Math 307 D **Final** **Summer 2015**

Your Name

Student ID #

- Do not open this exam until you are told to begin. You will have 60 minutes for the exam.
- Check that you have a complete exam. There are 6 questions for a total of 50 points.
- You are allowed to have one handwritten note sheet. An **equation sheet** is provided on the last page. No calculators are allowed.
- Cheating will result in a zero and be reported to the Dean's Academic Conduct Committee.
- **Show all your work**. Unless explicitly stated otherwise in a particular question, if there is no work supporting your answer, you will not receive credit for the problem. If you need more space to answer a question, continue on the back of the page, and indicate that you have done so.

Question	Points	Score
1	11	
2	10	
3	10	
4	11	
5	8	
6	0	
Total:	50	

1. (a) (3 points) Suppose

$$f(t) = \begin{cases} \cos t & 0 \le t < 5 \\ 5 & 5 \le t < 15 \\ 0 & 15 \le t < 20 \\ e^{-t} & 20 \le t \end{cases}.$$

Write $f(t)$ as a sum involving step functions.

Solution: Using "pulse functions", we get

$$\begin{aligned} f(t) &= \cos(t)(u_0(t) - u_5(t)) + 5(u_5(t) - u_{15}(t)) + e^{-t}(u_{20}(t)) \\ &= \cos(t)u_0(t) + (5 - \cos(t))u_5(t) - 5u_{15}(t) + e^{-t}u_{20}(t). \end{aligned}$$

(b) (3 points) Show that $\mathcal{L}\{tu_3(t) + 5u_{10}(t)\} = e^{-3s}\left(\frac{1}{s^2} + \frac{3}{s}\right) + \frac{5e^{-10s}}{s}$. (Hint: You may use the table at the end of the exam.)

Solution: Using the table gives $e^{-3s}\mathcal{L}\{t+3\} + 5e^{-10s}\mathcal{L}\{1\}$, which gives the answer since $\mathcal{L}\{1\} = 1/s$ and $\mathcal{L}\{t+3\} = \mathcal{L}\{t\} + 3\mathcal{L}\{1\} = 1/s^2 + 3/s$.

(c) (5 points) Find the solution of the initial value problem

$$y'' - 2y' + y = 5u_{10}(t), \qquad y(0) = 0, y'(0) = 1.$$

Solution: Using the second half of the previous computation and the identities in the table, we find

$$(s^2 - 2s + 1)\mathcal{L}\{y\} = \frac{5e^{-10s}}{s} + 1$$

so that

$$\begin{aligned} \mathcal{L}\{y\} &= 5e^{-10s}\frac{1}{s(s-1)^2} + \frac{1}{(s-1)^2} \\ &= 5e^{-10s}\left(\frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2}\right) + \frac{1}{(s-1)^2} \end{aligned}$$

where $1 = A(s-1)^2 + Bs(s-1) + Cs$ gives $C = 1$ (plug in $s = 1$), $A = 1$ (plug in $s = 0$), and $B = -1$ (compare s^2 coefficients on either side). Taking the inverse Laplace transform (remembering to shift by 10) gives

$$y = 5u_{10}(t)(1 - e^{t-10} + (t-10)e^{t-10}) + te^t.$$

2. (a) (1 point) Is $\mathcal{L}\left\{t^2\right\} = \mathcal{L}\left\{t\right\}^2$?

> **Solution:** No; from the table, this would say $\frac{2}{s^3} = \left(\frac{1}{s}\right)^2$, which is very false.

(b) (2 points) Write down some $f(t)$ for which one could **not** immediately compute $\mathcal{L}\left\{f(t)\right\}$ by the identities at the end of the exam.

> **Solution:** We are more or less limited to computing the Laplace transforms of polynomial combinations of exponentials times step functions. There are an enormous number of functions not of this form, like $f(t) = \sqrt{t}, \sin(t)/t, e^{-t^2}, \ln t, \ldots$.

(c) (2 points) Describe the difference between variation of parameters and reduction of order.

> **Solution:** Variation of parameters requires two inputs while reduction of order requires one input. Variation of parameters is only useful for finding particular solutions of inhomogeneous equations, while reduction of order can do that as well as give a second solution to a homogeneous equation. Reduction of order sometimes leads to uglier computations than variation of parameters in situations where both are applicable.

(d) (2 points) If $y' = e^{-yt}$, can y ever be decreasing? If so, give an example; if not, explain why not.

> **Solution:** No, it cannot ever be decreasing, since the exponential function is always positive, so $y' > 0$ always.

(e) (3 points) Write down your own differential equation which **cannot** be solved using the methods from this class. Roughly describe what prevents our methods from working. (Hint: make sure your equation is not separable.)

