Your Preferred Name

Student ID #

- Do not open this exam until you are told to begin. You will have 50 minutes for the exam.
- Check that you have a complete exam. There are 5 questions for a total of 44 points.
- You are allowed to have one handwritten note sheet. An **equation sheet** is provided on the last page. No calculators are allowed.
- Cheating will result in a zero and be reported to the Dean's Academic Conduct Committee.
- **Show all your work**. Unless explicitly stated otherwise in a particular question, if there is no work supporting your answer, you will not receive credit for the problem. If you need more space to answer a question, continue on the back of the page, and indicate that you have done so.

Question	Points	Score
1	9	
2	9	
3	9	
4	7	
5	10	
Total:	44	

1. Consider the differential equation

$$(2-t)y = \frac{e^t}{t(t^2+1)} - ty'.$$

(a) (2 points) Check all that apply. The differential equation is...

○ Autonomous √ **Linear** ○ Separable
√ **First order** ○ Second order ○ Constant-coefficient

(b) (5 points) Find the general solution of the differential equation. You may assume $t > 0$.

Solution: The equation in standard form is

$$y' + \frac{2-t}{t}y = \frac{e^t}{t^2(t^2+1)}$$

so the integrating factor is

$$\mu(t) = \exp\left(\int \frac{2-t}{t}\,dt\right) = \exp(2\ln t - t) = t^2e^{-t}.$$

The general solution is then

$$\begin{aligned} y(t) &= \frac{1}{\mu(t)}\left(\int \frac{1}{t^2+1}\,dt + c\right) \\ &= \frac{e^t}{t^2}\left(\arctan(t) + c\right). \end{aligned}$$

(c) (2 points) Find the particular solution with $y(1) = 0$. (If you did not solve (b), you may use a reasonable guess instead.)

Solution: We have

$$y(1) = 0 = e(\arctan(1) + c),$$

so $c = -\arctan(1) = -\pi/4$, giving

$$y(t) = \frac{e^t}{t^2}(\arctan(t) - \pi/4).$$

2. (a) (4 points) Find an explicit solution of the differential equation

$$y' = \frac{1-y^2}{2y}.$$

Solution: The equation is separable, giving

$$\int \frac{2y}{1-y^2}\,dy = \int dt.$$

We can use u-substitution with $u = 1 - y^2$ for the integral. We can also use partial fractions. The numerator has lower degree than the denominator, so polynomial division is not necessary. Since $1 - y^2 = (1-y)(1+y)$,

$$\frac{2y}{(1-y)(1+y)} = \frac{A}{1-y} + \frac{B}{1+y}.$$

We find $A = 1$, $B = -1$. Hence we have an implicit solution

$$-\ln|1-y| - \ln|1+y| = t + c.$$

Solving for y, we have

$$\ln|(1-y)(1+y)| = -t - c$$

so that

$$|1-y^2| = De^{-t}$$

and hence

$$y = \pm\sqrt{1 - De^{-t}}.$$

(b) (5 points) **Classify** the equilibrium solutions of the differential equation

$$y' = (e^{1-y} - e^2)y$$

as stable, semistable, or unstable. Draw the **phase line** and some **sample solutions**. If $y(0) = -1/2$, **what is** $\lim_{t\to\infty} y(t)$?

Solution: We have $e^{1-y} = e^2$ when $y = -1$, so the equilibrium solutions are $y = 0$ and $y = -1$. We further find

$$y' > 0 \qquad \text{when} \qquad -1 < y < 0$$

and

$$y' < 0 \qquad \text{when} \qquad y < -1, \qquad y > 0.$$

$y(t) = 0$ is then a stable equilibrium while $y(t) = -1$ is unstable. If $y(0) = -1/2$, we see $\lim_{t\to\infty} y(t) = 0$. (The phase line and sample solutions have been omitted.)

3. (a) (3 points) Given

$$y' = ty + y^2, \qquad \text{and} \qquad y(0) = 1,$$

estimate $y(3)$ using Euler's method with $h = 1$.

Step k	t_k	y_k	$f(t_k, y_k)$

Solution: Here $f(t, y) = ty + y^2$ and $y_{k+1} = y_k + f(t_k, y_k)$.

Step k	t_k	y_k	$f(t_k, y_k)$
0	0	1	1
1	1	$1 + 1 = 2$	$2 + 4 = 6$
2	2	$2 + 6 = 8$	$16 + 64 = 80$
3	3	$8 + 80 = 88$	

Hence $y(3) \approx 88$.

(b) (3 points) Determine the largest interval on which the IVP

$$\cos(t/2)y' - \sin(t/2)y = \ln|t + 1|, \qquad y(0) = 5$$

is guaranteed to exist. *Do not attempt to solve* the differential equation.

Solution: In standard form, the equation is

$$y' - \frac{\sin(t/2)}{\cos(t/2)} y = \frac{\ln|t + 1|}{\cos(t/2)}.$$

The coefficient on y is continuous except when $\cos(t/2) = 0$; the closest points of discontinuity to 0 are $t = \pm\pi$. The right-hand side is continuous except when $t = -1$ or $\cos(t/2) = 0$; the closest points of discontinuity to 0 are $t = -1, +\pi$. Hence the interval is $-1 < t < \pi$.

