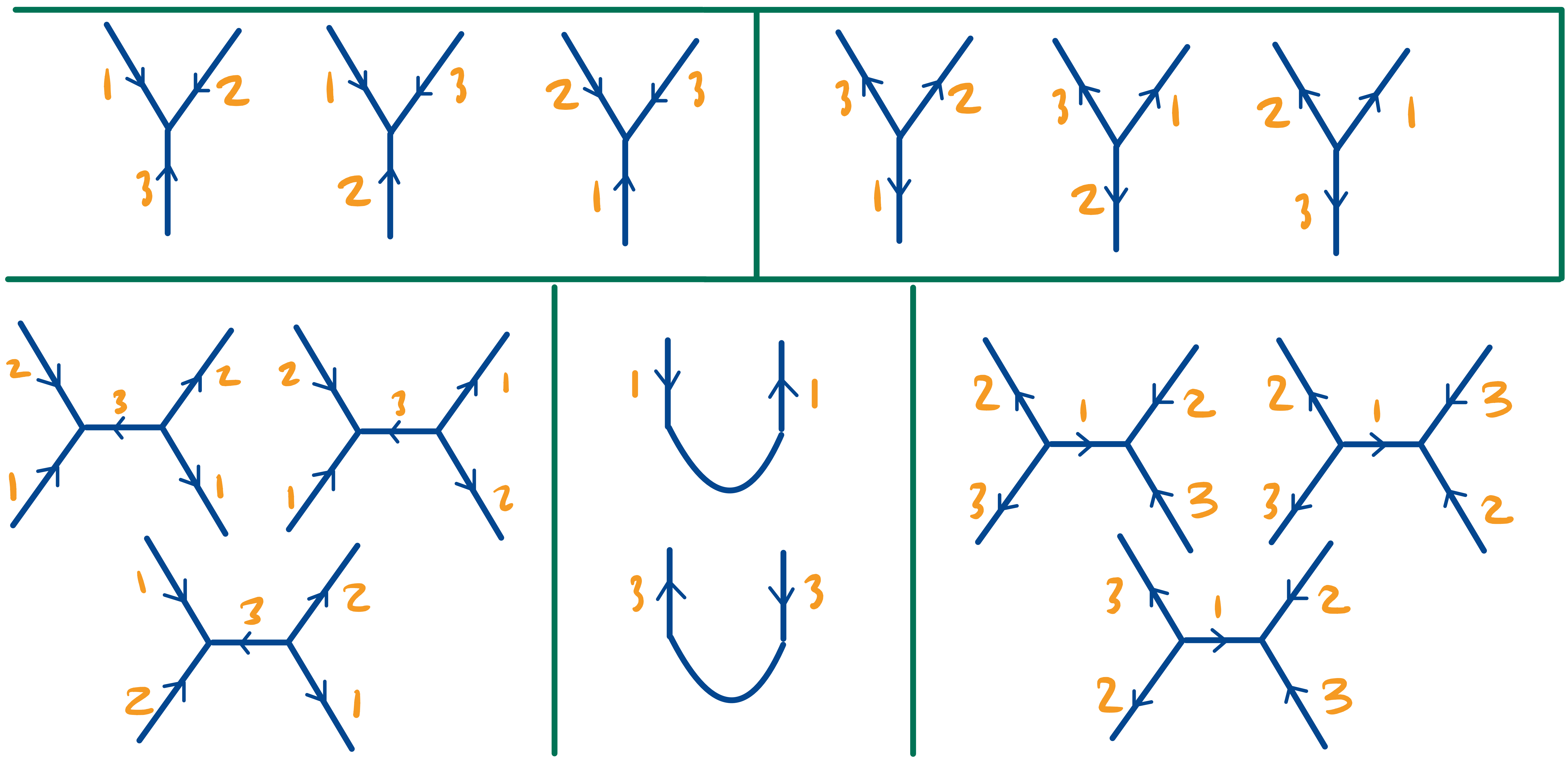
Will use directed notation here:



→

SL_3-growth rules

Kuperberg-Khovanov growth rules:



