Math 126 Challenge Problems/Solutions

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1. *(The War of the Quaternions.)*

> *"Quaternions came from Hamilton after his really good work had been done; and, though beautifully ingenious, have been an unmixed evil to those who have touched them in any way, including Clerk Maxwell."*
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> —Lord Kelvin, 1892.

Sir William Rowan Hamilton (1805 to 1865) was, roughly speaking, a standard early 19th century genius. He made important contributions to mathematics, astronomy, mechanics, etc., some of which are very widely remembered today. In particular, Hamiltonian/Lagrangian Mechanics and *quaternions* are still widely taught in undergraduate and graduate physics and pure mathematics curricula.

The quaternions have an enigmatic history. Their defining relations are said to have come to Hamilton in a flash of inspiration as he was walking over a bridge—and he couldn't resist carving the relations into the bridge that very day. You have been learning basic *vector analysis*, for instance when you compute $\mathbf{r}'(t)$. Some of you will certainly apply these and similar ideas to physical phenomena like electromagentism. However, this is not how the subject was first taught—in the mid-1800's, quaternions ruled. Maxwell's equations in Maxwell's original (re)formulation were written using quaternions, and only afterwards were these in turn written using vectors. Quaternions eventually lost the war for describing physical phenomena, but today they are well cared-for by pure mathematicians because of their stunning algebraic properties.

This problem will briefly introduce the quaternions. This presentation is not historical or even particularly common, but it serves our purposes, chiefly brevity. Define a quaternion as a pair consisting of a scalar and a three-dimensional vector, eg. $(3, \langle 7, 0, -1\rangle)$. We have some basic operations on quaternions:

- Addition: $(c, \mathbf{v}) + (d, \mathbf{u}) = (c + d, \mathbf{v} + \mathbf{u})$.
- Pure multiplication: $(0, \mathbf{v})(0, \mathbf{u}) = (-\mathbf{v} \cdot \mathbf{u}, \mathbf{v} \times \mathbf{u})$. This one is important, and frankly is a little bizarre; it is a reformulation of Hamilton's stroke of brilliance.
- Multiplication by scalars: $(r, \mathbf{0})(d, \mathbf{u}) = (rd, r\mathbf{u})$. Here $(r, \mathbf{0})$ is thought of as the scalar r and this formula is often written $r(d, \mathbf{u}) = (rd, r\mathbf{u})$.
- General multiplication: combine the previous three operations, assuming multiplication and addition are distributive. Explicitly,

$$\begin{aligned}(c, \mathbf{v})(d, \mathbf{u}) &= [(c, \mathbf{0}) + (0, \mathbf{v})][(d, \mathbf{0}) + (0, \mathbf{u})] \\ &= cd + c(0, \mathbf{u}) + d(0, \mathbf{v}) + (0, \mathbf{v})(0, \mathbf{u}) \\ &= (cd - \mathbf{v} \cdot \mathbf{u}, c\mathbf{u} + d\mathbf{v} + \mathbf{v} \times \mathbf{u}).\end{aligned}$$

(a) Show that, unlike complex numbers, the order of the factors of a product of two quaternions matters, i.e. PQ is not generally QP. Also show that, just like complex numbers, quaternion multiplication is associative, i.e. $(PQ)R = P(QR)$, so we can write lots of products without worrying about parentheses. It's probably easiest to do this for "pure" quaternions first, i.e. those with scalar part 0. If you get stuck, it might be worthwhile to skip this part and assume associativity for now.

(b) Like complex numbers, any quaternion has a *conjugate*, denoted by a *, which is defined in almost the same way: $(c, \mathbf{v})^* = (c, -\mathbf{v})$. We can think of the vector part of the quaternion as a sort of three-dimensional imaginary piece. Show that, while the conjugate of complex numbers z_1, z_2 satisfies $(z_1 z_2)^* = z_1^* z_2^*$, the conjugate of quaternions involves an order reversal, $(PQ)^* = Q^* P^*$.

