Your Name

Student ID #

- Do not open this exam until you are told to begin. You will have 60 minutes for the exam.
- Check that you have a complete exam. There are 5 questions for a total of 55 points.
- You are allowed to have one handwritten note sheet. An **equation sheet** is provided on the last page. No calculators are allowed.
- Cheating will result in a zero and be reported to the Dean's Academic Conduct Committee.
- **Show all your work**. Unless explicitly stated otherwise in a particular question, if there is no work supporting your answer, you will not receive credit for the problem. If you need more space to answer a question, continue on the back of the page, and indicate that you have done so.

Question	Points	Score
1	10	
2	11	
3	10	
4	11	
5	13	
Total:	55	

1. Consider the initial value problem

$$y'' - 2y' + 2y = 0, \qquad y(0) = 2, y'(0) = 2.$$

(a) (4 points) Give the general solution of this differential equation.

(b) (4 points) Solve the given initial value problem.

(c) (2 points) Roughly describe how your solution to (b) would change if $2y$ above were replaced by y.

2. A pond is fed by a small, polluted stream, and is drained for irrigation. The pond starts at 999 million gallons of pure water and is drained for irrigation at a constant rate of 1 million gallons per hour. The polluted stream has 2 metric ton of nitric acid per million gallons of water. The stream initially flows into the pond at a rate of 2 million gallons per hour, though it dries up over time so that after t hours it flows at a rate of $2/(1+t)^2$ million gallons per hour.

   (a) (3 points) When will the pond be emptied? (You may leave your answer as the solution of an explicit quadratic equation $at^2 + bt + c = 0$.)

   (b) (4 points) Suppose $V(t)$ is the volume of the pond after t hours. Write down a differential equation for the amount (in metric tons) of nitric acid in the pond after t hours in terms of $V(t)$.

   (c) (4 points) Solve the differential equation from (b). Note: you will be unable to evaluate all the integrals you encounter, so you may leave unevaluated integrals in your answer.

3. This problem concerns the differential equation

$$y' = (y^3 - y)2^{-y}.$$

(a) (1 point) Is the equation autonomous? Why or why not?

(b) (6 points) Provide the following:

(i) the phase line;

(ii) an *approximate* slope field for $-2 \le y \le 4$ and $0 \le t \le 6$;

(iii) all equilibrium solutions;

(iv) some sample solution curves;

(v) label stable, unstable, and semistable equilibria.

(c) (3 points) If $y(0) = 3$, estimate $y(2)$ using Euler's method with $h = 1$. (You do not need to simplify your expressions.)

Step k	t_k	y_k	$f(t_k, y_k)$

4. Consider the differential equation

$$t(1+t)v' + (2+t)v = 0.$$

   (a) (3 points) Explain why there is a unique solution on $I = (0, \infty)$ for each initial condition $v(1) = v_0$.

   (b) (6 points) Show that $v(t) = v_0 \frac{1+t}{2t^2}$ is the unique solution from (a).

   (c) (2 points) Find all equilibrium solutions. (Note: this question makes sense despite the equation being non-autonomous.)

5. (a) (3 points) Explain why $y_1(t) = t$ is always a solution to the differential equation

$$m(t)y'' - ty' + y = 0,$$

where $m(t)$ is any given function.

(b) (7 points) Use reduction of order to find a second solution (i.e. besides $y_1(t) = t$) to

$$t(1+t)y'' - ty' + y = 0, \qquad t > 0.$$

(Hint: you may use the result of Problem 4 even if you did not solve it.)

(c) (3 points) Is the Wronskian of t and $t \ln t - 1$ ever zero? Why or why not? (Here $t > 0$.)

Equation Sheet

- $y' = f(t, y)$
- $y' = f(t)g(y)$
- $\int \frac{dy}{g(y)} = \int f(t)\,dt$
- $y' + p(t)y = g(t)$, $y(t_0) = y_0$
- $\mu(t) = e^{\int p(t)\,dt}$
- $y = \frac{1}{\mu(t)}(\int \mu(t)g(t)\,dt + c)$
- $y' = ry + k$, $T' = k(T - T_S)$, $v' = \pm g - \frac{m}{k}v$
- $t_{n+1} = t_n + h$, $y_{n+1} = y_n + f(t_n, y_n)h$
- $\phi_0(t) = 0$, $\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))\,ds$
- $y'' = f(t, y, y')$
- $y'' + p(t)y' + q(t) = g(t)$, $y(t_0) = y_0$, $y'(t_0) = y'_0$
- $ay'' + by' + cy = 0$
- $ar^2 + br + c = 0$
- $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, $r = \lambda \pm i\mu$
- $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = y_1y'_2 - y'_1y_2$
- $y = c_1e^{r_1t} + c_2e^{r_2t}$, $y = c_1e^{rt} + c_2te^{rt}$, $y = c_1e^{\lambda t}\sin(\mu t) + c_2e^{\lambda t}\cos(\mu t)$
- $y_2(t) = u(t)y_1(t)$, $v = u'$