(c) (3 points) Compute the second Picard iterate $\phi_2(t)$ for the IVP

$$y' = y^3 + 14t, \qquad y(0) = 0.$$

Solution: We have $\phi_0(t) = 0$, and by definition $\phi_{k+1}(t) = \int_0^t f(s, \phi_k(s))\, ds$ where $f(t, y) = y^3 + 14t$. We compute

$$\phi_0(t) = 0$$
$$\phi_1(t) = \int_0^t 14s\, ds = 7t^2$$
$$\phi_2(t) = \int_0^t (7^3 s^6 + 14s)\, ds = 49t^7 + 7t^2.$$

4. You are enjoying a cool 32 ounce drink on a warm spring day. You drink through a straw at a constant rate of 1 ounce per minute. Sadly, liquid is dripping through a hole in the bottom of the cup, losing liquid at a rate of 1/10th of an ounce per minute for each ounce of liquid remaining.

   (a) (3 points) **Write down** (but do not solve) an initial value problem for the volume $V(t)$ of liquid left after t minutes.

   Solution: Drinking through the straw contributes -1 to V'. The hole contributes $-V(t)/10$ to V', so

   $$V' = -1 - \frac{V}{10}, \qquad V(0) = 32.$$

   (b) (4 points) The outside temperature is 70° F, and the drink is initially 40° F. The drink warms as a rate of 0.5° F per minute per degree of temperature difference. **Write down** (but do not solve) an initial value problem for the temperature $T(t)$ of the liquid after t minutes. **What is** $\lim_{t\to\infty} T(t)$?

   Solution: From Newton's law of cooling, we have

   $$T' = -0.5(T - 70), \qquad T(0) = 40.$$

   From physical intuition, $\lim_{t\to\infty} T(t) = 70$, which is also a stable equilibrium solution.

5. (a) (3 points) Compute the **Wronskian** of $\{y_1, y_2\} = \{\cos e^t, \sin e^t\}$. In fact, the differential equation

$$y'' = y' - e^{2t} y$$

has $\{y_1, y_2\}$ as a **fundamental solution set**—what does this mean for applications?

Solution: The Wronskian is

$$\begin{aligned} W(y_1, y_2)(t) &= y_1 y_2' - y_1' y_2 = \cos e^t \cdot (e^t \cos e^t) - (-e^t \sin e^t) \cdot \sin e^t \\ &= e^t(\cos^2 e^t + \sin^2 e^t) = e^t. \end{aligned}$$

Practically speaking, any IVP with the given DE and initial conditions $y(t_0) = y_0$, $y'(t_0) = y_0'$ has a unique solution of the form $y(t) = c_1 y_1 + c_2 y_2$.

(b) (3 points) Find the general solution of the differential equation

$$y'' + 10y' + 25y = 0.$$

Solution: The characteristic equation is $r^2 + 10r + 25 = (r + 5)^2$. A fundamental solution set is then $\{e^{-5t}, te^{-5t}\}$, so the general solution is

$$y = c_1 e^{-5t} + c_2 t e^{-5t}.$$

(c) (4 points) Find two different **fundamental solution sets** of the differential equation

$$y'' + 4y' + 13y = 0.$$

What can you say about the **limiting behavior** $\lim_{t\to\infty} y$?

Solution: The characteristic equation is $r^2 + 4r + 13 = 0$, which has roots $r = -2 \pm 3i$. Hence two fundamental solution sets are

$$\{e^{(-2+3i)t}, e^{(-2-3i)t}\}, \qquad \{e^{-2t}\cos(3t), e^{-2t}\sin(3t)\}.$$

Since $e^{-2t} \to 0$ as $t \to \infty$, any solution tends to 0 as well, regardless of initial conditions.

Equation Sheet

- $y' = f(t, y)$
- $y' = f(t)g(y)$
- $\int \frac{dy}{g(y)} = \int f(t)\, dt$
- $y' + p(t)y = g(t)$, $y(t_0) = y_0$
- $\mu(t) = e^{\int p(t)\, dt}$
- $y = \frac{1}{\mu(t)}(\int \mu(t)g(t)\, dt + c)$
- $y' = ry \pm k$, $T' = -k(T - T_S)$, $mv' = \pm mg - kv$
- $t_{n+1} = t_n + h$, $y_{n+1} = y_n + f(t_n, y_n)h$
- $\phi_0(t) = 0$, $\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))\, ds$
- $y'' = f(t, y, y')$
- $y'' + p(t)y' + q(t) = g(t)$, $y(t_0) = y_0$, $y'(t_0) = y_0'$
- $ay'' + by' + cy = 0$
- $ar^2 + br + c = 0$
- $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, $r = \lambda \pm i\omega$
- $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$
- $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, $y = c_1 e^{rt} + c_2 t e^{rt}$, $y = d_1 e^{\lambda t} \cos(\omega t) + d_2 e^{\lambda t} \sin(\omega t)$