Worksheet 1 Solutions

1. *Derivatives*

   (a) $f'(x) = \frac{6}{1+3x}$
   (b) $f'(x) = \frac{15}{1+3x}$
   (c) $f'(x) = \frac{3b}{1+3x}$
   (d) $\frac{dy}{dx} = 1$
   (e) $\frac{du}{dv} = \frac{1}{1+a^2v}\frac{a}{2\sqrt{v}} = \frac{a}{2\sqrt{v}(1+a^2v)}$
   (f) $\frac{dy}{dt} = \frac{-\sin(t+c)}{\cos(t+c)} = -\tan(t+c)$, so $\frac{d^2y}{dt^2} = -\sec^2(t+c)$.

2. *Integrals*

   (a) This was practice with substitution:

      i. $p = 0$ gives $\int_e^{e^2} \frac{1}{x}\, dx = \ln(e^2) - \ln(e) = 2 - 1 = 1$.

      ii. $p = 1$ gives $\int_e^{e^2} \frac{1}{x\ln(x)}\, dx$. Substitution with $u = \ln(x)$ gives $\int_1^2 \frac{1}{u}\, du = \ln(2) - \ln(1) = \ln(2)$.

      iii. any other values of p, $\int_e^{e^2} \frac{1}{x\ln(x)^p}\, dx$. You still can do substitution, $u = \ln(x)$. To get $\int_1^2 \frac{1}{u^p}\, du = \int_1^2 u^{-p}\, du = \frac{1}{-p+1}2^{-p+1} - \frac{1}{-p+1}(1)^{-p+1} = -\frac{1}{(p-1))2^{p-1}} + \frac{1}{p-1} = \frac{1}{p-1}\left(1 - \frac{1}{2^{p-1}}\right)$. Notice that if you plug in $p = 0$ you get the same answer as the first part. However, this formula doesn't work for $p = 1$ because the integration of $\int_1^2 \frac{1}{u^p}\, du$ was different for $p = 1$ as you say in the second part above.

   (b) This is integration by parts practice: Use $u = x$ and $dv = e^{-2x}dx$, which gives you $\int xe^{-2x}\, dx = -\frac{1}{2}xe^{-2x} + \int \frac{1}{2}e^{-2x}\, dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$

   (c) This can be done by substitution and using the known result for arctangent or it can be done by trig substitution: First we write it as a limit, $\lim_{t\to\infty} \int_{2/3}^t \frac{dx}{9x^2+4}$. Now we factor out the 4 (we are getting ready to use the arctangent result), this gives $\lim_{t\to\infty} \frac{1}{4}\int_{2/3}^t \frac{dx}{9x^24+1} = \lim_{t\to\infty} \frac{1}{4}\int_{2/3}^t \frac{dx}{(3x/2)^2+1}$. Now substitute, $u = 3x/2$ to get $\lim_{t\to\infty} \frac{1}{4}\frac{2}{3}\int_1^{3t/2} \frac{dx}{u^2+1} = \lim_{t\to\infty} \frac{1}{6}(\tan^{-1}(3t/2) - \tan^{-1}(1)) = \frac{1}{6}(\pi/2 - \pi/4) = \frac{\pi}{24}$

3. *Tangent Lines*

   The equation for the tangent line at $x = 0$ is given by $y = f(0)+f'(0)(x-0)$. Since $f'(x) = -\frac{1}{2\sqrt{1-x}}$, we have $f'(0) = -\frac{1}{2}$ and we have $f(0) = 1$. So the tangent line is given by $y = 1 - \frac{1}{2}x$.

   (a) Sketch graph

   (b) Remember, near $x = 0$, we have $\sqrt{1-x} \approx 1 - \frac{1}{2}x$, so $\sqrt{0.99} \approx 1 - \sqrt{1}2(0.01) = 0.995$ (Actual value = 0.99498743...)

   (c) Using the linear approximation, we approximate the solution to $\sqrt{1-x} = 0.8$ by solving $1 - \frac{1}{2}x \approx 0.8$ which gives $x = 0.4$.
   This particular equation can also be solved for the exact answer, by first squaring both sides to get $1 - x = 0.8^2 = 0.64$ and so $x = 0.36$.

   (d) Using the linear approximation, $\int_0^1 \sqrt{1-x}dx \approx \int_0^1 1 - \frac{1}{2}xdx = 1 - \frac{1}{4}(1)^2 = \frac{3}{4}$.
   This particular integral can also be evaluated exactly, let $u = 1 - x$ to get $\int_0^1 \sqrt{1-x}dx = -\int_1^0 \sqrt{u}du = \frac{2}{3}(1)^{3/2} - \frac{2}{3}(0)^{3/2} = \frac{2}{3}$.