Math 126 Challenge Problems/Solutions

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1. Find the surface area of the part of the plane $z = \alpha x + \beta y + \gamma$ for which $a \leq x \leq b$, $c \leq y \leq d$.

Let $f(x, y) = \alpha x + \beta y + \gamma$. We are computing the area between

$$\begin{aligned} P_{ac} &= (a, c, f(a, c)), \qquad & P_{ad} &= (a, d, f(a, d)) \\ P_{bc} &= (b, c, f(b, c)), \qquad & P_{bd} &= (b, d, f(b, d)). \end{aligned}$$

Note that

$$\begin{aligned} P_{ac} - P_{bc} &= \langle a - b, 0, \alpha(a - b) \rangle \\ P_{ad} - P_{bd} &= \langle a - b, 0, \alpha(a - b) \rangle \\ P_{ac} - P_{ad} &= \langle 0, c - d, \beta(c - d) \rangle \\ P_{bc} - P_{bd} &= \langle 0, c - d, \beta(c - d) \rangle, \end{aligned}$$

which is to say these vertices form a parallelogram. By dropping a perpendicular, it's easy to see that the area of a parallelogram with sides A, B and angle θ between them is $|A||B| \sin\theta$. Since here $0 \leq \theta \leq \pi$, $\sin\theta \geq 0$, so $\sin\theta = +\sqrt{1 - \cos^2\theta}$. That is, the area is $\sqrt{(|A||B|)^2 - (A \cdot B)^2}$.

Here, $A = (a - b)\langle 1, 0, \alpha \rangle$, $B = (c - d)\langle 0, 1, \beta \rangle$, so $A \cdot B = (a - b)(c - d)\alpha\beta$. Also, $|A| = (b - a)\sqrt{1 + \alpha^2}$, $|B| = (d - c)\sqrt{1 + \beta^2}$, so the area is

$$\begin{aligned} &(b - a)(d - c)\sqrt{(1 + \alpha^2)(1 + \beta^2) - \alpha^2\beta^2} \\ &\qquad = (b - a)(d - c)\sqrt{1 + \alpha^2 + \beta^2}. \end{aligned}$$

Note that $\alpha = f_x$ and $\beta = f_y$, and also that $(b - a)(d - c)$ is the area of the rectangle $R = [a, b] \times [c, d]$ in the xy-plane. A more suggestive way to write this is then

$$\iint_R \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA.$$

(This derivation generalizes immediately to different dimensions, which is why I used the dot product to compute the area rather than the cross product, which only works in 3D.) ∎

2. Use (1) to guess a formula for the surface area of a general graph $f(x, y)$ above a region R in the xy-plane in terms of a double integral. (Hint: Tangent plane approximation.)

See the end of (1). The geometric interpretation of that formula is that we cut up R into tiny pieces, approximate the curve by its tangent plane at some sample points on the piece, compute the surface area of that tangent plane over the tiny piece, add up the results, and compute the limit as the tiny pieces get arbitrarily fine. Note that it essentially contains the familiar arc length formula for a graph $y = f(x)$. ∎