Your Name

Student ID #

- Do not open this exam until you are told to begin. You will have 60 minutes for the exam.
- Check that you have a complete exam. There are 6 questions for a total of 50 points.
- You are allowed to have one handwritten note sheet. An **equation sheet** is provided on the last page. No calculators are allowed.
- Cheating will result in a zero and be reported to the Dean's Academic Conduct Committee.
- **Show all your work**. Unless explicitly stated otherwise in a particular question, if there is no work supporting your answer, you will not receive credit for the problem. If you need more space to answer a question, continue on the back of the page, and indicate that you have done so.

Question	Points	Score
1	11	
2	10	
3	10	
4	11	
5	8	
6	0	
Total:	50	

1. (a) (3 points) Suppose

$$f(t) = \begin{cases} \cos t & 0 \le t < 5 \\ 5 & 5 \le t < 15 \\ 0 & 15 \le t < 20 \\ e^{-t} & 20 \le t \end{cases}.$$

Write $f(t)$ as a sum involving step functions.

(b) (3 points) Show that $\mathcal{L}\{tu_3(t) + 5u_{10}(t)\} = e^{-3s}\left(\frac{1}{s^2} + \frac{3}{s}\right) + \frac{5e^{-10s}}{s}$. (Hint: You may use the table at the end of the exam.)

(c) (5 points) Find the solution of the initial value problem

$$y'' - 2y' + y = 5u_{10}(t), \qquad y(0) = 0, y'(0) = 1.$$

2. (a) (1 point) Is $\mathcal{L}\left\{t^2\right\} = \mathcal{L}\left\{t\right\}^2$?

   (b) (2 points) Write down some $f(t)$ for which one could **not** immediately compute $\mathcal{L}\left\{f(t)\right\}$ by the identities at the end of the exam.

   (c) (2 points) Describe the difference between variation of parameters and reduction of order.

   (d) (2 points) If $y' = e^{-yt}$, can y ever be decreasing? If so, give an example; if not, explain why not.

(e) (3 points) Write down your own differential equation which **cannot** be solved using the methods from this class. Roughly describe what prevents our methods from working. (Hint: make sure your equation is not separable.)

3. A spring is suspended from the ceiling of an elevator. A mass of 1 kg is attached to the end of the spring and the spring provides a resistance of 1 N when stretched 1 meter beyond its unstretched (**not equilibrium**) length. When the spring moves at 5 m/s, viscous damping exerts a force of 0.1 N. Furthermore, a malfunction in the elevator causes it to oscillate in its shaft in such a way that the mass feels an external force of $20\cos(t/10)$ N upward. The mass starts at rest (relative to the ceiling of the elevator) at its equilibrium position.

   (a) (4 points) Write down an initial value problem describing the motion of the mass (relative to the ceiling of the elevator). **Do not solve your IVP.**

   (b) (2 points) What is the natural frequency of the spring-mass system? Do you expect to see significant resonance? Why or why not?

   (c) (4 points) If you were to solve the IVP from (a) using the methods discussed in class, you would have to solve a differential equation which is... (check all that apply)

   ◯ first order ◯ second order ◯ linear ◯ non-linear ◯ constant coefficient
   ◯ non-constant coefficient ◯ homogeneous ◯ non-homogeneous

   Along the way, you would have to solve a system of ____________ linear equations in ____________ unknowns.

4. In this problem, you are told that the homogeneous equation

$$y'' - 4y' + 13y = 0$$

has general solution $y = e^{2t}(c_1 \cos(3t) + c_2 \sin(3t))$. Consider solving the initial value problem

$$y'' - 4y' + 13y = 120\cos(t), \qquad y(0) = 0, y'(0) = 0.$$

(a) (6 points) Which of the following techniques from class could be used to solve the IVP? Fill in the blanks for each technique that applies. **Do not solve** the IVP at this stage.