(c) Using the complex numbers as inspiration, we would like to define the *norm* of a quaternion P by $|P| = \sqrt{PP^*}$. Show that PP^* has $\mathbf{0}$ vector part, so it makes sense to treat PP^* as a scalar. Show that its scalar part is non-negative, so it makes sense to take the square root. Finally, show that $|PQ| = |P||Q|$; this is sometimes called Euler's four-square identity.

(d) Using the conjugate, show that every non-zero quaternion has an inverse, i.e. given $(c, \mathbf{v})$, find some $(d, \mathbf{u})$ such that their product is $(1, \mathbf{0})$. You might find inspiration in the complex numbers. Thus we can both multiply and divide quaternions—something we were never able to do with vectors!

One last comment: quaternions are intimately related to rotations in three dimensional space. In particular, suppose we have a vector $\mathbf{v}$ and an axis $\mathbf{u}$ (a unit vector) where we want to rotate $\mathbf{v}$ about the axis some angle α. Encode $\mathbf{v}$ as a quaternion $V = (0, \mathbf{v})$. We can also encode the rotation in a (unit) quaternion $U = (\cos(\alpha/2), \mathbf{u}\sin(\alpha/2))$. It turns out that the result of rotating $\mathbf{v}$ by α about the $\mathbf{u}$-axis is given by the vector part of UVU^*; the scalar part is 0. This is actually used by some computer programs to perform "axis and angle" rotations on three dimensional vectors.

(a) Using the pure multiplication rule, we have $(0, \mathbf{i})(0, \mathbf{j}) = (-\mathbf{i} \cdot \mathbf{j}, \mathbf{i} \times \mathbf{j}) = (0, \mathbf{k})$. However, since $\mathbf{i} \times \mathbf{j} = k = -\mathbf{j} \times \mathbf{i}$, reversing the order gives $(0, \mathbf{j})(0, \mathbf{i}) = (0, -\mathbf{k})$.

Next, we have directly that

$$\begin{aligned}
[(0, \mathbf{u})(0, \mathbf{v})](0, \mathbf{w}) &= (-\mathbf{u} \cdot \mathbf{v}, \mathbf{u} \times \mathbf{v})(0, \mathbf{w}) \\
&= (-(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}, (-\mathbf{u} \cdot \mathbf{v})\mathbf{w} + (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}) \\
(0, \mathbf{u})[(0, \mathbf{v})(0, \mathbf{w})] &= (0, \mathbf{u})(-\mathbf{v} \cdot \mathbf{w}, \mathbf{v} \times \mathbf{w}) \\
&= (-\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}), \mathbf{u}(-\mathbf{v} \cdot \mathbf{w}) + \mathbf{u} \times (\mathbf{v} \times \mathbf{w}))
\end{aligned}$$

An identity from the text gives $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$, so the scalar parts indeed agree. Another identity from the text (the "vector triple product") gives $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ and $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = -(\mathbf{w} \cdot \mathbf{v})\mathbf{u} + (\mathbf{w} \cdot \mathbf{u})\mathbf{v}$. Combining these gives $[-(\mathbf{u} \cdot \mathbf{v})\mathbf{w} + (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}] - [\mathbf{u}(\mathbf{v} \cdot \mathbf{w}) + \mathbf{u} \times (\mathbf{v} \times \mathbf{w})] = -(\mathbf{u} \cdot \mathbf{v})\mathbf{w} - (\mathbf{w} \cdot \mathbf{v})\mathbf{u} + (\mathbf{w} \cdot \mathbf{u})\mathbf{v} + \mathbf{u}(\mathbf{v} \cdot \mathbf{w}) - (\mathbf{u} \cdot \mathbf{w})\mathbf{v} + (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = 0 - 0 + 0 = 0$, so indeed the vector parts agree.

