Math 307 E - Summer 2011
Practice Midterm 2
August 17, 2011

Name: ______________________________ Student number: ____________________

1	10	
2	10	
3	10	
4	10	
5	10	
6	3*	
Total	50	

- Complete all questions.
- You may use a scientific calculator during this examination. Other electronic devices (e.g. cell phones) are not allowed, and should be turned off for the duration of the exam.
- You may use one hand-written 8.5 by 11 inch page of notes.
- Show all work for full credit.
- You have 60+ minutes to complete the exam.

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1. Find the general solution to the differential equations:

   (a) (5 points)
$$y'' - 2y' - 3y = te^t.$$
   The characteristic equation is $r^2 - 2r - 3 = (r-3)(r+1) = 0$, so $r = 3, -1$. Then the homogeneous solution is $y_h = c_1 e^{3t} + c_2 e^{-t}$.
   We guess a particular solution: $y_p = (a_0 + a_1 t)e^t$. Then $y_p' = [(a_0 + a_1) + a_1 t]e^t$ and $y_p'' = [(a_0 + 2a_1) + a_1 t]e^t$. Plugging this into the left-hand side, we get
$$e^t\left[(a_0 + 2a_1 - 2(a_0 + a_1) - 3a_0) + (a_1 - 2a_1 - 3a_1)t\right] = te^t.$$
   So that we have two equations: $-4a_1 = 1$ and $-4a_0 = 0$. So $y_p = \frac{-t}{4}e^t$, and
$$y(t) = y_h + y_p = c_1 e^{3t} + c_2 e^{-t} - \frac{t}{4}e^t.$$

   (b) (5 points)
$$y'' - 2y' - 3y = g(t)$$
   Hint: Express your answer using integrals.

   Using the homogeneous solution above, we have $y_1(t) = e^{3t}$ and $y_2(t) = e^{-t}$. The variation of parameters formula tells us that $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ where $u_1(t) = -\int_0^t y_2(s)\frac{g(s)}{W(y_1,y_2)(s)}ds$, and $u_2(t) = \int_0^t y_1(s)\frac{g(s)}{W(y_1,y_2)(s)}ds$. We compute the wronskian:
$$W(e^{3t}, e^{-t})(t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 3e^{3t} & -e^{-t} \end{vmatrix} = -4e^{2t}.$$
   Then
$$u_1(t) = -\int_0^t e^{-s}\frac{g(s)}{-4e^{2s}}ds = \frac{1}{4}\int_0^t e^{-3s}g(s)ds,$$
$$u_2(t) = \int_0^t e^{3s}\frac{g(s)}{-4e^{2s}}ds = -\frac{1}{4}\int_0^t e^{s}g(s)ds,$$
   and so $y(t) = y_h(t) + y_p(t)$ gives
$$y(t) = c_1 e^{3t} + c_2 e^{-t} + \frac{e^{3t}}{4}\int_0^t e^{-3s}g(s)ds - \frac{e^{-t}}{4}\int_0^t e^{s}g(s)ds$$

2. (10 points) Suppose that the motion of a spring-mass system satisfies

$$u'' + u' + 1.5u = \sin(2t)$$

and that the mass starts ($t = 0$) at the equilibrium position from rest. Find the the position $u(t)$ at any time t.

We first solve the homogeneous equation. We get $r^2 + r + 1.5 = 0$, so $r = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}i$. Then $e^{rt} = e^{-t/2}(\cos(t\sqrt{5}/2) + i\sin(t\sqrt{5}/2))$; taking real and imaginary parts we get

$$u_h(t) = c_1 e^{-t/2}\cos(t\sqrt{5}/2) + c_2 e^{-t/2}\sin(t\sqrt{5}/2).$$

Then we need to find a particular solution. We might as well use undetermined coefficients. So we replace $\sin(2t)$ with e^{2ti}, since $Im(e^{2ti}) = \sin(2t)$. Then we guess that $v_p(t) = Ae^{2ti}$. Plugging into the equation, we get:

