1. (10 points) Solve the following initial value problem explicitly. Your answer should be a function in the form $y = g(t)$, where there is no undetermined constant in g.

$$\frac{dy}{dt} = \frac{t}{y + t^2 y}, \quad y(0) = -3.$$

This equation is separable. After factoring the right hand side of the DE we have it in standard form:

$$\frac{dy}{dt} = \frac{1}{y} \cdot \frac{t}{1+t^2}$$

Putting all the y-stuff on the left and all the t-stuff on the right then yields

$$y\,dy = \frac{t}{1+t^2}\,dt$$

Both integrals are straightforward: on the left we get $\frac{1}{2}y^2$, and on the right (via u-sub if necessary) we get $\frac{1}{2}\ln(1+t^2) + C$; we may omit absolute value signs, since $t^2 + 1$ is always positive. Hence

$$\frac{1}{2}y^2 = \frac{1}{2}\ln(1+t^2) + C,$$

or, after solving for y,

$$y = \pm\sqrt{\ln(1+t^2) + C}$$

(where we've absorbed the factor of 2 into the C when multiplying through).

Applying the initial condition $t = 0, y = -3$ give us

$$-3 = \pm\sqrt{\ln(1+0^2) + C} = \pm\sqrt{C},$$

which means that $C = 9$ and we must choose the negative square root. Thus the solution to the IVP is

$$y = -\sqrt{\ln(1+t^2) + 9}.$$

2. (10 total points) Consider the following linear initial value problem:

$$(t^2-4t-5)\frac{dy}{dt}+2(t+1)y-1=0, \quad y(0)=y_0.$$

   (a) (3 points) Using the existence and uniqueness theorem for linear differential equations, state the maximum interval on which a unique solution to the above IVP is guaranteed to exist.

   First we put the linear equation in standard form by dividing through by (t^2-4t-5):

$$\frac{dy}{dt}+\frac{2(t+1)}{t^2-4t-5}y=\frac{1}{t^2-4t-5}.$$

   Note that $(t^2-4t-5)=(t+1)(t-5)$, so we get cancellation in the term in front of y, so the above equation simplifies to

$$\frac{dy}{dt}+\left(\frac{2}{t-5}\right)y=\frac{1}{(t+1)(t-5)}.$$

   From this we see that $f(t)=\left(\frac{2}{t-5}\right)$ is continuous on the intervals $t<5$ and $t>5$, while $g(t)=\frac{1}{(t+1)(t-5)}$ is continuous on the intervals $t<-1$, $-1<t<5$ and $t>5$.
   The largest interval surrounding the initial condition time $t=0$ on which both f and g are continuous is thus $-1<t<5$. Hence we are guaranteed a unique solution to the IVP for $-1<t<5$.

(b) (7 points) Solve the above differential equation for the case $y_0 = -\frac{1}{5}$.

Our integrating factor is

$$\mu(t) = e^{\int f(t)\,dt} = e^{\int \frac{2}{t-5}\,dt} = e^{2\ln|t-5|} = (t-5)^2.$$

here we may omit the absolute value signs at the end as anything square is always at least zero. The general solution to the DE is then

$$\begin{aligned}
y(t) &= \frac{1}{\mu(t)}\left(\int \mu(t)g(t)\,dt + C\right) \\
&= (t-5)^{-2}\left(\int \frac{(t-5)^2}{(t+1)(t-5)}\,dt + C\right) \\
&= (t-5)^{-2}\left(\int \frac{(t-5)}{(t+1)}\,dt + C\right) \\
&= (t-5)^{-2}\left(\int 1 - \frac{6}{(t+1)}\,dt + C\right) \\
&= (t-5)^{-2}\left(t - 6\ln(t+1) + C\right) \\
&= \frac{t - 6\ln(t+1) + C}{(t-5)^2}.
\end{aligned}$$