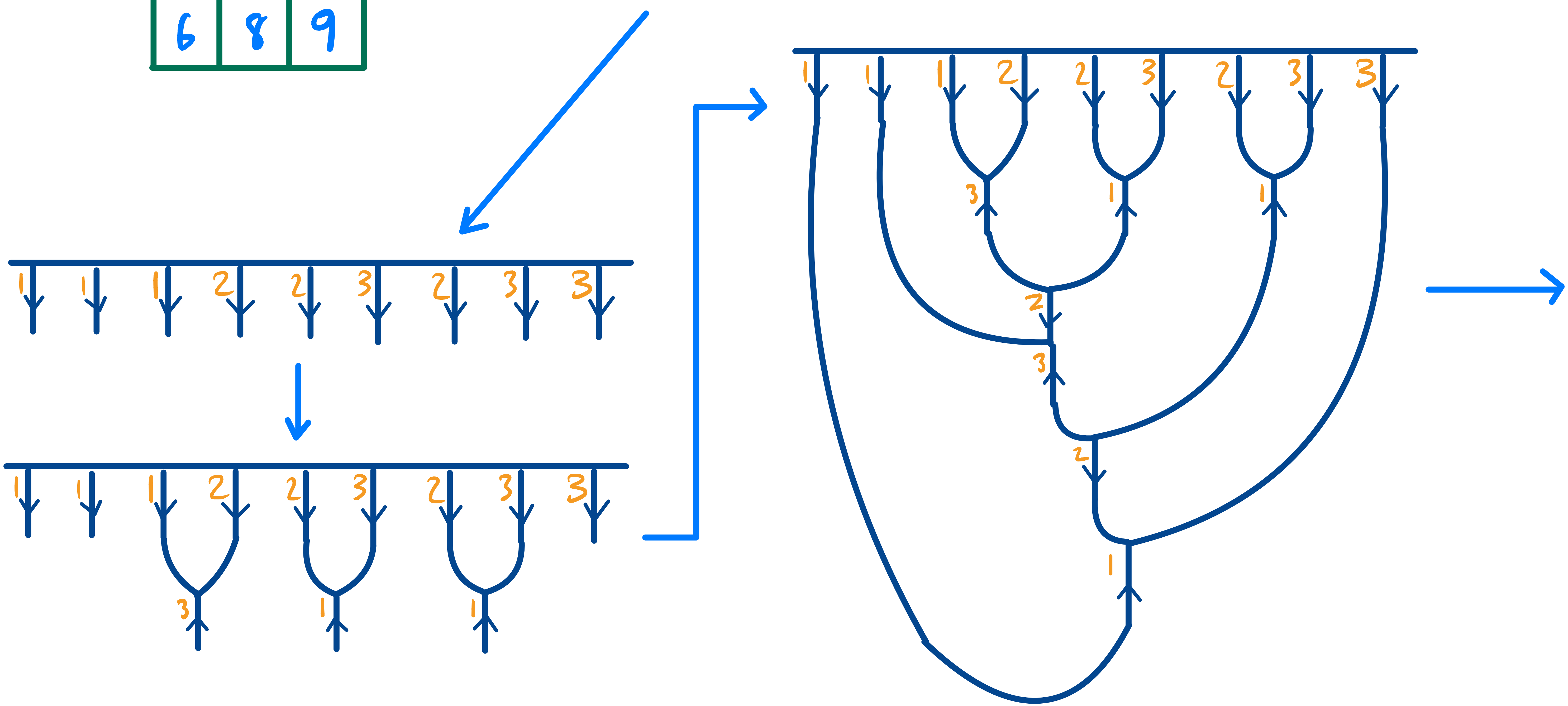
SL_3-growth rules

Ex

$T =$

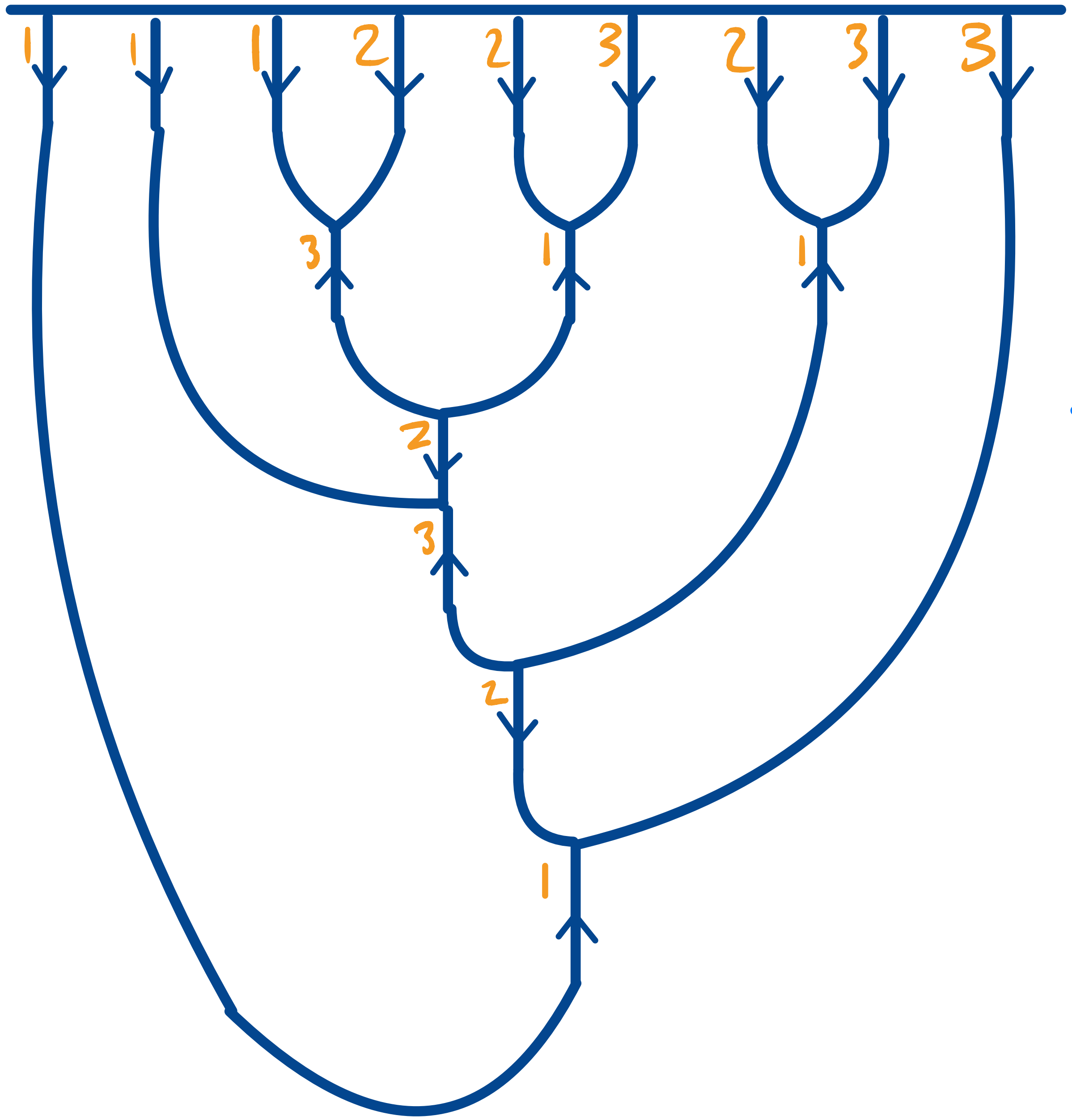
1	2	3
4	5	7
6	8	9

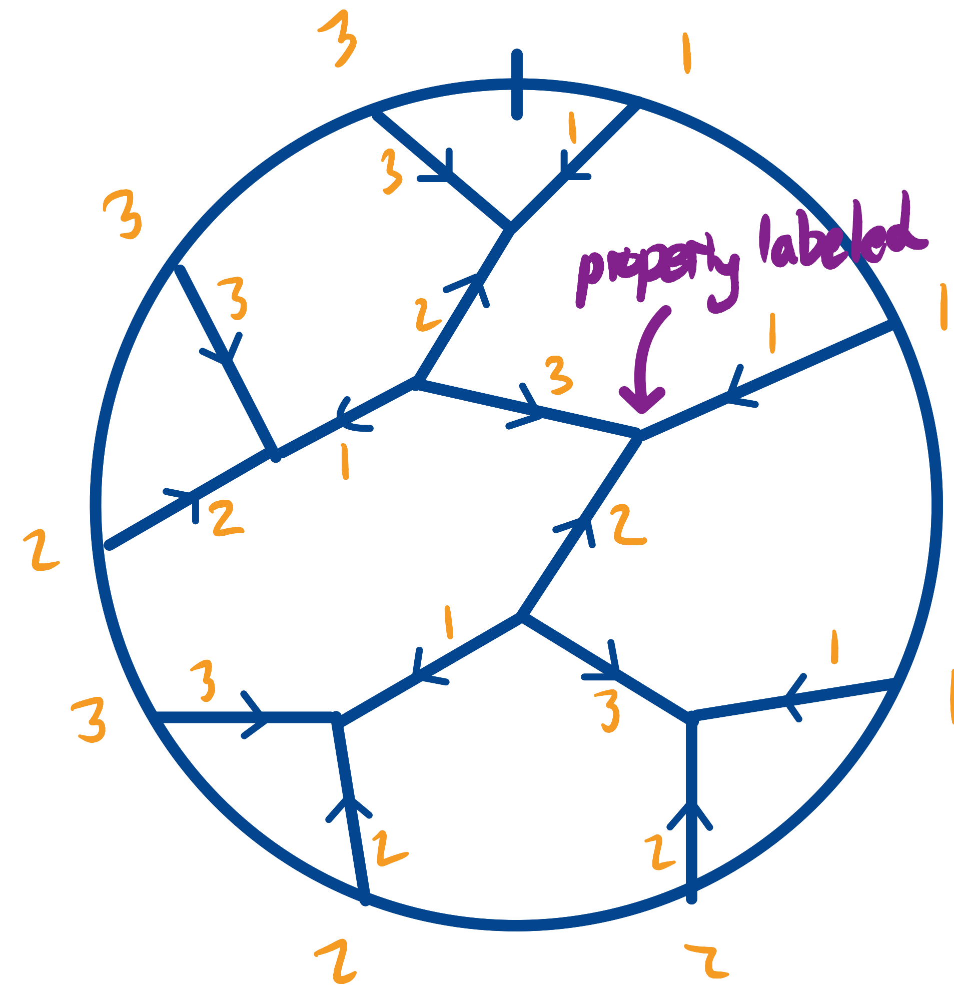
→ 111223233



SL_3-growth rules

($T \longrightarrow 111223233$)





Now just erase labels!

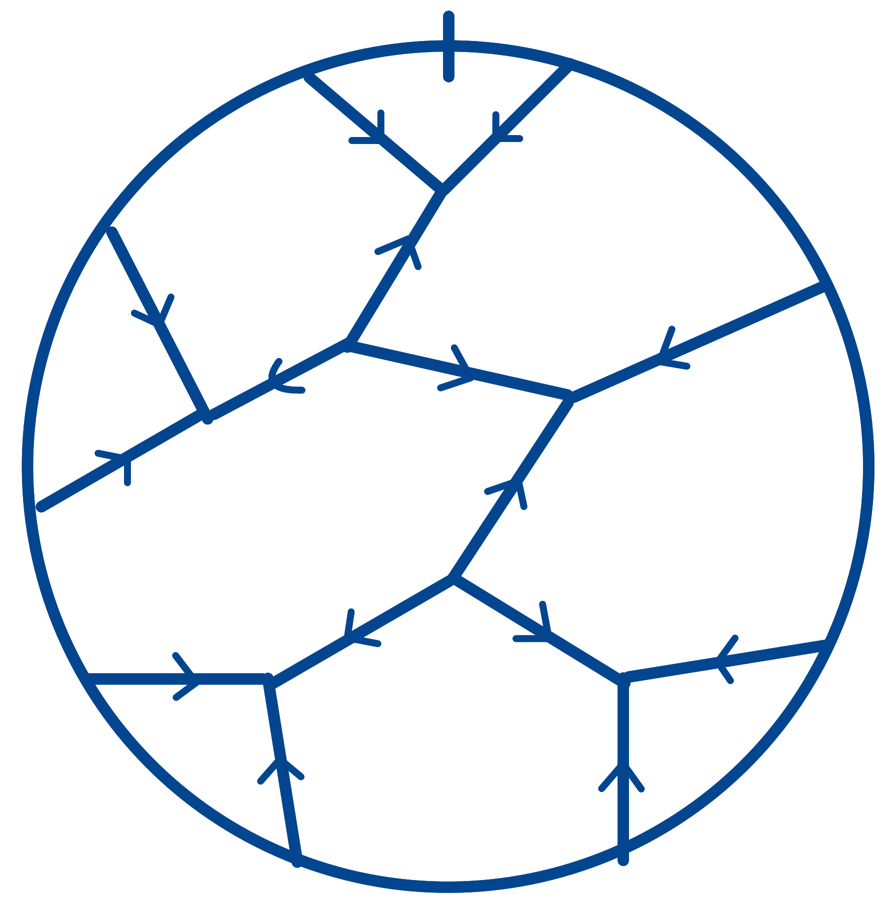
SL_3-growth rules

In all:

$T =$

1	2	3
4	5	7
6	8	9

$\longmapsto$



SL_3-growth rules

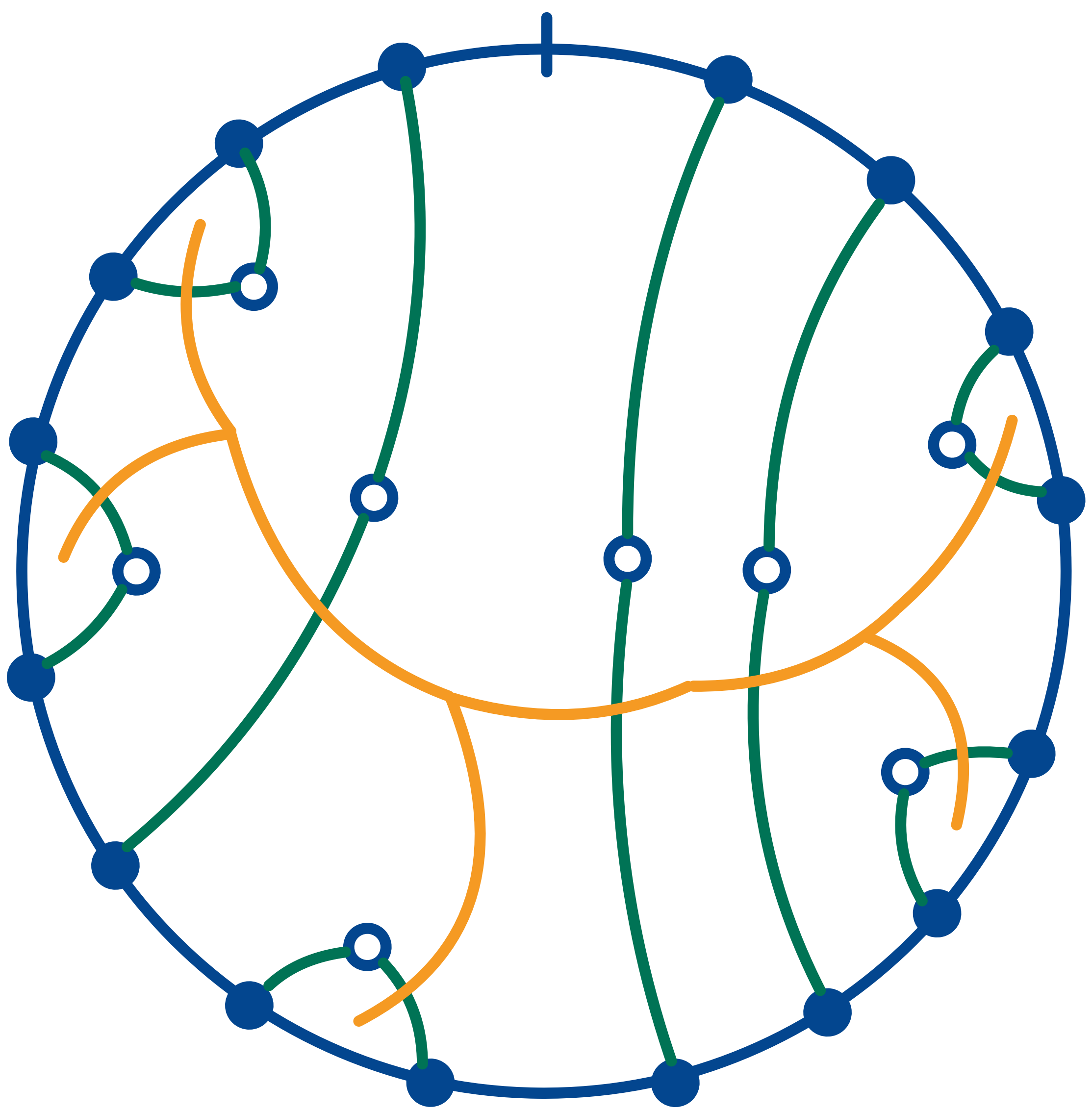
Fact

1. Growth algorithm is well-defined: output is independent of choices
2. Growth algorithm surjects onto non-elliptic $SL(3)$ webs ☒

Q What's really going on here??

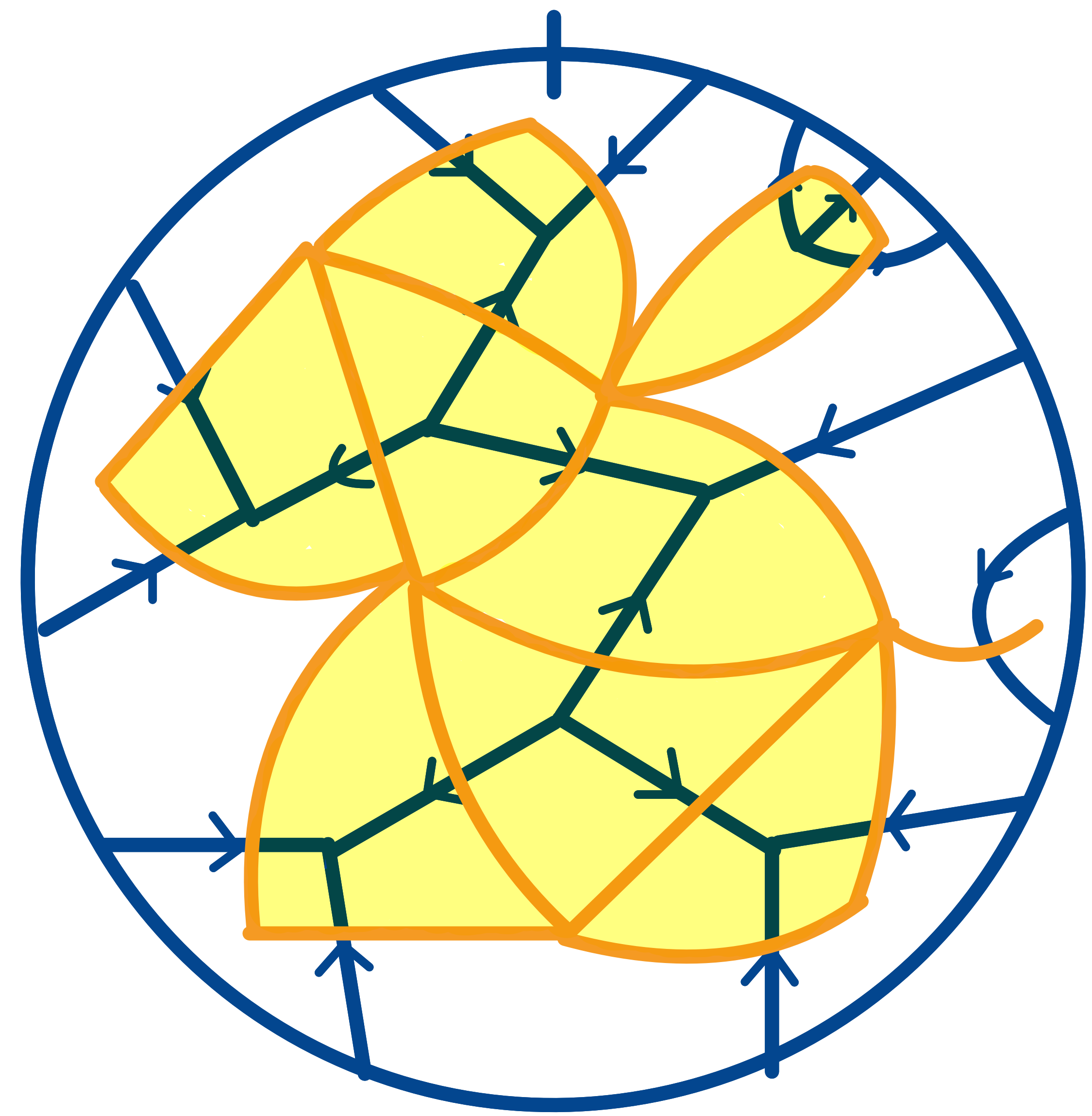
SL_3-web duals

Obs $D(W)$ for $SL(3)$ is a triangulation:



$D(W)$ is 1D for $SL(2)$

vs.



