Your Name

Student ID #

- Do not open this exam until you are told to begin. You will have 60 minutes for the exam.
- Check that you have a complete exam. There are 5 questions for a total of 55 points.
- You are allowed to have one handwritten note sheet. An **equation sheet** is provided on the last page. No calculators are allowed.
- Cheating will result in a zero and be reported to the Dean's Academic Conduct Committee.
- **Show all your work**. Unless explicitly stated otherwise in a particular question, if there is no work supporting your answer, you will not receive credit for the problem. If you need more space to answer a question, continue on the back of the page, and indicate that you have done so.

Question	Points	Score
1	10	
2	11	
3	10	
4	11	
5	13	
Total:	55	

1. Consider the initial value problem

$$y'' - 2y' + 2y = 0, \qquad y(0) = 2, y'(0) = 2.$$

(a) (4 points) Give the general solution of this differential equation.

Solution: The characteristic equation is $r^2 - 2r + 2 = 0$ which has roots at $r = 1 \pm i$. Hence the general solution is

$$y = c_1 e^{(1+i)t} + c_2 e^{(1-i)t}$$

or

$$y = d_1 e^t \cos t + d_2 e^t \sin t.$$

(b) (4 points) Solve the given initial value problem.

Solution: Using the first form above gives the system of equations

$$\begin{aligned} y(0) &= 2 = c_1 + c_2 \\ y'(0) &= 2 = (1+i)c_1 + (1-i)c_2 = (c_1 + c_2) + i(c_1 - c_2). \end{aligned}$$

Hence $i(c_1 - c_2) = 0$, so $c_1 = c_2 = 1$. Using the second form gives

$$\begin{aligned} y(0) &= 2 = d_1 \\ y'(0) &= 2 = d_1 + 0 + d_2, \end{aligned}$$

so that $d_1 = 2, d_2 = 0$. Hence

$$\begin{aligned} y &= e^{(1+i)t} + e^{(1-i)t} \\ y &= 2e^t \cos t. \end{aligned}$$

(c) (2 points) Roughly describe how your solution to (b) would change if $2y$ above were replaced by y.

Solution: The resulting characteristic equation is $r^2 - 2r + 1 = (r-1)^2$, which has a repeated root, so the new fundamental solution set is $\{e^t, te^t\}$. The rest is essentially the same—solve a system of two equations.

2. A pond is fed by a small, polluted stream, and is drained for irrigation. The pond starts at 999 million gallons of pure water and is drained for irrigation at a constant rate of 1 million gallons per hour. The polluted stream has 2 metric ton of nitric acid per million gallons of water. The stream initially flows into the pond at a rate of 2 million gallons per hour, though it dries up over time so that after t hours it flows at a rate of $2/(1+t)^2$ million gallons per hour.

   (a) (3 points) When will the pond be emptied? (You may leave your answer as the solution of an explicit quadratic equation $at^2 + bt + c = 0$.)

   Solution: Let $V(t)$ be the volume of the pond in millions of gallons, so $V(0) = 999$. Water flows in at a rate of $2/(1+t)^2$ and out at a rate of 1, so

   $$V' = \frac{2}{(1+t)^2} - 1, \qquad V(0) = 999.$$

   Hence $V(t) = 1001 - \frac{2}{1+t} - t$. Setting this to 0 and multiplying by $1+t$ gives $-t^2 + 1000t + 999 = 0$. (The numeric answer is slightly less than 1001, while the explicit answer is ugly.)

   (b) (4 points) Suppose $V(t)$ is the volume of the pond after t hours. Write down a differential equation for the amount (in metric tons) of nitric acid in the pond after t hours in terms of $V(t)$.

   Solution: Let $A(t)$ be the amount of nitric acid in the lake at time t, so $A(0) = 0$. The stream adds $2/(1+t)^2 \cdot 2$ metric tons per hour, while irrigation subtracts $1 \cdot A(t)/V(t)$ metric tons per hour. Hence

   $$A' = \frac{4}{(1+t)^2} - \frac{1}{V(t)}A = \frac{4}{(1+t)^2} - \frac{1+t}{999 + 1000t - t^2}A.$$

   (c) (4 points) Solve the differential equation from (b). Note: you will be unable to evaluate all the integrals you encounter, so you may leave unevaluated integrals in your answer.