Applying the IC $t = 0, y = -\frac{1}{5}$ gives us

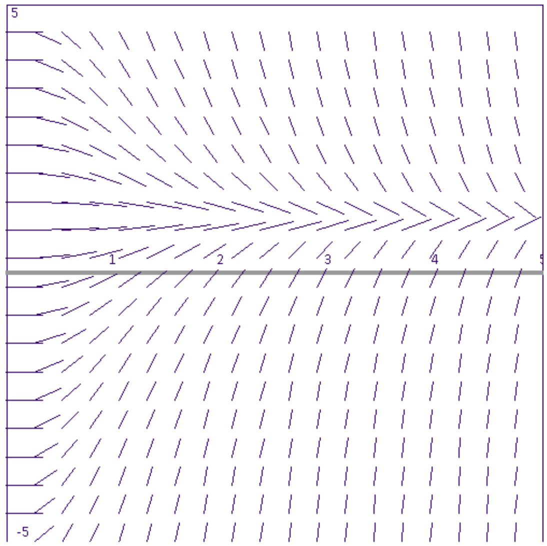
$$-\frac{1}{5} = \frac{0 - 6\ln(0+1) + C}{(0-5)^2} = \frac{C}{25},$$

so $C = -5$. The solution to the IVP is therefore

$$y = \frac{t - 6\ln(t+1) - 5}{(t-5)^2}.$$

Note that if you plot the above function you'll see that it has vertical asymptotes at $t = -1$ and $t = 5$, thereby confirming that the solution is valid on the interval $-1 < t < 5$.

3. (10 total points)
The slope field to the differential equation $\frac{dy}{dx} = f(x,y)$ is plotted below for $0 \le x \le 5, -5 \le y \le 5$:



(a) (4 points) Circle the differential equation that corresponds to the above slope field.

$$\frac{dy}{dx} = x(y+1) \qquad \frac{dy}{dx} = -x(y+1) \qquad \frac{dy}{dx} = x(y-1) \qquad \frac{dy}{dx} = -x(y-1)$$

We see that $\frac{dy}{dx}$ has an equilibrium solution at $y = 1$. This is the case for $\frac{dy}{dx} = x(y-1)$ and $\frac{dy}{dx} = -x(y-1)$ but not so for the other two equations, ruling them out. Moreover we see from the slope field that $\frac{dy}{dx}$ is negative for $t > 0$ and $y > 1$ and positive for $t > 0$ and $y < 1$. This is the case when $f(x,y) = -x(y-1)$ but not so for $f(x,y) = x(y-1)$. We conclude that the slope field is that of the differential equation

$$\frac{dy}{dx} = -x(y-1).$$

(b) (6 points) Let $y = \phi(x)$ be the solution to the differential equation you circled above that satisfies the initial condition $y(0) = 0$. Use Euler's method with a step size of $h = 0.5$ to estimate the value of the solution at $x = 1.5$. You may use decimal approximations in your final answer (but keep at least 4 digits precision at all points).

Recall that Euler's method for the IVP $\frac{dy}{dx} = f(x,y),\ \ y(x_0) = y_0$ with step size h is given by the scheme

- Set x_0 and y_0 to be the given initial conditions
- for $n \ge 0$ set $x_{n+1} = x_n + h$ and $y_{n+1} = y_0 + h \cdot f(x_n, y_n)$.

For us we have $h = 0.5$ and $f(x,y) = -x(y-1)$, so we have

- $x_0 = 0$ and $y_0 = 0$
- $x_1 = \frac{1}{2}$ and $y_1 = y_0 + h \cdot f(x_0, y_0) = 0 + \frac{1}{2} \cdot [-0(0-1)] = 0$
- $x_2 = 1$ and $y_2 = y_1 + h \cdot f(x_1, y_1) = 0 + \frac{1}{2} \cdot \left[-\frac{1}{2}(0-1)\right] = \frac{1}{4}$
- $x_2 = \frac{3}{2}$ and $y_3 = y_2 + h \cdot f(x_2, y_2) = \frac{1}{4} + \frac{1}{2} \cdot \left[-1(\frac{1}{4} - 1)\right] = \frac{5}{8}$

At this point we stop, as we've reached $x_3 = 1.5$. Our estimate for $\phi(1.5)$ is thus $y_3 = \frac{5}{8} = 0.625$.

4. (10 total points) Consider the autonomous differential equation

$$\frac{dy}{dt} = sin^2(y) - K,$$

where K is a constant such that $y = \frac{\pi}{3}$ is an equilibrium solution.

(a) (5 points) Find K, and state whether $y = \frac{\pi}{3}$ is a stable, unstable or semistable equilibrium solution. Be sure to justify your answer.

Recall that an autonomous equation $\frac{dy}{dt} = f(y)$ has an equilibrium solution $y = C$ if $f(C) = 0$. So here we know

$$f\left(\frac{\pi}{3}\right) = sin^2\left(\frac{\pi}{3}\right) - K = 0.$$

Hence

$$K = sin^2\left(\frac{\pi}{3}\right) = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}.$$

Now recall that to ascertain whether $y = C$ is a stable, unstable or semistable equilibrium solution to $\frac{dy}{dt} = f(y)$ we can evaluate $\frac{df}{dy}$ at C; if the derivative is negative we have a stable solution, if the derivative is positve we have an unstable solution, and if the derivative is zero we need to do more work.