$D(W)$ is "2D" for $SL(3)$

SL_3-web duals

- Fontaine-Kamnitzer-Kuperberg showed duals of SL_3 basis webs live inside the affine building $\Delta(SL_3^{\vee})$
  - Growth rules build one triangle at a time
  - Labels encode distance information
  - Non-elliptic condition ⟺ CAT(0)

next!

Double cosets

Obs Given groups $H \leq G$, have a "distance" on G: (or G/H)

$$d : G \times G \to H \backslash G / H$$
$$d(p,q) = Hp^{-1}qH$$

a **double coset**

Ex For $\mathbb{R}_{>0} \leq \mathbb{C}^\times$, (double) cosets are represented by polar angles. Then $d(re^{i\theta}, se^{i\phi}) = e^{i(\phi-\theta)}\mathbb{R}_{>0}$.

Basically $(x,y) \mapsto \arg\left(\frac{y}{x}\right) = \operatorname{atan2}(y,x)$!

Double cosets

Ex For $\mathbb{C}[[t]]^\times < \mathbb{C}((t))$, coset reps are $\{t^n \mid n \in \mathbb{Z}\}$

Ex For $GL_r(\mathbb{C}[[t]]) < GL_r(\mathbb{C}((t)))$, double coset reps are

$$t^\lambda = \begin{pmatrix} t^{\lambda_1} & & & \\ & t^{\lambda_2} & & \\ & & \ddots & \\ & & & t^{\lambda_r} \end{pmatrix}$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ integers.

] $GL(r)$ dominant weights

Ex For $PGL_r(\mathbb{C}[[t]]) < PGL_r(\mathbb{C}((t)))$, now have

$$t^\lambda = t^{\lambda + c(1^r)}$$

] $SL(r)$ dominant weights

Affine Grassmannians

Def The affine Grassmannian of $SL(r)^\vee$ is

$$\mathrm{AffGr}_r = PGL_r(\mathbb{C}((t)))/PGL_r(\mathbb{C}[[t]]).$$

- Have "distance"

$$d : \mathrm{AffGr}_r \times \mathrm{AffGr}_r \to \mathbb{Z}^r_{dec}/\langle 1,\dots,1\rangle$$

$\big]$ SL(r) dominant weights

where $d(pH, qH) = d(p,q) = \lambda \iff p^{-1}q \in H t^\lambda H$

- Have $d(p,p) = 0$ $\quad d(p,q) = d(gp, gq)$ $\quad d(q,p) = \underline{-\mathrm{rev}(d(p,q))}$

not quite (anti)symmetric

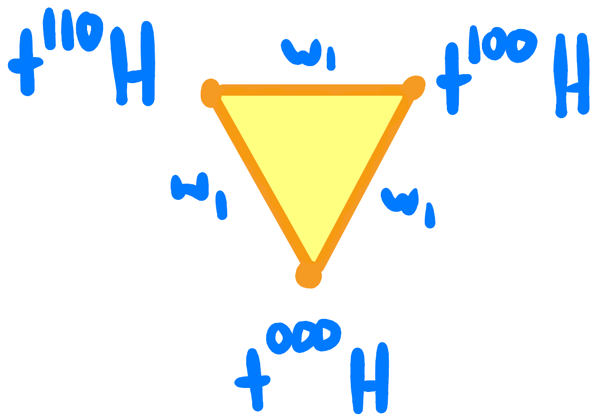
Affine Buildings

- The fundamental weights of $SL(r)$ are $\omega_i = (1^i, 0^{r-i})$

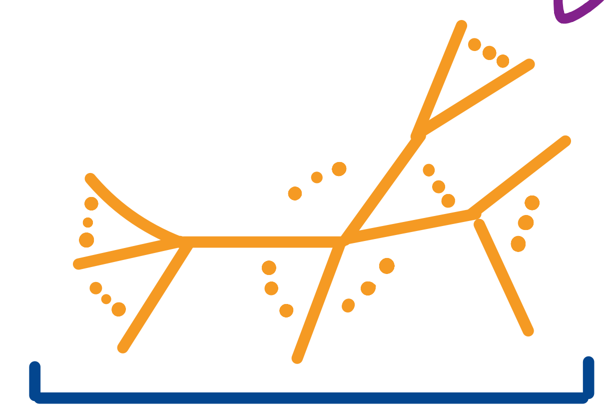
  $\omega_i^* = (0^{r-i}, -1^i) \equiv \omega_{r-i}$.
- Corresponds to $\Lambda^i V$ and $\Lambda^i V^*$.
- Essentially generates $SL(r)$-representation theory (e.g. Karoubi envelope...)

Affine Buildings

Def The **affine building** on AffGr_r is the simplicial complex Δ_r whose vertices are the points of AffGr_r and whose simplices are collections of points all of whose distances are fundamental weights.

Ex 

$\in \Delta_3$

Ex $\Delta_2 =$ 

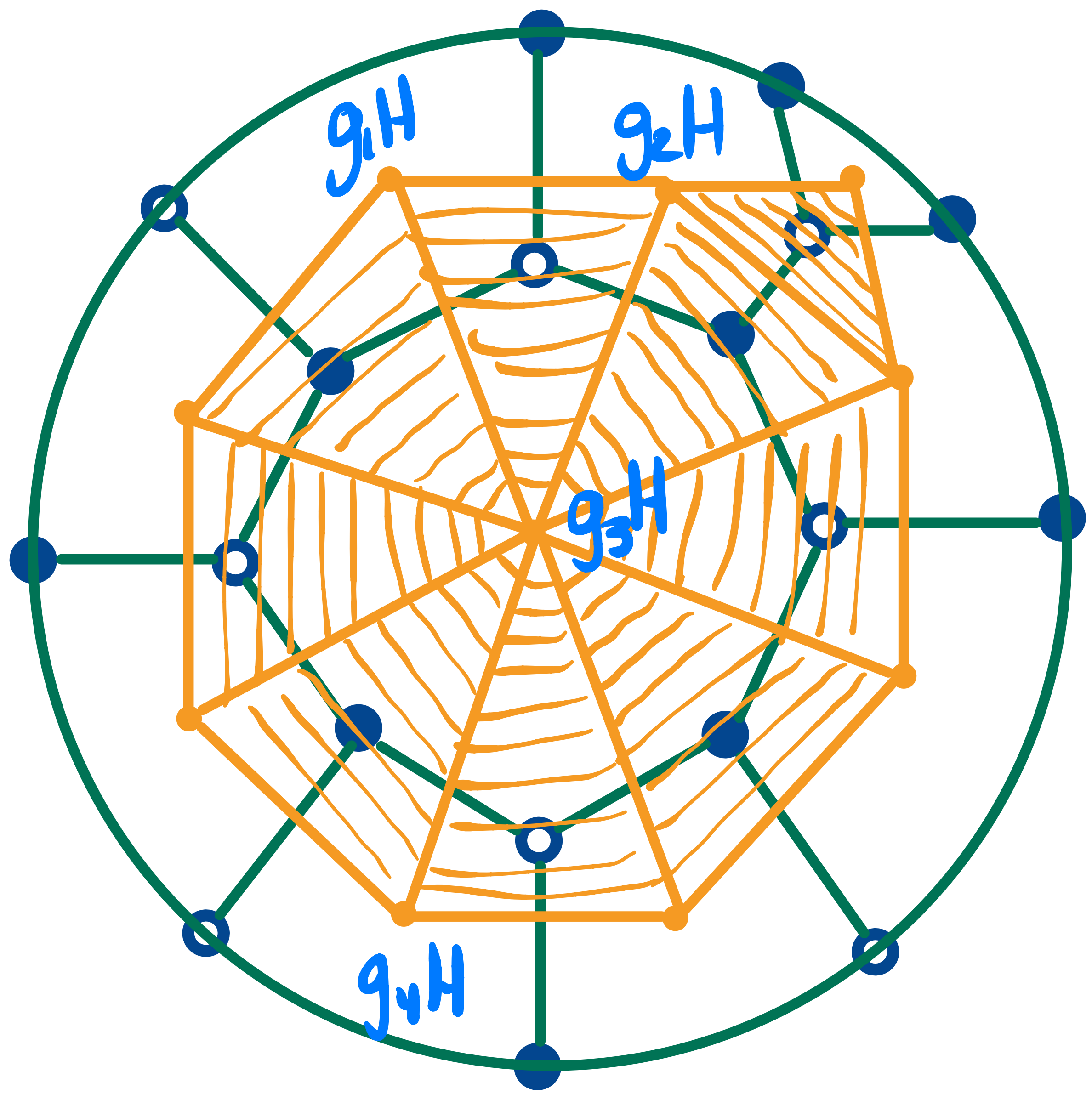
tree where every vertex has uncountable degree

Affine Buildings

Thm (Fontaine–Kamnitzer–Kuperberg '13)
The duals of non-elliptic $SL(3)$ basis webs can be embedded in Δ_3. For faces F_1, F_2, the distance between the corresponding vertices in Δ is the <u>geodesic distance</u> in Δ (or the embedding).

Affine Buildings

Ex There exist $g_1, g_2, g_3, g_4, \ldots$ s.t.