To amplify this to all quaternions, first note that the product of scalar quaternions is definitely associative: $(a, \mathbf{0})(b, \mathbf{0})(c, \mathbf{0}) = (abc, \mathbf{0})$. Similarly the product of a scalar quaternion and two pure quaternions is quickly seen to be associative, and the product of two scalar quaterions and one pure quaternion is also associative.

Now recall that $(a, \mathbf{u})(b, \mathbf{v}) = [(a, \mathbf{0}) + (0, \mathbf{u})][(b, \mathbf{0}) + (0, \mathbf{v})]$, which we can expand as usual (the "FOIL" method). If we expand out $(a, \mathbf{u})(b, \mathbf{v})(c, \mathbf{w})$ in this manner, each term will be the product of some pure quaternions and some scalar quaternions, each of which is associative by the above reasoning, so the whole thing is associative.

(b) We compute directly

$$\begin{aligned}
((a, \mathbf{u})(b, \mathbf{v}))^* &= (ab - \mathbf{u} \cdot \mathbf{v}, -a\mathbf{v} - b\mathbf{u} - \mathbf{u} \times \mathbf{v}) \\
(b, \mathbf{v})^*(a, \mathbf{u})^* &= (b, -\mathbf{v})(a, -\mathbf{u}) \\
&= (ba - \mathbf{v} \cdot \mathbf{u}, -b\mathbf{u} - a\mathbf{v} + \mathbf{v} \times \mathbf{u}).
\end{aligned}$$

These are the same since $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$. In fact, the order reversal is done just so the sign of the cross product term works out.

(c) We have

$$\begin{aligned}(a, \mathbf{u})(a, \mathbf{u})^* &= (a, \mathbf{u})(a, -\mathbf{u}) \\ &= (a^2 + \mathbf{u} \cdot \mathbf{u}, -a\mathbf{u} + a\mathbf{u} - \mathbf{u} \times \mathbf{u} \\ &= (a^2 + |\mathbf{u}|^2, \mathbf{0}).\end{aligned}$$

The scalar part is $a^2 + |\mathbf{u}|^2$, which is indeed non-negative. Showing $|PQ| = |P||Q|$ is equivalent to showing $|PQ|^2 = |P|^2|Q|^2$, i.e. to showing $(PQ)(PQ)^* = PP^*QQ^*$. From the previous part, we have $(PQ)(PQ)^* = PQQ^*P^*$. Since QQ^* has zero vector part, it commutes with P^*, so we have $PQQ^*P^* = PP^*QQ^*$, as desired. Note we made heavy use of associativity here since we ignored parentheses repeatedly.

(d) For a non-zero complex number z, recall that $|z|^2 = zz^*$, so $1 = (zz^*)/|z|^2 = z(z^|z|^2)$. Since $1 = zz^{-1}$, this forces $z^{-1} = z^*/|z|^2$. Similarly, we can hope this formula defines an inverse for non-zero quaternions. From the explicit formula in the previous part, we see that $PP^* = (0, \mathbf{0})$ if and only if $P = (0, \mathbf{0})$, since $a^2 + |\mathbf{u}|^2 = 0$ if and only if $a = 0, \mathbf{u} = \mathbf{0}$. So, a non-zero quaternion P has non-zero norm $|P|$. Now let $Q = P^*/|P|^2$. We have $PQ = PP^*/|P|^2 = |P|^2/|P|^2 = 1 = (1, \mathbf{0})$, so $Q = P^{-1}$—right?

There's one small thing to check—we would like $QP = 1$, but we only showed $PQ = 1$. For complex numbers, this follows from commutativity, but we don't have commutativity, from part (a). Thankfully, this verification is also easy. We compute $QP = P^*P/|P|^2$, and now we ask, is P^*P the same as $|P|^2 = PP^*$? Yes; redo the explicit computation in (c) to show this. So, P indeed has an inverse, namely $P^*/|P|^2$, just like with complex numbers.

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