Math 126 Spring 2010

Work with projections, dot products, and cross-products.

1. Decide for each expression below whether it is a vector (**V**), a scalar (**S**), or nonsense (**N**). Note that $\mathbf{a}, \mathbf{b}, \mathbf{u}$, and $\mathbf{v}$ are vectors, while c and d are scalars.

(SEE ATTACHED FOR MORE DETAILED SOL'NS)

		Circle one:			
(a)	$\mathbf{a}\cdot(\mathbf{u}-c\mathbf{v})$	**V**	**S**	**N**	
(b)	$\mathbf{a}\cdot(\mathbf{b}+c)$	**V**	**S**	**N**	← BECAUSE OF $\vec{b}+c$
(c)	$(c+d)\cdot\mathbf{a}$	**V**	**S**	**N**	← BECAUSE OF "·" SYMBOL
(d)	$\mathbf{uv}$	**V**	**S**	**N**	← NO SYMBOL BETWEEN
(e)	$\frac{\mathbf{a}}{c}$	**V**	**S**	**N**	
(f)	$\frac{c}{\mathbf{a}}$	**V**	**S**	**N**	← DIVIDING BY VECTOR NOT DEFINED
(g)	$(\mathbf{a}\cdot\mathbf{b})\times\mathbf{u}$	**V**	**S**	**N**	← SCALAR × VECTOR
(h)	$\mathbf{a}\times(\mathbf{b}\times\mathbf{u})$	**V**	**S**	**N**	
(i)	$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{u})$	**V**	**S**	**N**	
(j)	$(c\mathbf{a})\times\mathbf{b}$	**V**	**S**	**N**	
(k)	$c(\mathbf{a}\cdot\mathbf{b})(\mathbf{u}\times\mathbf{v})$	**V**	**S**	**N**	

SEE ATTACHED

2. Determine whether each of the following is true or false. If it is true, prove it. If it is false, give a counterexample. Note that $\mathbf{a}$ and $\mathbf{b}$ are vectors and c is a scalar.

   (a) Suppose $\mathbf{a}\cdot\mathbf{b}=\mathbf{0}$. Then it must be true that at least one of $\mathbf{a}$ or $\mathbf{b}$ must be the zero vector. FALSE!!!

   (b) Suppose $c\mathbf{a}=\mathbf{0}$. Then it must be true that either $c=0$ or $\mathbf{a}=\mathbf{0}$ (or both). TRUE!

3. Suppose that $\mathbf{a}$ and $\mathbf{b}$ are nonzero vectors.

SEE ATTACHED

   (a) Show by examples that $\text{comp}_{\mathbf{a}}\mathbf{b}$ and $\text{comp}_{\mathbf{b}}\mathbf{a}$ can be the same and can be different. What conditions on $\mathbf{a}$ and $\mathbf{b}$ will guarantee they are the same? $\left\{ \vec{a}\cdot\vec{b}=0 \text{ or } |\vec{a}|=|\vec{b}| \right.$

   (b) Your friend who skips class frequently says, "I'm confused. Isn't $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$? If that is true, how can $\text{comp}_{\mathbf{a}}\mathbf{b}$ and $\text{comp}_{\mathbf{b}}\mathbf{a}$ be different?" What is your answer?

   (c) Show by examples that $\text{proj}_{\mathbf{a}}\mathbf{b}$ and $\text{proj}_{\mathbf{b}}\mathbf{a}$ can be the same and can be different. What conditions on $\mathbf{a}$ and $\mathbf{b}$ will guarantee they are the same? $\left\{ \vec{a}=\vec{b} \right.$

WORKSHEET 2a NOTES AND SOLUTIONS

(1) (SEE PAGE 1 FOR ANSWERS)

NOTE WE HAVE DEFINED

SUM/DIFFERENCE $\vec{u}+\vec{v}=\vec{w}$, $\vec{u}-\vec{v}=\vec{w}$

vectors

SCALAR MULTIPLE $c\vec{u}=\vec{v}$ NO SYMBOL HERE!

scalar, vectors

DOT PRODUCT $\vec{u}\cdot\vec{v}=c$

vectors, scalar

CROSS PRODUCT $\vec{u}\times\vec{v}=\vec{w}$

vectors

SPECIFIC COMMENTS

(a) $\underbrace{\vec{a}}_{\text{vector}}\cdot(\underbrace{\vec{u}-c\vec{v}}_{\text{vector}}) = \text{scalar}$

(b) $\vec{a}\cdot(\underbrace{\vec{b}+c}_{\text{NONSENSE}})$

FOR EXAMPLE
$\langle 1,-2,4\rangle+10$ IS NONSENSE
(WE DID NOT DEFINE THIS)

(c) $(\underbrace{c+d}_{\text{scalar}})\cdot\vec{a}$ vector

THIS "DOT" SYMBOL IS RESERVED FOR DOT PRODUCTS OF TWO VECTORS SO THIS IS NONSENSE.
FOR A SCALAR PRODUCT, PUT NO SYMBOL

(d) $\vec{u}\,\vec{v}$ vectors

NONSENSE, WE DID NOT DEFINE THIS.
SOME SYMBOL IS NEEDED ("$\cdot$" OR "$\times$")

(e) $\dfrac{\vec{a}}{c}=\dfrac{1}{c}\vec{a}=\text{vector}$ scalar, vector

EXAMPLE:
$\dfrac{\langle 1,2,4\rangle}{10}=\dfrac{1}{10}\langle 1,2,4\rangle=\langle\frac{1}{10},\frac{2}{10},\frac{4}{10}\rangle$