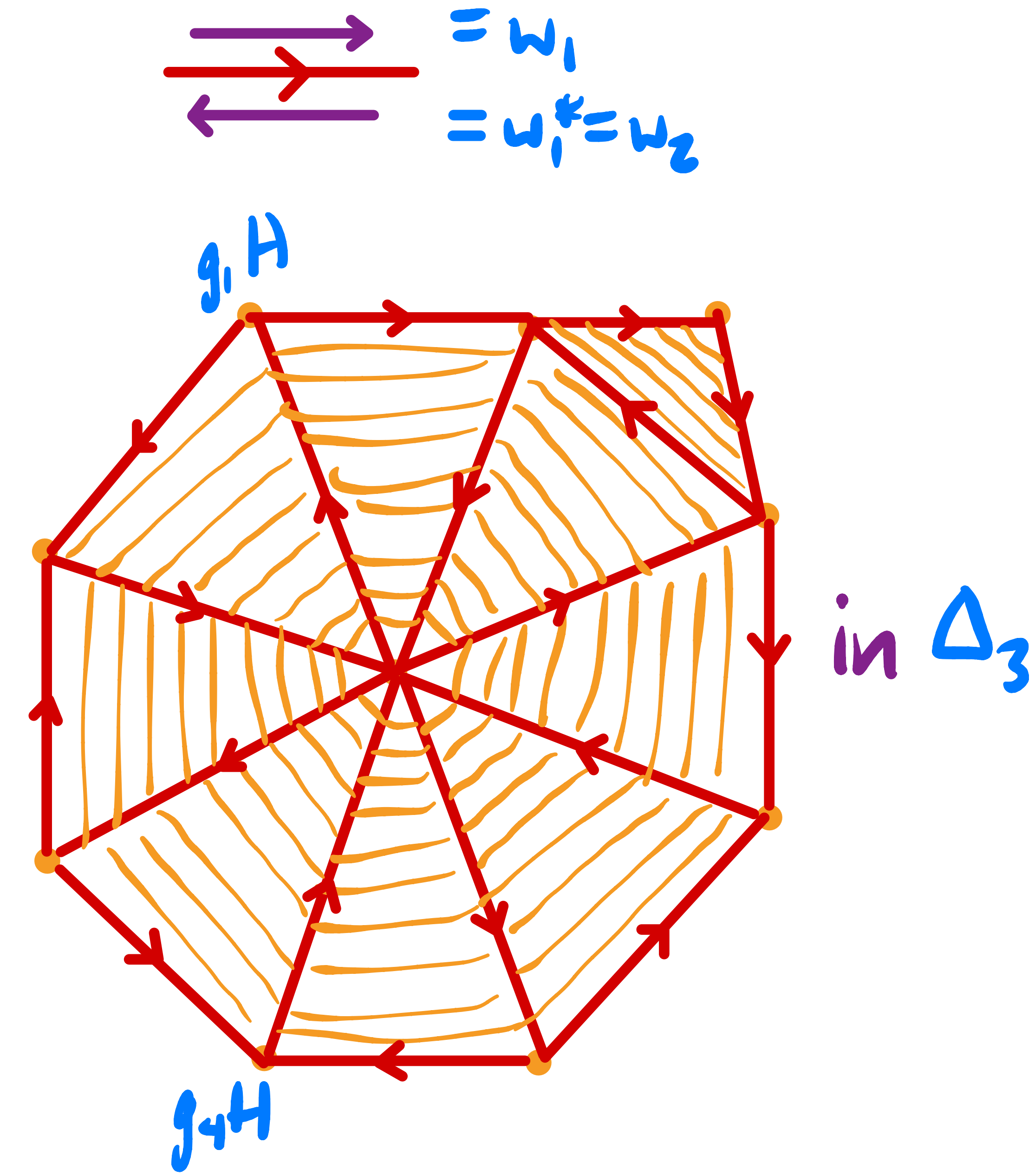
$d(g_1, g_2) = w_1$

$d(g_2, g_3) = w_1$

$\vdots$

$d(g_4, g_1) = 2w_1$ geodesic distance

$\vdots$

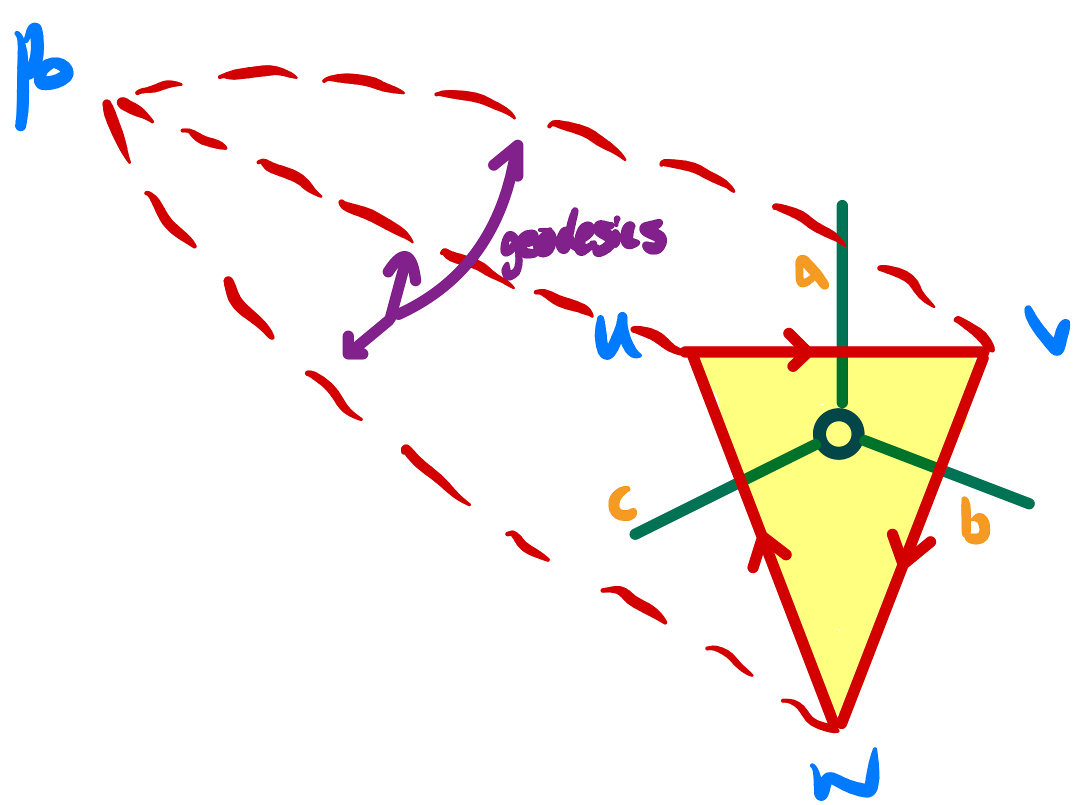


Affine Buildings

Q | Growth rule label meaning?

A | Let p_0 be base face.

Then



Fact | $d(p_0,w)-d(p_0,v)\in S_{r'w_1}$

Hence

$$d(p_0,w)-d(p_0,v)=e_a$$
$$d(p_0,v)-d(p_0,w)=e_b$$
$$d(p_0,w)-d(p_0,u)=e_c$$

$$0=e_a+e_b+e_c$$

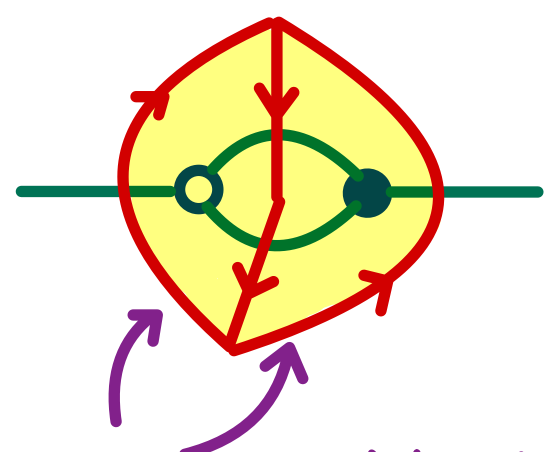
$$\Leftrightarrow \{a,b,c\}=\{1,2,3\}$$

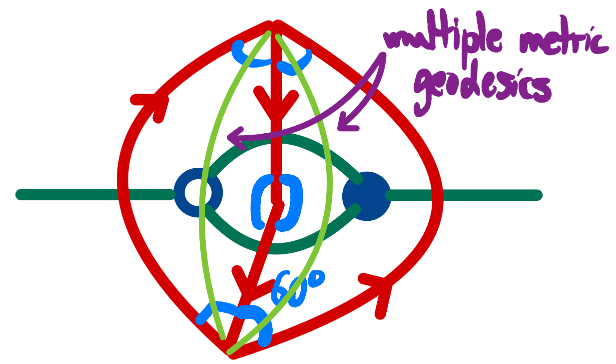
$\Leftrightarrow$ proper labeling!

Affine Buildings

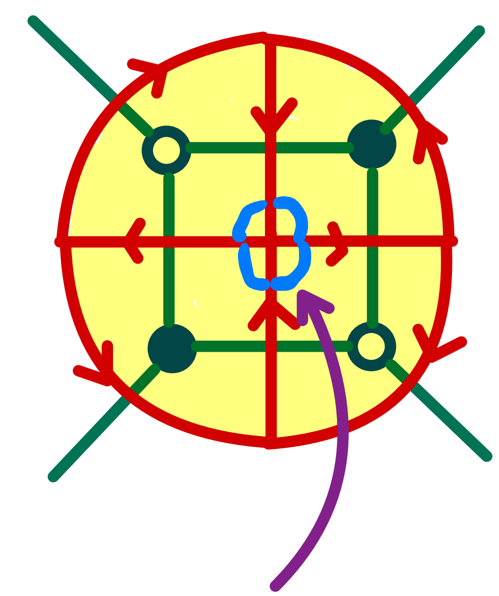
Q Non-elliptic condition meaning?

A





Not CAT(0)!



Angle $4 \cdot 60° = 240° < 360°$!
Also not CAT(0)!

The web basis problem

Problem (Khovanov-Kuperberg '96)

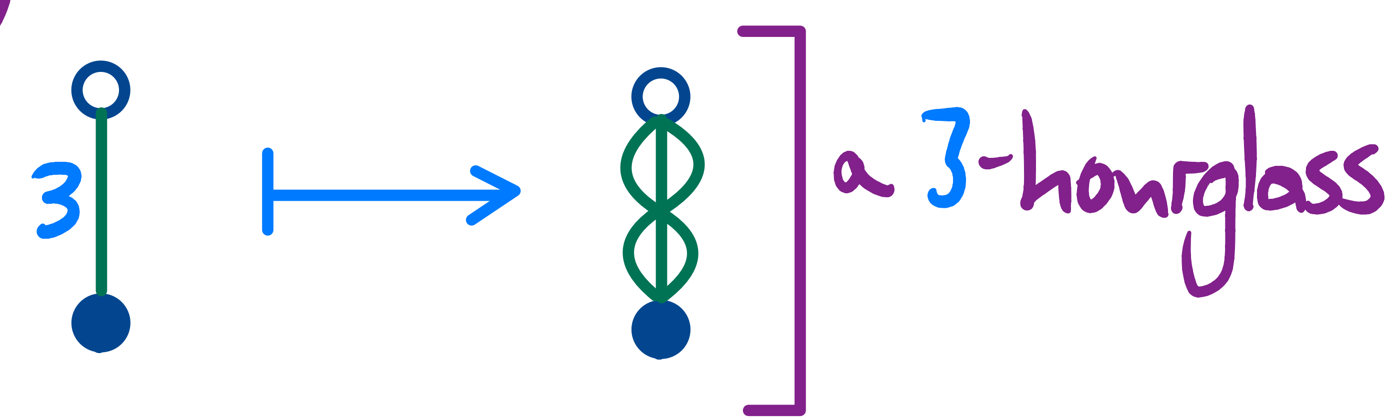
Give a web basis* for $\mathrm{Inv}_{SL_r}(V_1 \otimes \cdots \otimes V_n)$ for $r \geq 4$.

*with desirable properties for use in applications:

- testability
- reduction rules
- <u>rotation invariance</u>

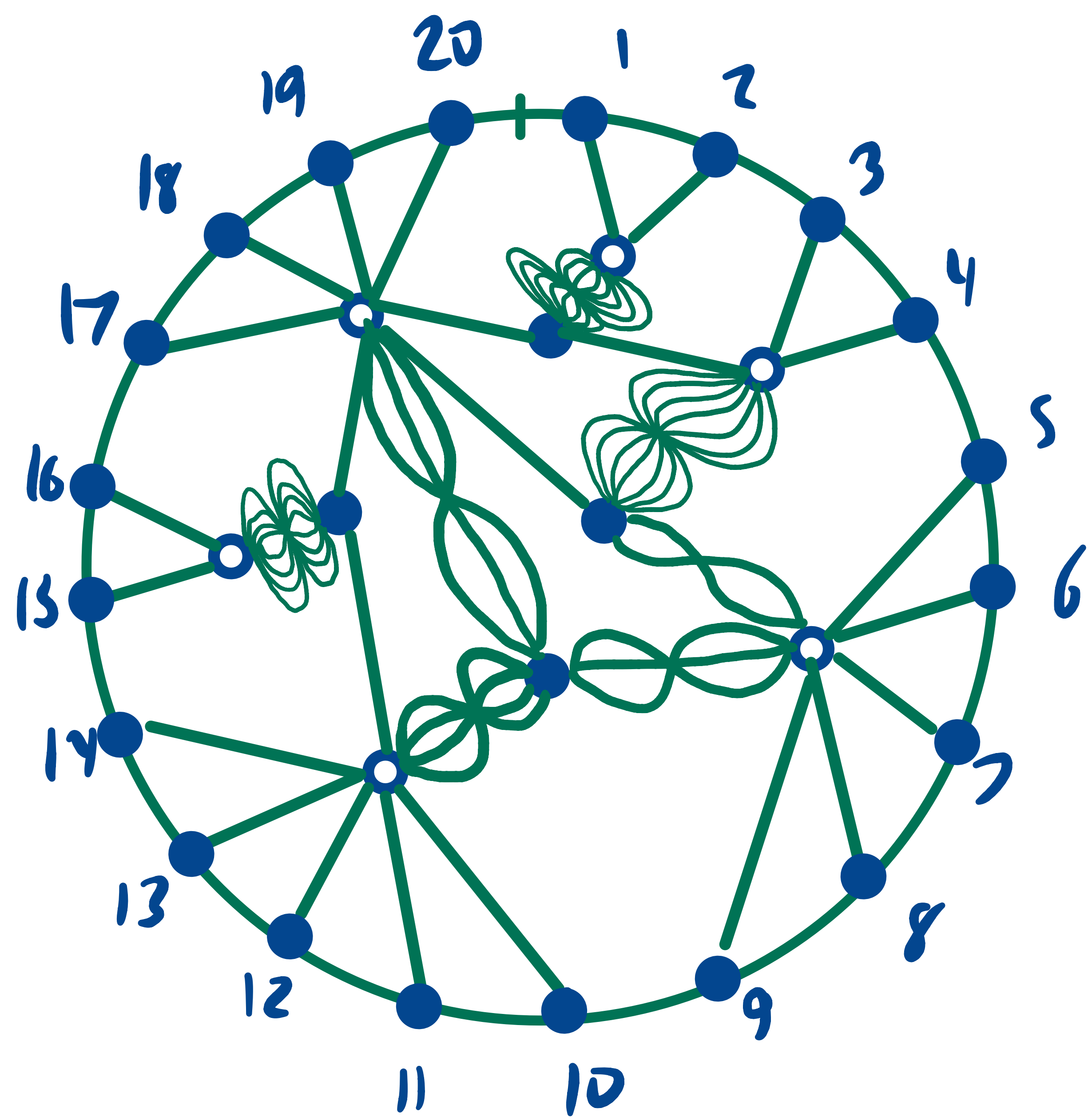
Hourglass plabic graphs

Def ([Gaetz–Pechenik–Pfannerer–Striker–S. '23])
An r-hourglass plabic graph (r-HPG) is a planar bipartite graph embedded in a disk with edge weights in $[r]$ which sum to r around internal vertices, and boundary vertices have degree 1.
An edge with weight m is drawn as an m-hourglass:



Hourglass plabic graphs

Ex

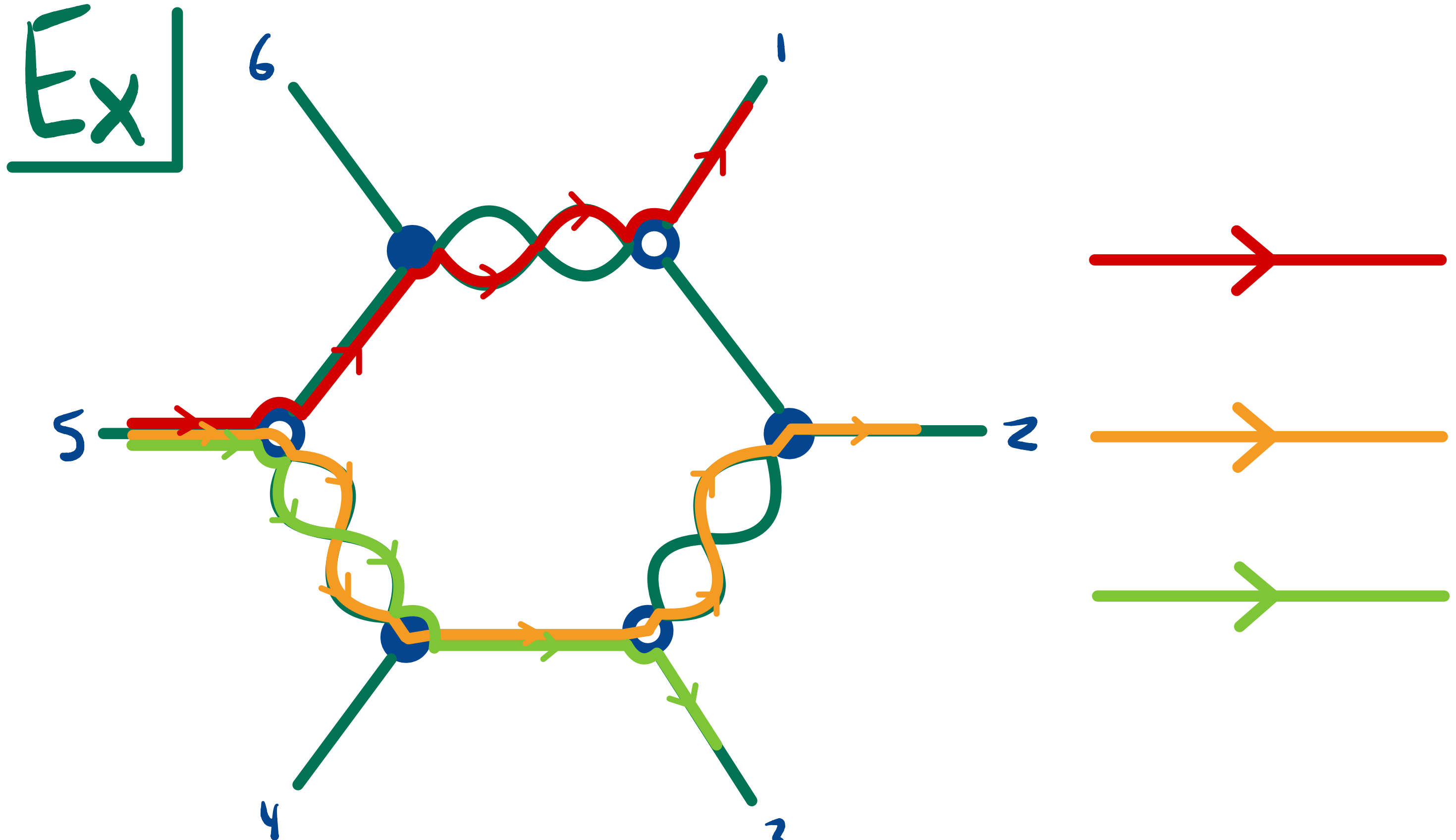


(r = 10)

Trip permutations

Def (GPPSS '23) An r-hourglass plabic graph has **trip permutations** $\mathrm{trip}_1, \ldots, \mathrm{trip}_{r-1}$ where trip_i takes the <u>ith left at white</u> and <u>ith right at black</u>:

Ex



$(r=4)$

$= \mathrm{trip}_1 = (135)(642)$

$= \mathrm{trip}_2 = (14)(25)(36)$

$= \mathrm{trip}_3 = (531)(246)$

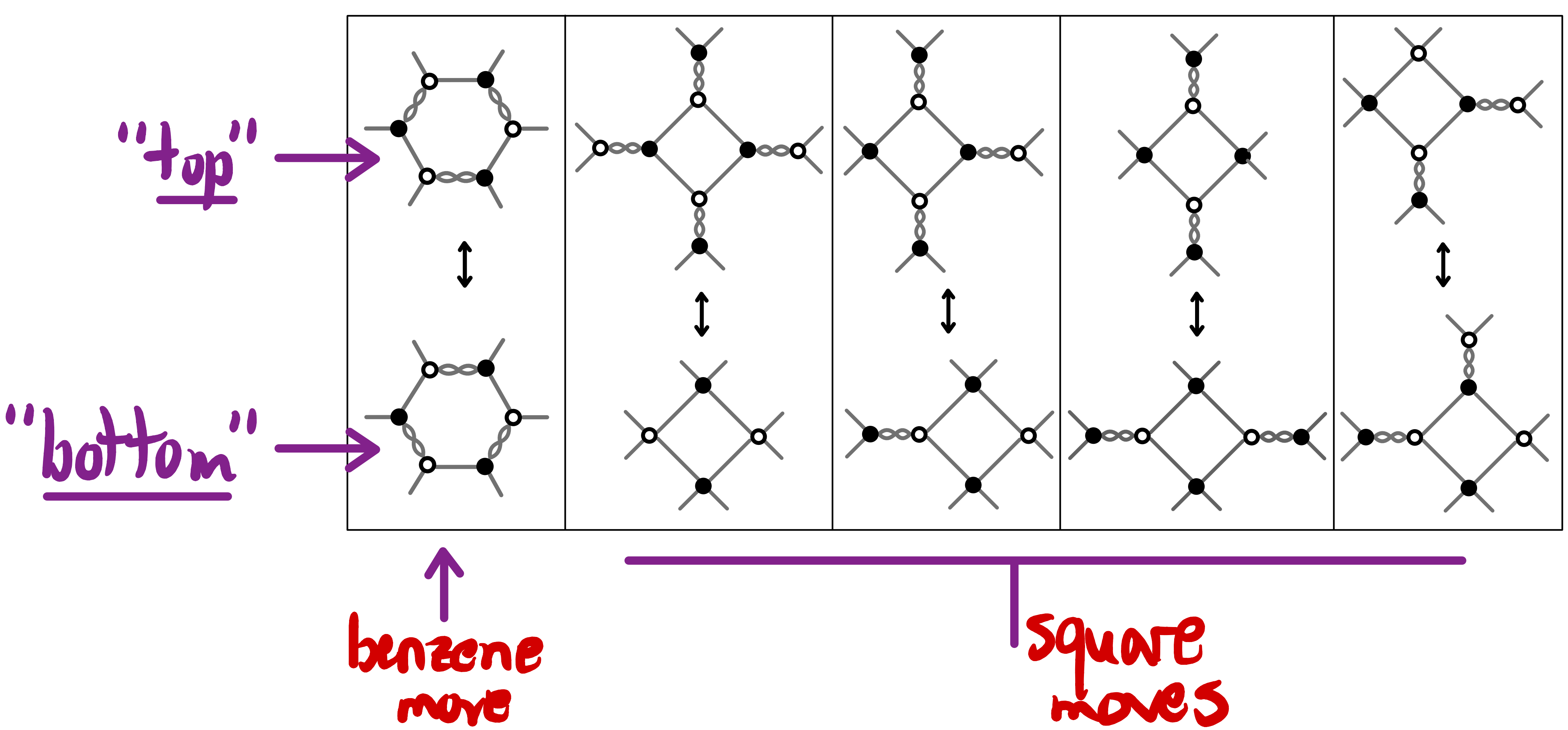
Note

$\mathrm{trip}_i = \mathrm{trip}_{r-i}^{-1}$!

r=4 moves

Thm (GPPSS '23) Two contracted, fully reduced 4-HPG's have the same $\mathrm{trip}_1, \mathrm{trip}_2, \mathrm{trip}_3$

$\Leftrightarrow$ they are related by moves:



r=4 moves

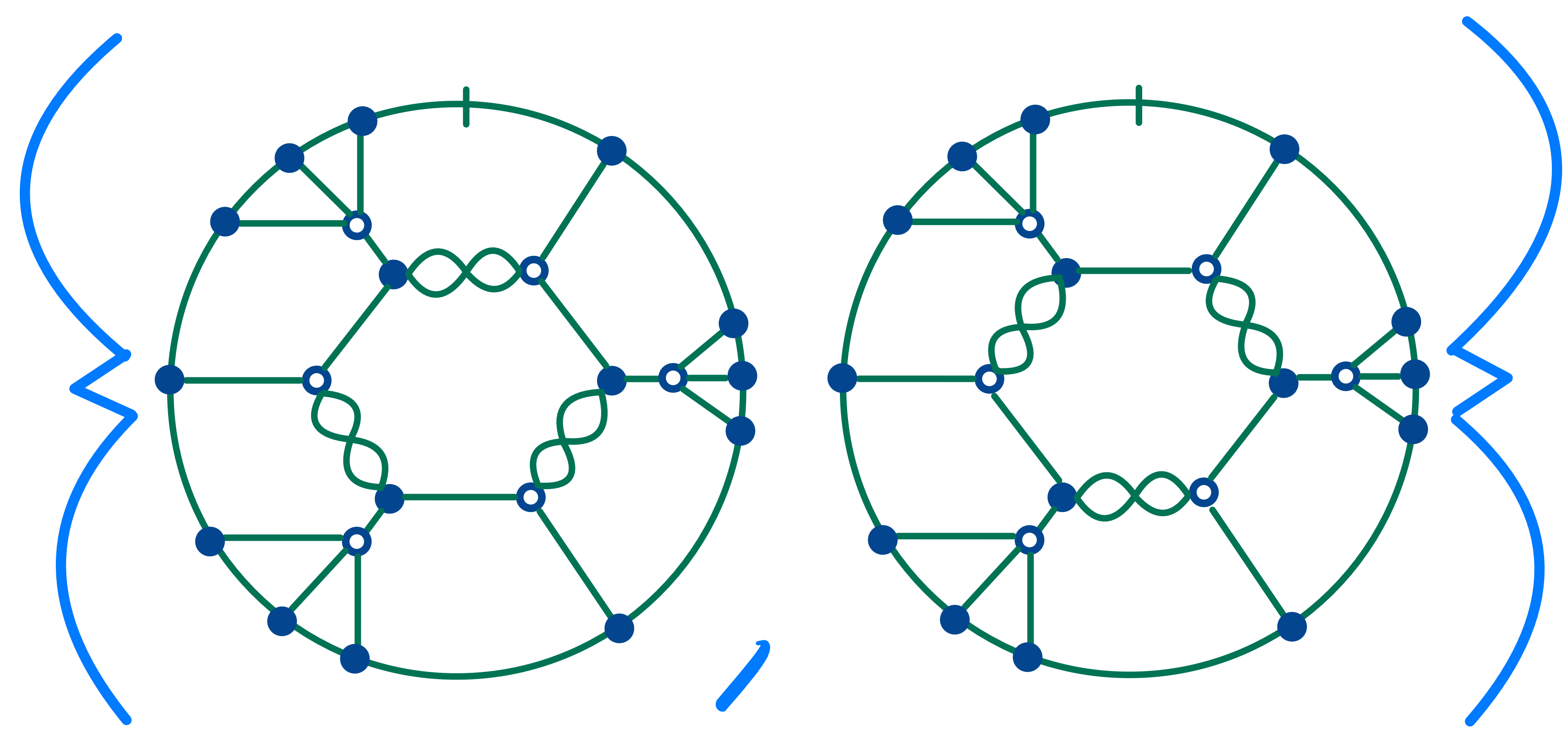
Thm (GPPSS '23) There is a bijection between SYT($4\times c$) and such move-classes. It sends $\mathrm{prom}_i(T)$ to $\mathrm{trip}_i(G)$.