Solution: There's an enormous variety of answers. A simple one is $y' = t + e^y$. This is not separable, and every other equation we've studied has been linear, but this is non-linear. (Technically undetermined coefficients can be applied to very general equations, but outside of the cases we studied, picking good guesses is very difficult or impossible. For this particular example, it turns out the magical substitution $u = e^{y-t^2/2}$ makes this equation separable, which allows one to solve it. The answer involves of an imaginary version of the error function which would be very difficult to simply guess.)

3. A spring is suspended from the ceiling of an elevator. A mass of 1 kg is attached to the end of the spring and the spring provides a resistance of 1 N when stretched 1 meter beyond its unstretched (**not equilibrium**) length. When the spring moves at 5 m/s, viscous damping exerts a force of 0.1 N. Furthermore, a malfunction in the elevator causes it to oscillate in its shaft in such a way that the mass feels an external force of $20\cos(t/10)$ N upward. The mass starts at rest (relative to the ceiling of the elevator) at its equilibrium position.

   (a) (4 points) Write down an initial value problem describing the motion of the mass (relative to the ceiling of the elevator). **Do not solve your IVP.**

   Solution: We have $m = 1$ kg, $k = 1$ N/m, $\gamma = 0.1/5$ N/(m/s), and $F(t) = -20\cos(t/10)$ (note the sign). The IVP

   $$my'' + \gamma y' + ky = F(t), \qquad y(t_0) = y_0, y'(t_0) = y'_0$$

   here is then

   $$y'' + 0.02y' + y = -20\cos(t/10), \qquad y(0) = 0, y'(0) = 0.$$

   (b) (2 points) What is the natural frequency of the spring-mass system? Do you expect to see significant resonance? Why or why not?

   Solution: The natural frequency is $\omega_0 = \sqrt{k/m} = 1$. Since γ is small (compared to k and m), resonance occurs quite close to ω_0, while the forcing frequency here is $1/10$, which is far from the natural frequency, so resonance does not occur.

   (c) (4 points) If you were to solve the IVP from (a) using the methods discussed in class, you would have to solve a differential equation which is... (check all that apply)

   ○ first order √ **second order** √ **linear** ○ non-linear √ **constant coefficient** ○ non-constant coefficient ○ homogeneous √ **non-homogeneous**

   Along the way, you would have to solve a system of ____**two**____ linear equations in ____**two**____ unknowns.

4. In this problem, you are told that the homogeneous equation

$$y'' - 4y' + 13y = 0$$

has general solution $y = e^{2t}(c_1 \cos(3t) + c_2 \sin(3t))$. Consider solving the initial value problem

$$y'' - 4y' + 13y = 120\cos(t), \qquad y(0) = 0, y'(0) = 0.$$

(a) (6 points) Which of the following techniques from class could be used to solve the IVP? Fill in the blanks for each technique that applies. **Do not solve** the IVP at this stage.

○ Separable DE, $y' = f(t)g(y)$ with $f(t) =$ __**N/A**__, $g(y) =$ __**N/A**__.

○ Integrating factors, with $\mu =$ __**N/A**__.

√ **Reduction of order, with** $y_1 =$ __$e^{2t}\cos(3t)$__.

√ **Variation of parameters, with** $y_1 =$ __$e^{2t}\cos(3t)$__, $y_2 =$ __$e^{2t}\sin(3t)$__.

√ **Undetermined coefficients, with** $Y =$ __$A\cos(t) + B\sin(t)$__.

√ **Laplace and inverse Laplace transform, where** $\mathcal{L}\{y\}(s) =$ __$120s/((s^2+1)(s^2-4s+13)$__.

(b) (5 points) Pick one of methods from (a) **except the Laplace transform** and use it to solve the initial value problem above. To save time, **you may skip** determining the constants c_1 and c_2.

Solution: Variation of parameters is a tempting choice, though the algebra is messy. Reduction of order is less appealing than variation of parameters since we have a full fundamental solution set. Undetermined coefficients is available, so we use it. (The Laplace transform would also be fine, but question 1 already tests that technique, so it has been disallowed; also, the algebra is again more involved than necessary.)

Using the above guess, one quickly finds $(-A-4B+13A)\cos t + (-B+4A+13B)\sin t = 120\cos t + 0\sin t$, so that $-12A - 4B = 120$, $-12B + 4A = 0$, so $-3B = A$, giving $-40B = 120$, so $B = -3$ and $A = 9$. Hence our particular solution is $y_p = 9\cos t - 3\sin t$, so the general solution is

$$y = e^{2t}(c_1\cos(3t) + c_2\sin(3t)) + 9\cos t - 3\sin t.$$

The question allows us to stop here, though after computing some easy (if lengthy) derivatives and plugging in the initial conditions, we find $c_1 = -9$ and $c_2 = 7$.