(f) $\dfrac{c}{\vec{a}}$ = nonsense

we did not define dividing by a vector

EXAMPLE:
$\dfrac{10}{\langle 1,2,4\rangle}$ = NONSENSE

(g) $(\underbrace{\vec{a}\cdot\vec{b}}_{\text{scalar}})\times\underbrace{\vec{u}}_{\text{vector}}$ = NONSENSE

(h) $\underbrace{\vec{a}}_{\text{vector}}\times(\underbrace{\vec{b}\times\vec{u}}_{\text{vector}})$ = VECTOR

(i) $\underbrace{\vec{a}}_{\text{vector}}\cdot(\underbrace{\vec{b}\times\vec{u}}_{\text{vector}})$ = SCALAR

(j) $\underbrace{(c\vec{a})}_{\text{vector}} \times \underbrace{\vec{b}}_{\text{vector}} = \text{vector}$

(k) $\underbrace{c(\vec{a}\cdot\vec{b})}_{\text{scalar}}\underbrace{(\vec{u}\times\vec{v})}_{\text{vector}} = \text{VECTOR}$

NO SYMBOL (GOOD)

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(2) (a) TRUE OR FALSE:
If $\vec{a}\cdot\vec{b}=0$, then $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$.

$\boxed{\text{FALSE!!!}}$

Remember $\vec{a}\cdot\vec{b}=0 \iff \vec{a}$ and $\vec{b}$ are orthogonal
that does not mean one of them must be zero.

Counter-example: $\vec{a}=\langle 1,-2\rangle$, $\vec{b}=\langle 2,1\rangle$
$\vec{a}\cdot\vec{b}=0$ (Hypothesis true)
and $\vec{a}$ is not the zero vector
and $\vec{b}$ is not the zero vector } (conclusion false)
Hence the statement is not always true.
(So it is a false statement)

(b) TRUE OR FALSE:
If $c\vec{a}=\vec{0}$, then $c=0$ or $\vec{a}=\vec{0}$.

$\boxed{\text{TRUE!}}$

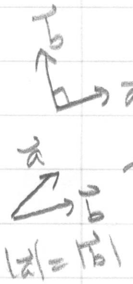
proof Assume $c\vec{a}=\vec{0}$.
Labeling the components of $\vec{a}=\langle a_1,a_2,a_3\rangle$
By definition of scalar multiplication,
$c\vec{a}=\langle ca_1,ca_2,ca_3\rangle$.
Since we are assuming this is the zero vector, we have $ca_1=0$ and $ca_2=0$ and $ca_3=0$.
So either $c=0$, or if it is not, a_1,a_2,a_3 are all zero. Thus, $c=0$ or $\vec{a}=\vec{0}$ //

③ (a) $\text{comp}_{\vec{a}}(\vec{b}) = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$ and $\text{comp}_{\vec{b}}(\vec{a}) = \dfrac{\vec{b}\cdot\vec{a}}{|\vec{b}|}$

So $\text{comp}_{\vec{a}}(\vec{b}) = \text{comp}_{\vec{b}}(\vec{a}) \iff \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \dfrac{\vec{b}\cdot\vec{a}}{|\vec{b}|}$ same

$$\iff |\vec{b}|(\vec{a}\cdot\vec{b}) = |\vec{a}|(\vec{a}\cdot\vec{b})$$
$$\iff |\vec{a}|(\vec{a}\cdot\vec{b}) - |\vec{b}|(\vec{a}\cdot\vec{b}) = 0$$
$$\iff (\vec{a}\cdot\vec{b})(|\vec{a}| - |\vec{b}|) = 0$$

which can only happen when i) $\vec{a}\cdot\vec{b} = 0$
or ii) $|\vec{a}| = |\vec{b}|$.



Thus, $\text{comp}_{\vec{a}}(\vec{b}) = \text{comp}_{\vec{b}}(\vec{a})$ exactly when
$\vec{a}\cdot\vec{b} = 0$ (i.e. $\vec{a}$ and $\vec{b}$ are orthogonal)
or $|\vec{a}| = |\vec{b}|$ (i.e. $\vec{a}$ and $\vec{b}$ have the same length)
Otherwise, the component projections are different.

(b) Remind your friend that component projections also depend on the vector length of the vector you are projecting onto.
Then give them an example.

(c) $\text{proj}_{\vec{a}}(\vec{b}) = \dfrac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\vec{a}$, $\text{proj}_{\vec{b}}(\vec{a}) = \dfrac{\vec{b}\cdot\vec{a}}{\vec{b}\cdot\vec{b}}\vec{b}$

Observe, for these vectors to be equal they must

① POINT IN THE SAME DIRECTION
② HAVE THE SAME LENGTH

FOR THESE TO BE THE SAME

① CAN ONLY HAPPEN IF $\vec{a}$ and $\vec{b}$ are already in the same direction. Thus, $\vec{a}$ and $\vec{b}$ must be parallel in the same direction.
② can only happen if $\text{comp}_{\vec{a}}(\vec{b}) = \text{comp}_{\vec{b}}(\vec{a})$
so $|\vec{a}| = |\vec{b}|$ (from above since not orthogonal)

$\vec{a}$ & $\vec{b}$ have to be the same