Ex

$T=$

1	2	6
3	5	10
4	7	11
8	9	12

$\longmapsto$

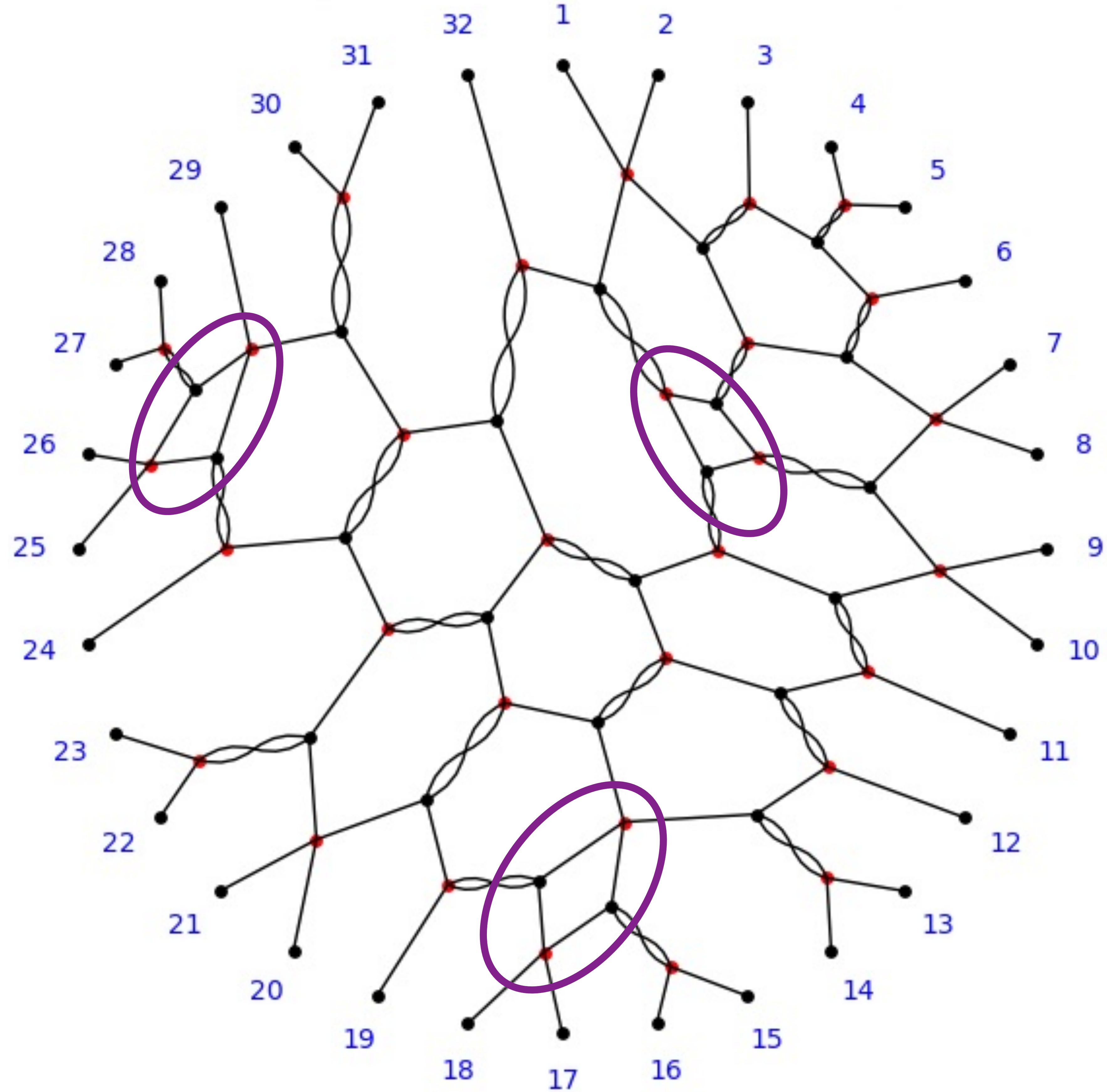


r=4 moves

Ex

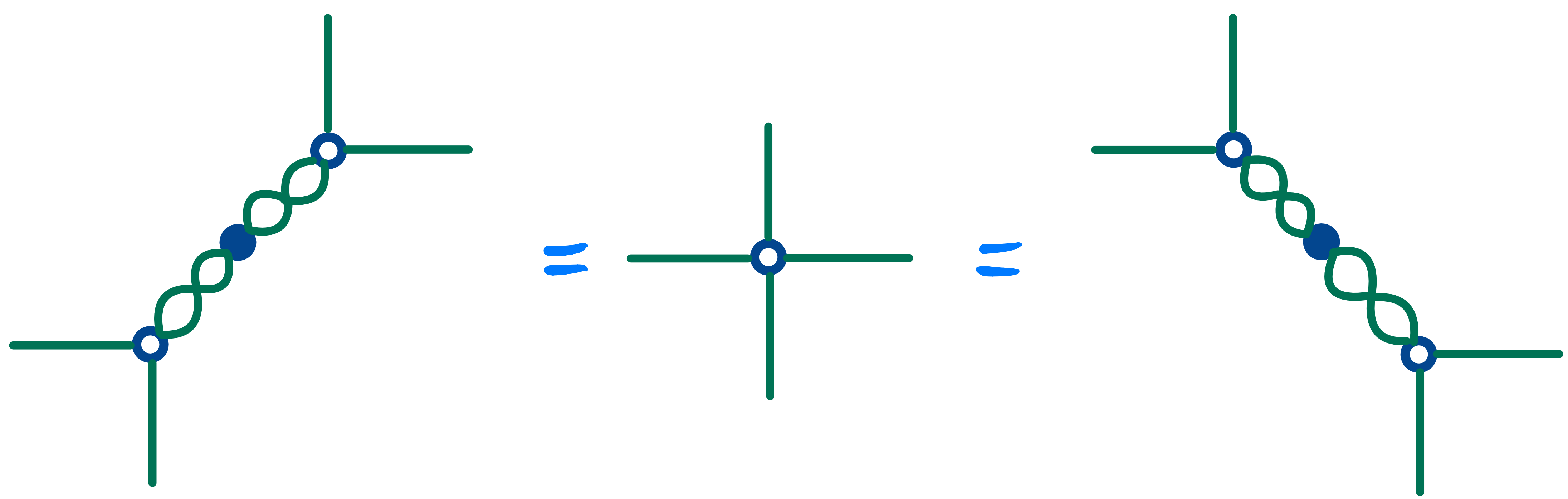
1	3	4	7	8	17	19	23
2	5	6	9	14	18	21	24
10	12	13	15	16	25	26	28
11	20	22	27	29	30	31	32

$\longmapsto$



Pockets

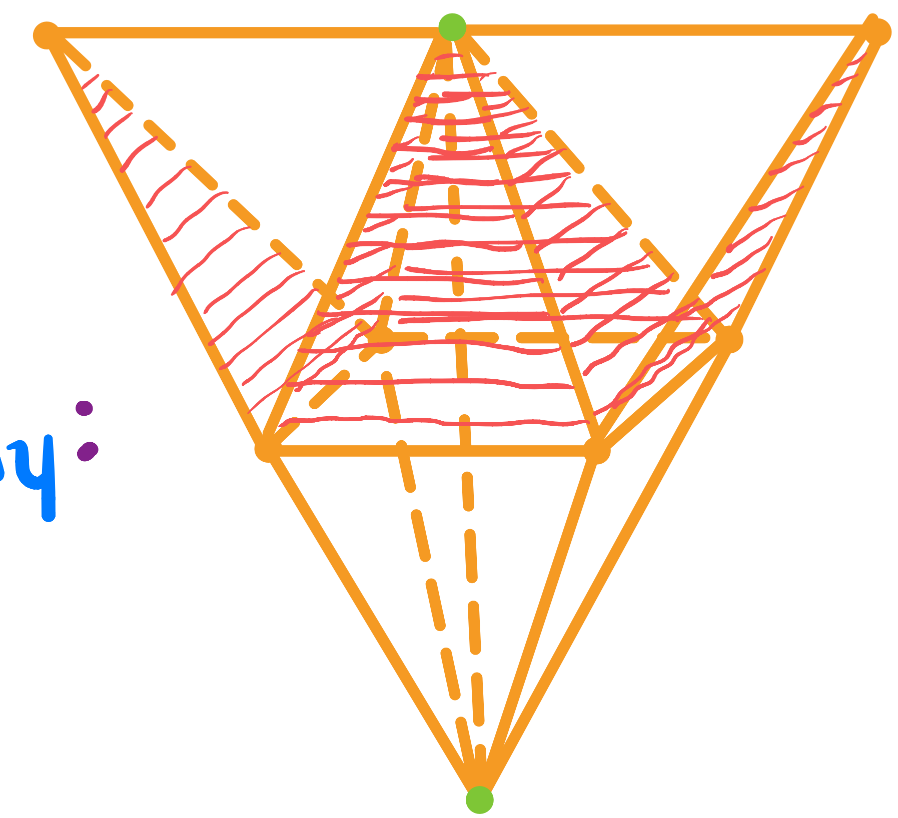
- We (GSSW) show there is a 3D analogue for the SL(4) web equivalence classes: pockets in Δ_4.
- Requires expanding sources/sinks via I=H moves:

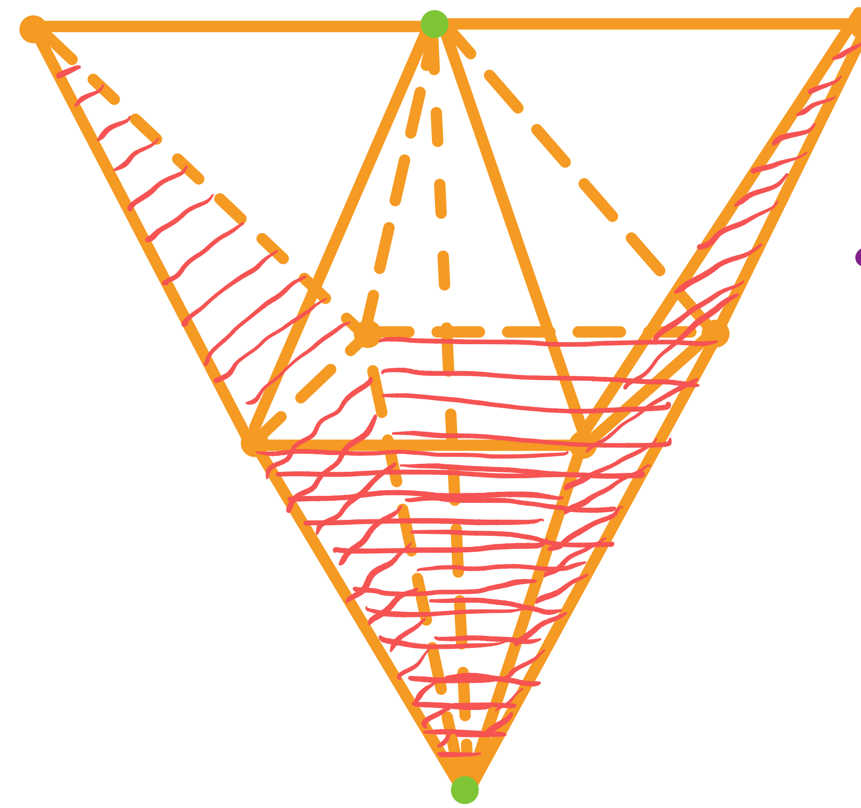


Pockets

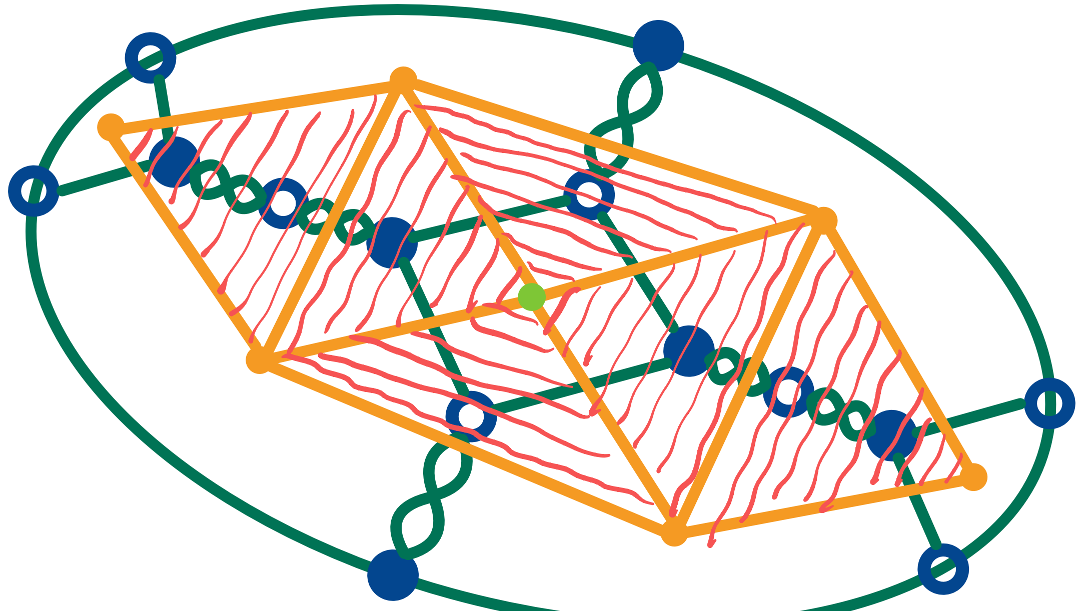
Ex

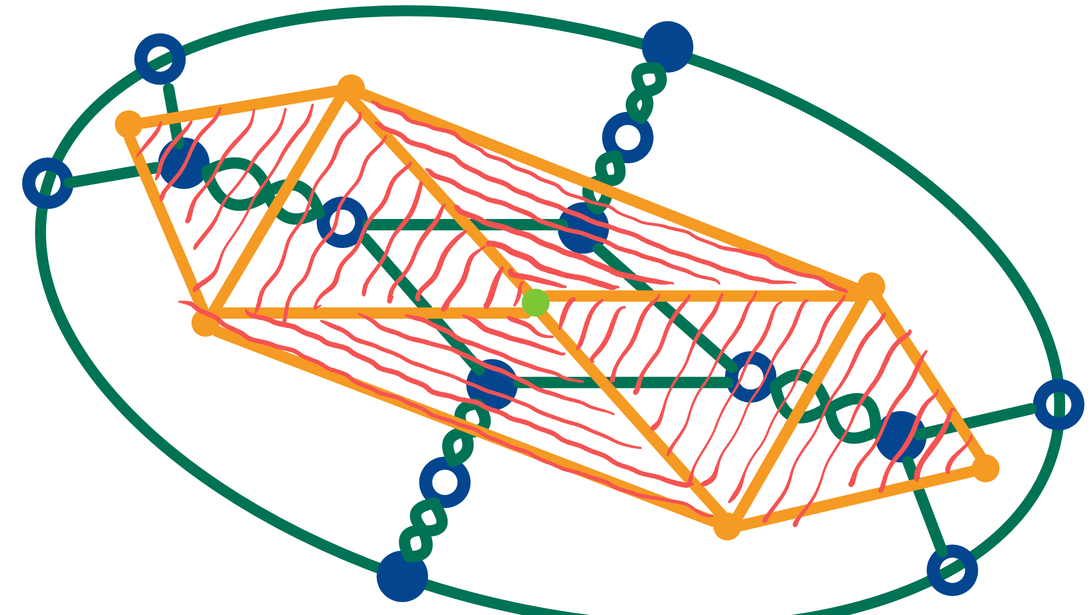
In Δ_4:





Pocket of square + benzene + IH class of $\bar{4}\{1,2\}\bar{3}\,\bar{2}\{3,4\}\bar{1}$

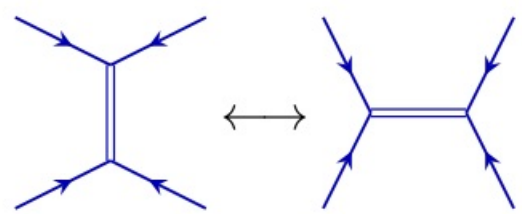




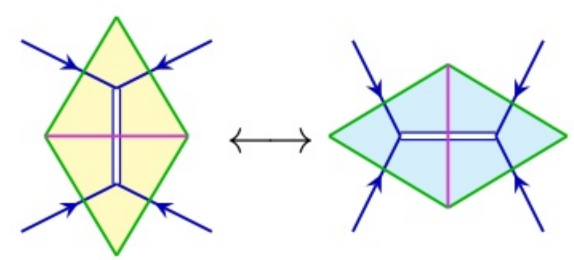
+6 more

Pockets

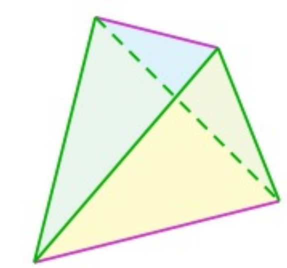
- Build pockets from 4-HPG basis web move classes:



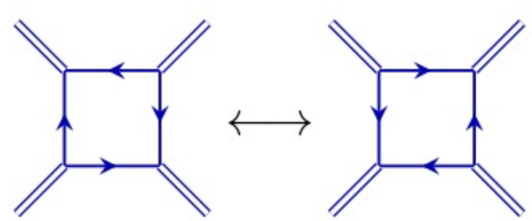
IH move between webs



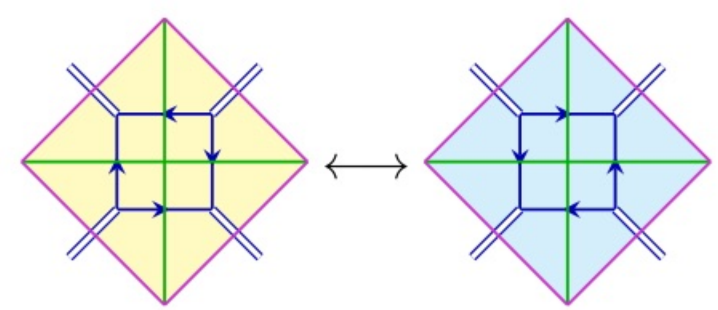
Dual diskoids



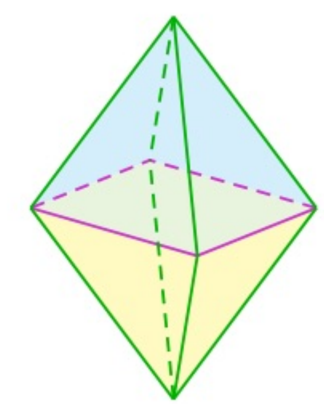
tetrahedron



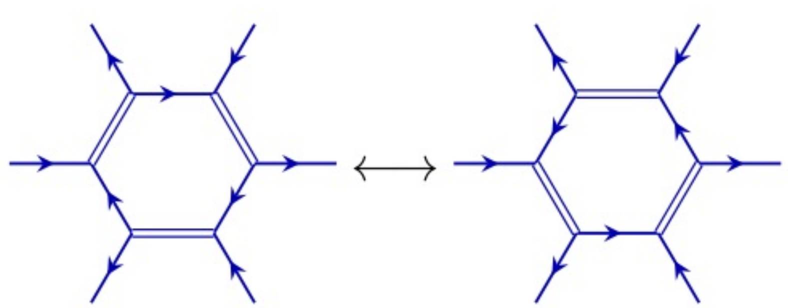
Square move between webs



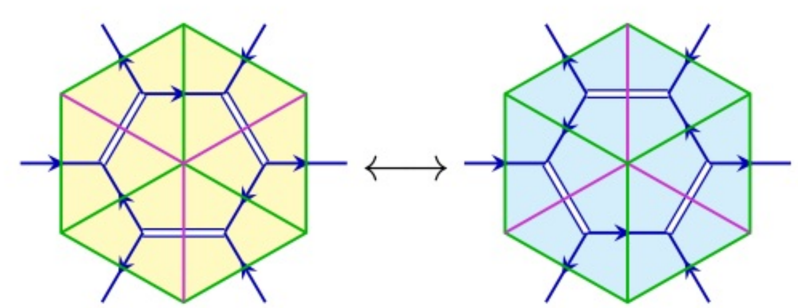
Dual diskoids



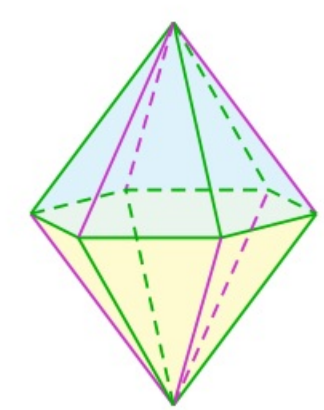
octahedron



Benzene move between webs



Dual diskoids



dodecahedron

Pockets

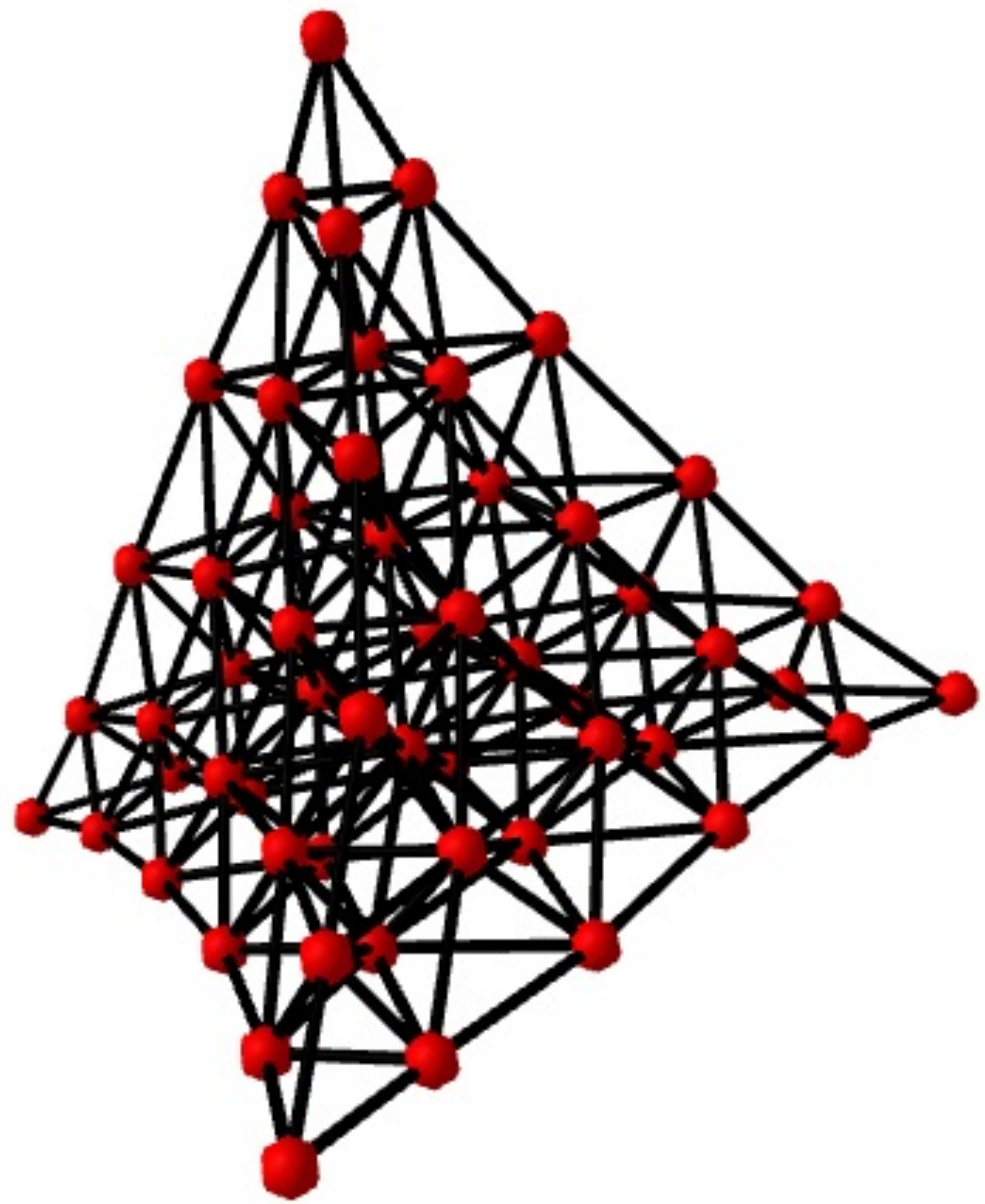
Thm (GSSW '24+) The pocket $P = P(T)$ is CAT(0) and a singular interval bundle over the closed disk. The simplicial sections of P are in bijection with the move-class corresponding to T.