○ Separable DE, $y' = f(t)g(y)$ with $f(t) =$ ____________, $g(y) =$ ____________.

○ Integrating factors, with $\mu =$ ____________.

○ Reduction of order, with $y_1 =$ ____________.

○ Variation of parameters, with $y_1 =$ ____________, $y_2 =$ ____________.

○ Undetermined coefficients, with $Y =$ ____________.

○ Laplace and inverse Laplace transform, where $\mathcal{L}\{y\}(s) =$ ____________.

(b) (5 points) Pick one of methods from (a) **except the Laplace transform** and use it to solve the initial value problem above. To save time, **you may skip** determining the constants c_1 and c_2.

5. This question deals with qualitative interpretations of differential equations, so it requires no computation.

   (a) Suppose an object's height y as a function of time t is described by the initial value problem

   $$(y')^2 + y^2 = e^{-t}, \qquad y(0) = y_0, y'(0) = y_0'.$$

   i. (2 points) Can one solve this initial value problem using the methods from class? Why or why not?

   ii. (2 points) What do you expect the object's height to be after a long time? Do you expect the answer to depend on the initial conditions?

   (b) Suppose the solution to an initial value problem is

   $$y(t) = \cos(\cos(t) + 1) + \frac{1}{t}\sin(t^2).$$

   i. (2 points) Identify the transient and steady state solutions.

   ii. (2 points) Is the steady state solution periodic? If so, what is its period?

6. (a) (1 point (bonus)) Briefly describe the purpose of the atan2 function.

   (b) (1 point (bonus)) Summarize the course using no more than four sentences. (Anything you write here has at least the possibility of being quoted to future students.)

Math 307 Final Equation Sheet

- $ay'' + by' + cy = 0$, $ar^2 + br + c = 0$, $\{e^{r_1 t}, e^{r_2 t}\}$, $\{e^{rt}, te^{rt}\}$, $\{e^{\lambda t}\cos(\mu t), e^{\lambda t}\sin(\mu t)\}$.
- $y = y_h + y_p = c_1 y_1 + c_2 y_2 + y_p$
- $y = uy_1$, $y = y_1 \left(\int \frac{\int y_1 e^{\int p(t)\,dt} g(t)\,dt + c}{y_1^2 e^{\int p(t)\,dt}}\,dt + d \right)$
- $y_p = u_1(t)y_1 + u_2(t)y_2$, $u_1(t) = -\int \frac{g(t)y_2(t)}{W(y_1,y_2)}\,dt$, $u_2(t) = \int \frac{g(t)y_1(t)}{W(y_1,y_2)}\,dt$, $W(y_1,y_2) = y_1 y_2' - y_1' y_2$.
- $A\cos(\omega t) + B\sin(\omega t) = R\cos(\omega t - \delta)$, $A = R\cos\delta$, $B = R\sin\delta$, $R = \sqrt{A^2 + B^2}$, $\delta = \text{atan2}(B, A)$, $\text{atan2}(B, A)$ is $\arctan(B/A)$ if $A > 0$ and is $\arctan(B/A) + \pi$ if $A < 0$.
- $my'' + \gamma y' + ky = F(t)$, $mg = kL$
- $\gamma^2 - 4mk$ vs. 0, $\omega_0 = \sqrt{\frac{k}{m}}$, $\lambda = -\frac{\gamma}{2m}$, $\mu = \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}} = \omega_0\sqrt{1 - \frac{\gamma^2}{4km}}$.

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
$af(t) + bg(t)$	$aF(s) + bG(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$e^{at}f(t)$	$F(s-a)$
t^n	$\frac{n!}{s^{n+1}}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{n-1}(0)$
$\delta(t)$	1
$\delta(t-c)$	e^{-cs}
$u_c(t)$	$\frac{e^{-cs}}{s}$
$u_c(t)f(t)$	$e^{-cs}\mathcal{L}\{f(t+c)\}(s)$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$