   Solution: Write

   $$A' + p(t)A = g(t)$$

   for $p(t) = \frac{1+t}{999+1000t-t^2}$ and $g(t) = \frac{4}{(1+t)^2}$. Use $\mu(t) = e^{\int_0^t p(s)\,ds}$; to integrate p, you can factor the denominator (which has two real roots) and use partial fractions. Note that $\mu(0) = e^0 = 1$ here. Integrating factors gives the solution as

   $$A(t) = \frac{1}{\mu(t)}\left(\int_0^t \mu(s)g(s)\,ds + c\right).$$

   where $c = A(0) = 0$. As it turns out, integrating μg requires special functions. In perhaps painful detail,

   $$A(t) = e^{-\int_0^t \frac{1+s}{999+1000s-s^2}\,ds}\left(\int_0^t e^{\frac{1+s}{999+1000s-s^2}} \frac{4}{(1+s)^2}\,ds\right).$$

3. This problem concerns the differential equation

$$y' = (y^3 - y)2^{-y}.$$

(a) (1 point) Is the equation autonomous? Why or why not?

Solution: An autonomous equation by definition is of the form $y' = f(y)$, so $f(y) = (y^3 - y)2^{-y}$ here.

(b) (6 points) Provide the following:

(i) the phase line;

(ii) an *approximate* slope field for $-2 \le y \le 4$ and $0 \le t \le 6$;

(iii) all equilibrium solutions;

(iv) some sample solution curves;

(v) label stable, unstable, and semistable equilibria.

Solution: We have $(y^3 - y)e^{-y} = y(y-1)(y+1)2^{-y}$, so equilibrium solutions occur at $y = -1, 0, 1$. Note that $2^{-y} > 0$ always, so this term does not affect either stability or equilibrium solutions. Indeed, $f(y)$ is negative on $(-\infty, -1), (0, 1)$ and positive on $(-1, 0), (1, \infty)$. Hence $y = -1$ and $y = 1$ are unstable while $y = 0$ is stable.

(The phase line, slope field, and sample solutions are not included here.)

(c) (3 points) If $y(0) = 3$, estimate $y(2)$ using Euler's method with $h = 1$. (You do not need to simplify your expressions.)

Step k	t_k	y_k	$f(t_k, y_k)$

Solution: Here $f(t, y) = (y^3 - y)2^{-y}$.

Step k	t_k	y_k	$f(t_k, y_k)$
0	0	3	3
1	1	$3 + 3 = 6$	$(6^3 - 6)/2^6 = 105/32$
2	2	$6 + (6^3 - 6)/2^6 = 297/32$	

4. Consider the differential equation

$$t(1+t)v' + (2+t)v = 0.$$

(a) (3 points) Explain why there is a unique solution on $I = (0, \infty)$ for each initial condition $v(1) = v_0$.

Solution: Here

$$v' + \frac{2+t}{t(1+t)}v = 0,$$

so $p(t) = \frac{2+t}{t(1+t)}$ is continuous away from $0, -1$, so in particular on $I = (0, \infty)$. The result then follows from the first order linear existence and uniqueness theorem from lecture and the text.

(b) (6 points) Show that $v(t) = v_0 \frac{1+t}{2t^2}$ is the unique solution from (a).

Solution: There are three approaches to this.

- We may verify the suggested solution is indeed a solution directly. Since the equation is linear homogeneous, we can just check the $v_0 = 2$ case.

$$\begin{aligned} v &= \frac{1+t}{t^2} = t^{-2} + t^{-1} = \frac{1}{t^2}(1+t) \\ v' &= -2t^{-3} - t^{-2} = -\frac{1}{t^3}(2+t) \\ t(1+t)v' + (2+t)v &= -(1+t)(2+t)t^{-2} + (2+t)(1+t)t^{-2} = 0. \end{aligned}$$

- We may recognize the equation as separable:

$$\begin{aligned} \frac{v'}{v} &= -\frac{2+t}{t(1+t)} \\ \ln|v| &= -\int \frac{2+t}{t(1+t)}\, dt \\ &= -\int \left(\frac{2}{t} - \frac{1}{1+t}\right) dt \\ &= -2\ln|t| + \ln|1+t| + c \end{aligned}$$

  so

$$v = t^{-2}(1+t)D.$$

- The equation is also susceptible to integrating factors. The computation is very similar to the separable method and is not included.

(c) (2 points) Find all equilibrium solutions. (Note: this question makes sense despite the equation being non-autonomous.)

Solution: Suppose $v(t) = v_0$, so $v' = 0$, and the differential equation reads $0 + (2+t)v_0 = 0$ for all t. Hence $v_0 = 0$ is the only equilibrium solution.

5. (a) (3 points) Explain why $y_1(t) = t$ is always a solution to the differential equation

$$m(t)y'' - ty' + y = 0,$$

where $m(t)$ is any given function.

Solution: We have $y_1' = 1$ and $y_1'' = 0$, so the differential equation reads

$$m(t) \cdot 0 - t \cdot 1 + t = 0.$$

(b) (7 points) Use reduction of order to find a second solution (i.e. besides $y_1(t) = t$) to

$$t(1+t)y'' - ty' + y = 0, \qquad t > 0.$$

(Hint: you may use the result of Problem 4 even if you did not solve it.)