Thm (GSSW '24+) Given an irreducible component of a Satake fiber of $\Delta = \Delta(SL_4^\vee)$ indexed by T, there is a dense open set U s.t. every point of U extends uniquely to a configuration $P(T) \hookrightarrow \Delta$ which preserves distances.

Pockets

Ex Pocket of 5×5 ASM:

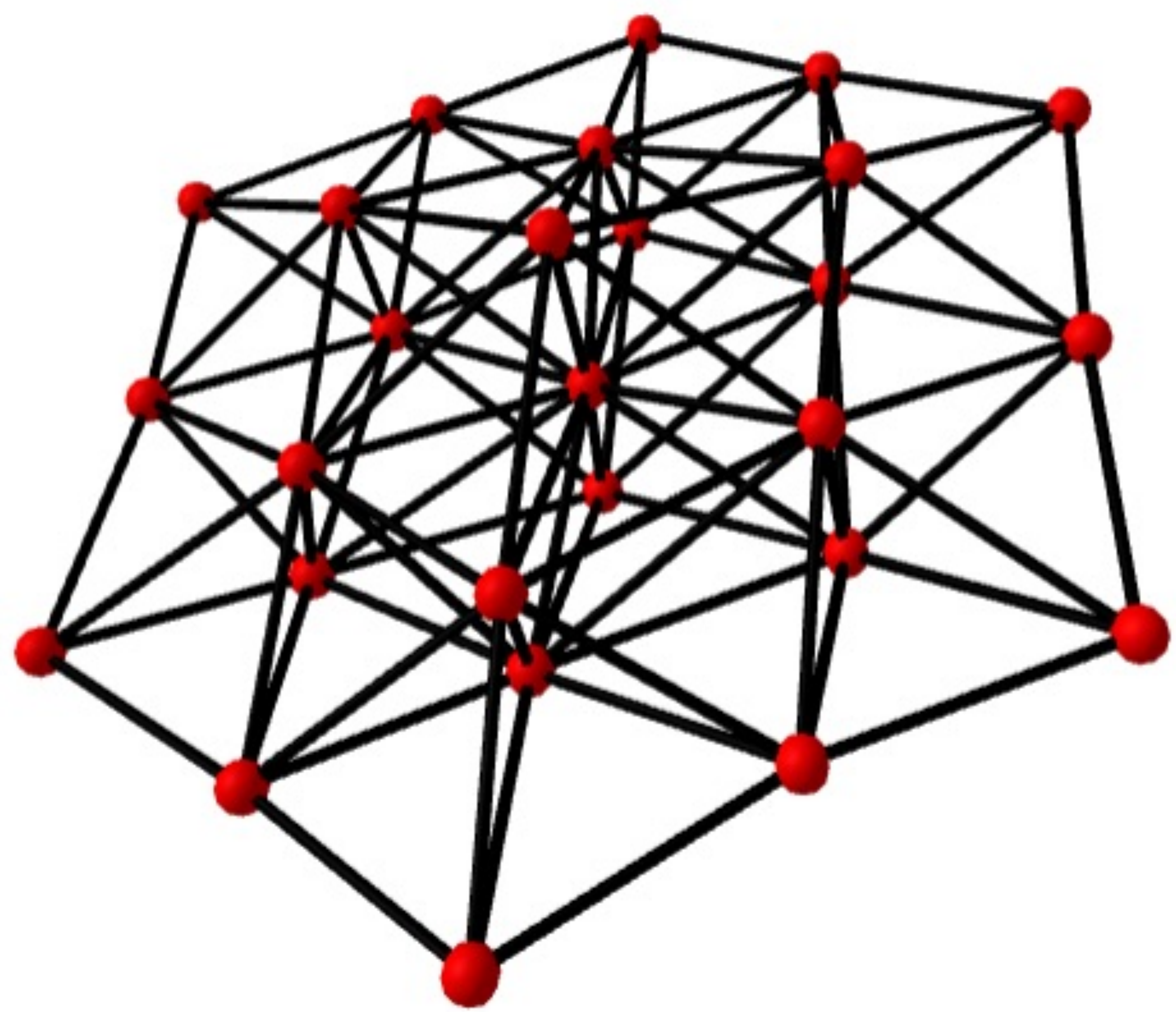
$L = 1^5 2^5 3^5 4^5$



Note Related to height functions, octahedral recurrence, tilings of the Aztec diamond, distributive lattices

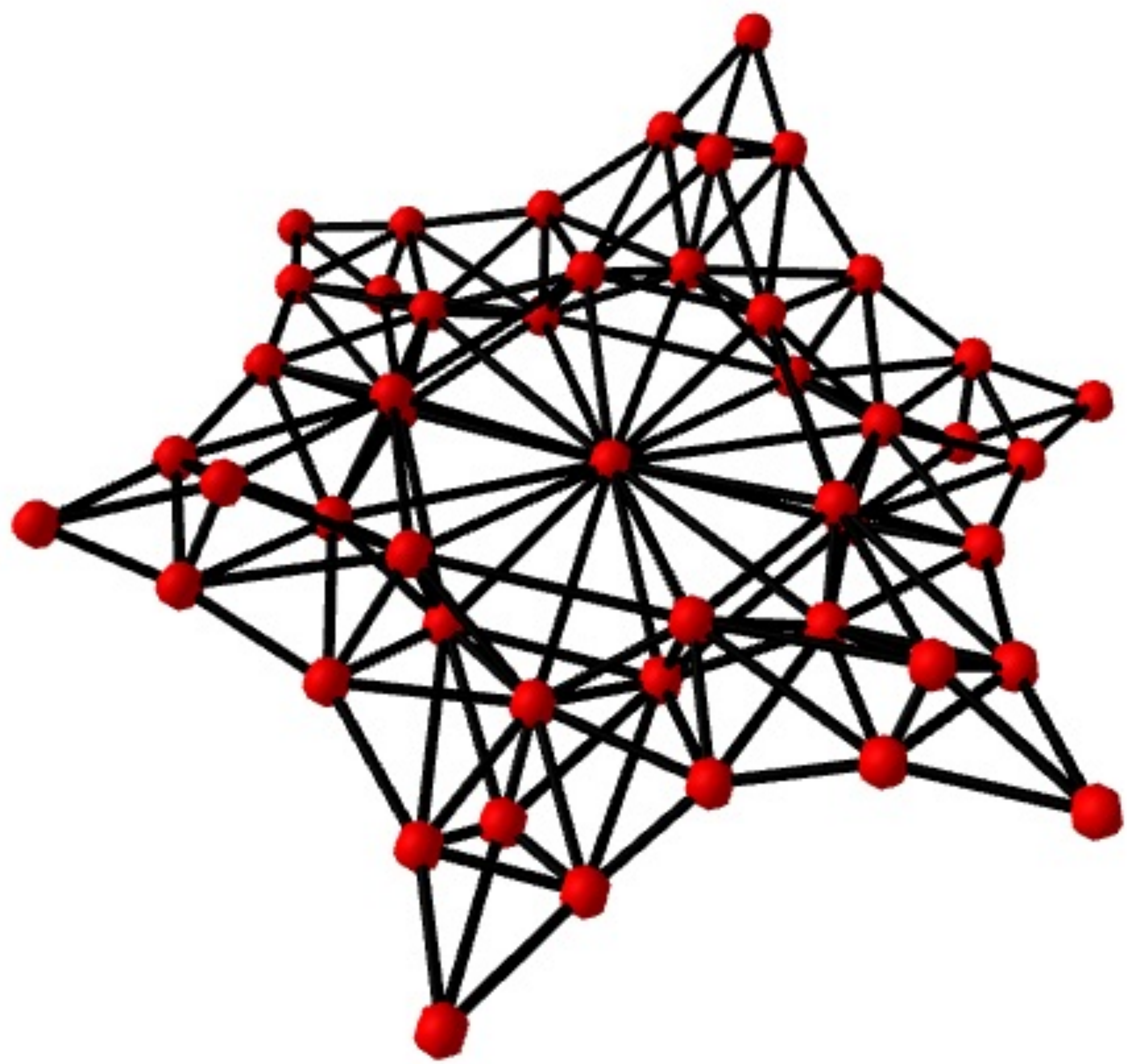
Pockets

Ex | Pocket of 2×2×2 PP:



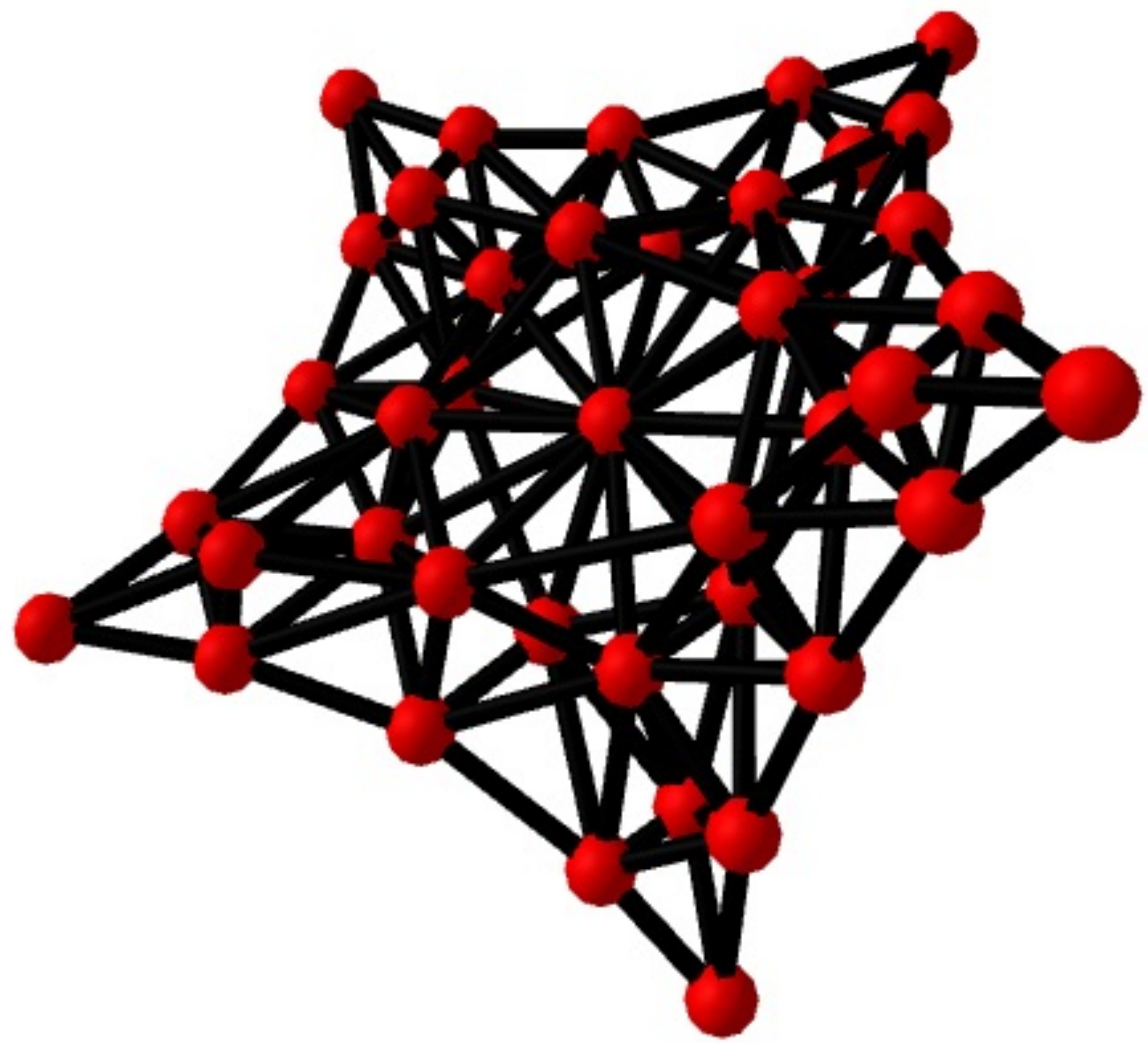
Pockets

Ex Pocket of a "chained hexagon":



Pockets

Ex Pocket of a "chained pentagon":



Note Not realizable in $\mathbb{R}^3$

THANKS!