$$e^{2ti}(-4 + 2i + 1.5)A = e^{2ti}.$$

This tells us that $A = \frac{1}{-2.5+2i} = \frac{-2.5-2i}{41/4} = -\frac{10}{41} - \frac{8}{41}i$. Then

$$\begin{aligned} v_p(t) &= -\left(\frac{10}{41} + \frac{8}{41}i\right)e^{2it} = -\left(\frac{10}{41} + \frac{8}{41}i\right)(\cos(2t) + i\sin(2t)) \\ &= \left(\frac{10}{41}\cos(2t) - \frac{8}{41}\sin(2t)\right) + i\left(\frac{-8}{41}\cos(2t) + \frac{-10}{41}\sin(2t)\right). \end{aligned}$$

Thus $u_p(t) = Im(v_p(t)) = -\left(\frac{8}{41}\cos(2t) + \frac{10}{41}\sin(2t)\right)$ is a particular solution to the original problem. We have then that

$$u(t) = u_h(t) + u_p(t) = c_1 e^{-t/2}\cos(t\sqrt{5}/2) + c_2 e^{-t/2}\sin(t\sqrt{5}/2) + -\left(\frac{8}{41}\cos(2t) + \frac{10}{41}\sin(2t)\right).$$

Since the initial conditions are $u(0) = 0 = u'(0)$, we get that

$$\begin{aligned} 0 &= u(0) = c_1 - 8/41 \\ 0 &= u'(0) = -\frac{c_1}{2} + c_2\frac{\sqrt{5}}{2} + \frac{20}{41}. \end{aligned}$$

Then $c_1 = 8/41$, and so $c_2 = -\frac{12}{41} \cdot \frac{2}{\sqrt{5}} = \frac{24\sqrt{5}}{205}$. Finally,

$$u(t) = \frac{8}{41}e^{-t/2}\cos(t\sqrt{5}/2) + \frac{24\sqrt{5}}{205}e^{-t/2}\sin(t\sqrt{5}/2) + -\left(\frac{8}{41}\cos(2t) + \frac{10}{41}\sin(2t)\right)$$

3. Compute the following Laplace transforms using the definition, or using only the numbers (1),(13),(14),(18), and (19) on the table.

   (a) (5 points)

$$\mathcal{L}\left\{t^2 e^{\pi t}\right\}$$

   We know from (1) that $\mathcal{L}\{1\} = \frac{1}{s}$. Then from (19), we know that $\mathcal{L}\{t^2 \cdot 1\} = \left(\frac{1}{s}\right)'' = \frac{2}{s^3}$. Finally, from (14), we know that $\mathcal{L}\{e^{\pi t} t^2\} = \frac{2}{(s-\pi)^3}$.

   (b) (5 points)

$$\mathcal{L}\left\{u_3(t)(t^2 - 2t - 1)\right\}$$

   If $g(t) = t^2 - 2t - 1$, then $f(t) := g(t+3) = (t+3)^2 - 2(t+3) - 1 = t^2 + 6t + 9 - 2t - 6 - 1 = t^2 + 4t + 2$. Observe that $f(t-3) = g(t)$, so that $\mathcal{L}\{u_3(t)g(t)\} = \mathcal{L}\{u_3(t)f(t-3)\}$ which is $e^{-3s}\mathcal{L}\{f(t)\}$ by (13).
   We use linearity to compute $\mathcal{L}\{f(t)\}$: since we computed $\mathcal{L}\{t\} = \frac{1}{s}$ and $\mathcal{L}\{t^2\} = \frac{2}{s^3}$ above, we only need to know $\mathcal{L}\{t\}$. For this we can use (18): $\mathcal{L}\{2t\} = \mathcal{L}\{(t^2)'\} = s\mathcal{L}\{t^2\} = \frac{2}{s^2}$. Dividing both sides by 2 (and using linearity), we get that $\mathcal{L}\{t\} = \frac{1}{s^2}$.
   Now by linearity: $\mathcal{L}\{f(t)\} = \frac{2}{s^3} + \frac{4}{s^2} + \frac{2}{s}$, so that

$$\mathcal{L}\{u_3(t)g(t)\} = e^{-3s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{2}{s}\right) = \frac{e^{-3s}}{s^3}\left(2s^2 + 4s + 2\right).$$