5. This question deals with qualitative interpretations of differential equations, so it requires no computation.

   (a) Suppose an object's height y as a function of time t is described by the initial value problem

   $$(y')^2 + y^2 = e^{-t}, \qquad y(0) = y_0, y'(0) = y_0'.$$

   i. (2 points) Can one solve this initial value problem using the methods from class? Why or why not?

   Solution: No, you could not solve it using the methods from class, since we dealt only with separable or linear equations, and this is neither. (Undetermined coefficients is hopeless, as it turns out.)

   ii. (2 points) What do you expect the object's height to be after a long time? Do you expect the answer to depend on the initial conditions?

   Solution: Over time, e^{-t} goes to 0, so $(y')^2 + y^2$ goes to 0, so certainly y^2, hence y, goes to zero. The object's height then tends to 0, regardless of the initial conditions.

   (b) Suppose the solution to an initial value problem is

   $$y(t) = \cos(\cos(t) + 1) + \frac{1}{t}\sin(t^2).$$

   i. (2 points) Identify the transient and steady state solutions.

   Solution: The transient solution is $\frac{1}{t}\sin(t^2)$ since it decays to zero. The steady state solution is $\cos(\cos(t) + 1)$ since it does not decay over time.

   ii. (2 points) Is the steady state solution periodic? If so, what is its period?

   Solution: The steady state solution $\cos(\cos(t)+1)$ is periodic with the same period as $\cos(t)$, namely 2π.

6. (a) (1 point (bonus)) Briefly describe the purpose of the atan2 function.

> **Solution:** Given A and B not both zero, $\delta := \text{atan2}(B, A)$ is defined to satisfy $A = R\cos(\delta)$, $B = R\sin(\delta)$, where $R = \sqrt{A^2 + B^2}$. More geometrically, it a polar angle for the point (A, B).

(b) (1 point (bonus)) Summarize the course using no more than four sentences. (Anything you write here has at least the possibility of being quoted to future students.)

> **Solution:** From the syllabus, "Math 307 is an introductory course in ordinary differential equations (ODEs) intended for students in engineering, mathematics, and the sciences. The course breaks into three pieces: first order DEs, second order DEs, and Laplace transforms.
>
> Specific topics include first-order linear and separable equations, autonomous equations and stability, Euler's numerical method, applications of first-order equations, second-order linear equations with constant coefficients, characteristic equations, connections between homogeneous and nonhomogeneous second-order linear equations, second-order linear equations with non-constant coefficients, variation of parameters, applications to mechanical systems, the Laplace transform and its inverse, formulas for Laplace transforms, the delta function and its Laplace transform, and second-order equations with discontinuous right-hand sides."

Math 307 Final Equation Sheet

- $ay'' + by' + cy = 0$, $ar^2 + br + c = 0$, $\{e^{r_1 t}, e^{r_2 t}\}$, $\{e^{rt}, te^{rt}\}$, $\{e^{\lambda t}\cos(\mu t), e^{\lambda t}\sin(\mu t)\}$.
- $y = y_h + y_p = c_1 y_1 + c_2 y_2 + y_p$
- $y = uy_1$, $y = y_1\left(\int \frac{\int y_1 e^{\int p(t)\,dt} g(t)\,dt + c}{y_1^2 e^{\int p(t)\,dt}}\,dt + d\right)$
- $y_p = u_1(t)y_1 + u_2(t)y_2$, $u_1(t) = -\int \frac{g(t)y_2(t)}{W(y_1,y_2)}\,dt$, $u_2(t) = \int \frac{g(t)y_1(t)}{W(y_1,y_2)}\,dt$, $W(y_1,y_2) = y_1 y_2' - y_1' y_2$.
- $A\cos(\omega t) + B\sin(\omega t) = R\cos(\omega t - \delta)$, $A = R\cos\delta$, $B = R\sin\delta$, $R = \sqrt{A^2+B^2}$, $\delta = \text{atan2}(B,A)$, $\text{atan2}(B,A)$ is $\arctan(B/A)$ if $A > 0$ and is $\arctan(B/A) + \pi$ if $A < 0$.
- $my'' + \gamma y' + ky = F(t)$, $mg = kL$
- $\gamma^2 - 4mk$ vs. 0, $\omega_0 = \sqrt{\frac{k}{m}}$, $\lambda = -\frac{\gamma}{2m}$, $\mu = \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}} = \omega_0\sqrt{1 - \frac{\gamma^2}{4km}}$.

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
$af(t) + bg(t)$	$aF(s) + bG(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$e^{at}f(t)$	$F(s-a)$
t^n	$\frac{n!}{s^{n+1}}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{n-1}(0)$
$\delta(t)$	1
$\delta(t-c)$	e^{-cs}
$u_c(t)$	$\frac{e^{-cs}}{s}$
$u_c(t)f(t)$	$e^{-cs}\mathcal{L}\{f(t+c)\}(s)$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$