Solution: Let $y_2 = uy_1 = ut$. Hence $y_2' = u't + u$ and $y_2'' = u''t + 2u'$. Plugging these into the differential equation and simplifying gives

$$u''t^2(1+t) + u'(2t + t^2) = 0.$$

Canceling t and letting $v = u'$ so $v' = u''$ gives exactly the differential equation from Problem 4. From that problem, $v(t) = v_0 \frac{1+t}{2t^2}$. Breaking this into pieces and integrating gives $u(t) = v_0 \frac{1}{t} + \frac{v_0}{2} \ln t + c$. Multiplying this by t to get y_2, we may choose $c = 0$ and $v_0 = 2$ to arrive at $y_2 = t \ln t - 1$. The general solution is

$$y_2 = c_1 t + c_2(t \ln t - 1).$$

(c) (3 points) Is the Wronskian of t and $t \ln t - 1$ ever zero? Why or why not? (Here $t > 0$.)

Solution: No, it is not. There are several ways to see this. By Abel's identity, the Wronskian in $t > 0$ is either always or never zero, so we can test it at $t = 1$. Then $y_1(1) = 1, y_1'(1) = 1$, $y_2(1) = -1$, $y_2'(1) = 1$, and $1 \cdot 1 - 1 \cdot -1 = 2 \neq 0$. Computing the Wronskian for general t is no more difficult and gives $W(t, t\ln t - 1) = t(\ln t + 1) - (t \ln t - 1) = t + 1$, which is non-zero for $t > 0$.

An abstract method is to show that two solutions to a homogeneous second order linear equation have zero Wronskian only if they are scalar multiples of each other. Here it is clear that $y_1 = t$ and $y_2 = t \ln t - 1$ are not scalar multiples. So, suppose w_1, w_2 are some fundamental solution set on $t > 0$ (which we know exists), so that $y_1 = c_1 w_1 + c_2 w_2$ and $y_2 = d_1 w_1 + d_2 w_2$. Now compute

$$\begin{aligned}
W(y_1, y_2) &= W(c_1 w_1 + c_2 w_2, d_1 w_1 + d_2 w_2) \\
&= \ldots = c_1 d_1 W(w_1, w_1) + c_1 d_2 W(w_1, w_2) + c_2 d_1 W(w_2, w_1) + c_2 d_2 W(w_2, w_2) \\
&= 0 + c_1 d_2 W(w_1, w_2) - d_1 c_2 W(w_1, w_2) + 0 \\
&= (c_1 d_2 - d_1 c_2) W(w_1, w_2).
\end{aligned}$$

By assumption, $W(w_1, w_2)$ is never zero, so we must only consider whether $c_1 d_2 - d_1 c_2$ can be zero. (This is itself a two-by-two determinant.) So, suppose $c_1 d_2 = d_1 c_2$. If

$c_2 = 0$, then either $c_1 = 0$ or $d_2 = 0$; if $c_1 = 0$, then $y_1 = 0w_1 + 0w_2 = 0 = t$, which is clearly false, while if $d_2 = 0$ then $y_1 = c_1 w_1$ and $y_2 = d_1 w_1$ are scalar multiples, which is again false. So, $c_1 \neq 0$. By interchanging the symbols and repeating this argument, we also have $d_1 \neq 0$. Hence $c_2/c_1 = d_2/d_1$, and again y_1 and y_2 are scalar multiples. So indeed their Wronskian can never be zero.

Equation Sheet

- $y' = f(t, y)$
- $y' = f(t)g(y)$
- $\int \frac{dy}{g(y)} = \int f(t)\, dt$
- $y' + p(t)y = g(t)$, $y(t_0) = y_0$
- $\mu(t) = e^{\int p(t)\, dt}$
- $y = \frac{1}{\mu(t)}(\int \mu(t)g(t)\, dt + c)$
- $y' = ry + k$, $T' = k(T - T_S)$, $v' = \pm g - \frac{m}{k}v$
- $t_{n+1} = t_n + h$, $y_{n+1} = y_n + f(t_n, y_n)h$
- $\phi_0(t) = 0$, $\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))\, ds$
- $y'' = f(t, y, y')$
- $y'' + p(t)y' + q(t) = g(t)$, $y(t_0) = y_0$, $y'(t_0) = y_0'$
- $ay'' + by' + cy = 0$
- $ar^2 + br + c = 0$
- $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, $r = \lambda \pm i\mu$
- $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$
- $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, $y = c_1 e^{rt} + c_2 t e^{rt}$, $y = c_1 e^{\lambda t} \sin(\mu t) + c_2 e^{\lambda t} \cos(\mu t)$
- $y_2(t) = u(t)y_1(t)$, $v = u'$