4. (10 points) Use the Laplace transform to solve the following IVP using the table:

$$y'' - y = \begin{cases} 1 & t < 2 \\ t/3 & 2 \le t \end{cases} \qquad \begin{cases} y(0) = 0 \\ y'(0) = 0. \end{cases}$$

We first re-write the driving function $g(t)$ (right-hand side) using the unit step functions. We get that $g(t) = 1 + u_2(t)(\frac{t}{3} - 1)$. Then we take the Laplace transform of both sides:

$$\mathcal{L}\{y''\} - \mathcal{L}\{y\} = \frac{1}{s} + \mathcal{L}\{u_2(t)(t/3 - 1)\}.$$

Use (13) to evaluate the last Laplace transform: first, observe that $t/3 - 1 = \frac{t-2}{3} - \frac{1}{3}$, so if $f(t) = t/3 - \frac{1}{3}$, we have that

$$\mathcal{L}\{u_2(t)(t/3 - 1)\} = \mathcal{L}\{u_2(t)f(t-2)\} = e^{-2s}\left(\frac{1}{3s^2} - \frac{1}{3s}\right).$$

For the left-hand side, we use the usual formulas (18 on the table):

$$(s^2 - 1)\mathcal{L}\{y\} = \frac{1}{s} + \frac{e^{-2s}}{3s^2} - \frac{e^{-2s}}{3s}.$$

Solve: $\mathcal{L}\{y\} = \frac{3 - e^{-2s}}{3s(s+1)(s-1)} + \frac{e^{-2s}}{3s^2(s+1)(s-1)}$.

Partial fractions expansion gives $\frac{1}{s(s+1)(s-1)} = \frac{-1}{s} + \frac{1/2}{s+1} + \frac{1/2}{s-1}$. Thus:

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)(s-1)}\right\} = -1 + \frac{1}{2}e^{-t} + \frac{1}{2}e^{t}.$$

Partial fractions expansion gives $F(s) := \frac{1}{s^2(s+1)(s-1)} = \frac{A}{s} + \frac{-1}{s^2} + \frac{-1/2}{s+1} + \frac{1/2}{s-1}$ if we just use the cover-up method. To solve for A, just pick a different value for s (other than $s = -1, 0, 1$); we pick $s = 2$, and get that $A = 0$. So it is easy now to compute

$$\mathcal{L}^{-1}\{F(s)\} = -t - \frac{1}{2}e^{-t} + \frac{1}{2}e^{t} =: f(t).$$

It follows that $\mathcal{L}^{-1}\left\{e^{-2s}\frac{F(s)}{3}\right\} = \frac{u_2(t)}{3}f(t-2)$ by (13); then finally,

$$\begin{aligned} y(t) &= -1 + \frac{1}{2}e^{-t} + \frac{1}{2}e^{t} - \frac{u_2(t)}{3}\left(-1 + \frac{1}{2}e^{-(t-2)} + \frac{1}{2}e^{t-2}\right) \\ &\quad + \frac{u_2(t)}{3}\left(-(t-2) - \frac{1}{2}e^{-t+2} + \frac{1}{2}e^{t-2}\right) \\ &= -1 + \cosh(t) + \frac{u_2(t)}{3}\left(3 - t - \cosh(t-2) + \sinh(t-2)\right). \end{aligned}$$

5. (10 points) A spring-mass system has a spring constant of 2N/m. A mass of 8kg is attached to the spring. Let γ be the damping constant of the system.

(a) (2 points) What is the *natural frequency* of the system?

$$w_0 = \sqrt{k/m} = \sqrt{2/8} = 1/2.$$

(b) (2 points) Suppose $\gamma = 9$. Is the (free) system under-damped, over-damped or critically damped?

The discriminant $\Delta = \gamma^2 - 4mk = 9^2 - 4(8)(2) > 0$. Thus the system is over-damped.

(c) (2 points) From now on, suppose $\gamma = 2$. Find the quasi-frequency of the (free) system.

$$\mu = w_0\sqrt{1 - \frac{\gamma^2}{4km}} = \frac{1}{2}\sqrt{1 - \frac{4}{4(8)(2)}} = \frac{1}{2}\sqrt{15/16}.$$

(d) (2 points) Suppose we apply an external force $F(t) = 5\cos(wt)$ N. What is the resonant frequency of this forced system?

$$w_{res} = w_0\sqrt{1 - \frac{\gamma^2}{2mk}} = \frac{1}{2}\sqrt{1 - \frac{4}{2(8)(2)}} = \frac{1}{2}\sqrt{7/8}.$$

(e) (2 points) Write down the initial value problem corresponding to this forced system where w is the resonant frequency, and the mass starts at rest from the equilibrium position.

$$8u'' + 2u' + 2u = 5\cos\left(t\frac{\sqrt{7/8}}{2}\right) \qquad \begin{cases} u(0) = 0 \\ u'(0) = 0. \end{cases}$$

6. (3 bonus points) Compute the laplace transform of $\ln(t)$ by following these steps.

(a) (1 point) Differentiate the formula

$$\mathcal{L}(t^p) = \int_0^\infty e^{-st} t^p dt = \frac{\Gamma(p+1)}{s^{p+1}}$$

with respect to p. For the the middle term, move the differential operator $\frac{d}{dp}$ inside the integral and apply it to the integrand.

$$\begin{aligned} \frac{\Gamma'(p+1)s^{p+1} - \Gamma(p+1)s^{p+1}\ln(s)}{s^{2p+2}} = \frac{d}{dp}\left(\frac{\Gamma(p+1)}{s^{p+1}}\right) &= \frac{d}{dp}\int_0^\infty e^{-st} t^p dt \\ &= \int_0^\infty e^{-st} \frac{d}{dp}(t^p)\, dt \\ &= \int_0^\infty e^{-st} t^p \ln(t) dt. \end{aligned}$$

(b) (1 point) Simplify as much as possible, and then evaluate the resulting expression at $p = 0$.

If we simplify the left-hand side, we get

$$\frac{\Gamma'(p+1) - \Gamma(p+1)\ln(s)}{s^{p+1}} = \int_0^\infty e^{-st} t^p \ln(t) dt.$$

Evaluating at $p = 0$, we get

$$\frac{\Gamma'(1) - \ln(s)}{s} = \int_0^\infty e^{-st} \ln(t) dt.$$

(c) (1 point) What is $\mathcal{L}(\ln(t))$?

By definition, $\mathcal{L}\{\ln(t)\} = \int_0^\infty e^{-st}\ln(t)dt$, which is

$$\frac{\Gamma'(1) - \ln(s)}{s}$$

by the previous formula.

Table of Laplace transforms:

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}, \quad s > 0$
2. e^{at}	$\frac{1}{s-a}, \quad s > a$
3. $t^n, \quad n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
4. $t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5. $\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
6. $\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
7. $\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \|a\|$
8. $\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \|a\|$
9. $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
10. $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
11. $t^n e^{at}, \quad n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}$
12. $u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$
14. $e^{ct}f(t)$	$F(s-c)$
15. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), c > 0$
16. $\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
17. $\delta(t-c)$	e^{-cs}
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19. $(-t)^n f(t)$	$F^{(n)}(s)$