Here we have $\frac{df}{dy} = 2\sin(y)\cos(y) = \sin(2y)$, which evaluated at $y = \frac{\pi}{3}$ is $\frac{\sqrt{3}}{2}$, ie. positive. Thus the equilibrium solution $y = \frac{\pi}{3}$ is unstable.

Alternately you can draw a picture of $f(y)$ versus y, and indicate that $f(y) < 0$ to the immediate left of $y = \frac{\pi}{3}$, and positive to the immediate right.

(b) (5 points) Suppose $y = \phi(t)$ is the unique solution to the differential equation satisfying the initial condition $y(0) = 0$. Find $\lim_{t\to\infty} \phi(t)$.

To answer this question one can draw a graph of $f(y)$ vs. y. We see that $f(y)$ is negative for $y = 0$, so $\phi(t)$ is thus decreasing initially. In fact, $\phi(t)$ will continue decreasing as it approaches the first negative y-value for which $f(y) = 0$. We are guaranteed that the solution $\phi(t)$ to the DE will never cross this y-value, since that corresponds to a (stable) equilibrium solution, and two separate solutions having the same y-value at the same time would violate uniqueness.

Our conclusion then is that $\phi(t)$ asymptotes to the first negative root of $f(y) = sin^2(y) - \frac{3}{4}$. To find this we set

$$sin^2(y) - \frac{3}{4} = 0$$

and solve for y. The first negative solution is $y = -\frac{\pi}{3}$.

Hence

$$\lim_{t\to\infty} \phi(t) = -\frac{\pi}{3}.$$

5. (10 total points + 3 bonus points) Water hyacinth is a particularly aggressive invasive plant species in lakes in the southern US. One of the reasons is that it grows very quickly: under good conditions a population will grow at a rate proportional to its own size, with its biomass increasing by a factor of $e = 2.71828\ldots$ every 14 days. Suppose a water hyacinth population establishes itself in a large lake in Florida where conditions are close to ideal. When ecologists discover the population it has a biomass of 750kg.

   Removal efforts begin immediately; however, because it takes some time to train local volunteers to remove the weed efficiently, the rate $R(t)$ at which water hyacinth can be removed from the lake is given by the function

$$R(t) = 600\left(1 - e^{-t}\right),$$

   where t is measured in **weeks** since the beginning of the removal effort, and $R(t)$ is in kg/week.

   (a) (7 points) Establish an initial value problem and solve it to find an explicit formula for the biomass of water hyacinth in the lake at time t.

Let $y(t)$ be the biomass in kg of water hyacinth in the lake at a time t, where t is measured in weeks, and $t = 0$ corresponds to the beginning of removal efforts.

In the absence of any removal effort, the above paragraph tells us that y would grow at a rate proportional to y, i.e.

$$\frac{dy}{dt} = ry$$

for some $r > 0$.

Solving this preliminary DE with $y(0) = y_0$ yields

$$y = y_0 e^{rt}$$

We know if that left unchecked biomass increases by a factor of e every 2 weeks. In mathematics this statement is

$$\frac{y(2)}{y(0)} = e$$

But

$$\frac{y(2)}{y(0)} = \frac{y_0 e^{2r}}{y_0 e^0} = e^{2r}$$

so $e^{2r} = e$. Solving for r by taking logs on both sides yields $r = \frac{1}{2}$.

Now in reality the rate at which y grows is reduced by the rate of removal $R(t)$. Hence the water hyacinth in the lake obeys the differential equation

$$\frac{dy}{dt} = \frac{1}{2}y - 600\left(1 - e^{-t}\right)$$

subject to the initial condition $y(0) = 750$.

[Solution continued on the next page]

To solve this IVP, note that it is linear. In standard form we have

$$\frac{dy}{dt} - \frac{1}{2}y = -600\left(1 - e^{-t}\right),$$

i.e. $f(t) = -\frac{1}{2}$ and $g(t) = -600\left(1 - e^{-t}\right)$. The integrating factor is thus

$$\mu(t) = e^{\int f(t)\,dt} = e^{\int -\frac{1}{2}\,dt} = e^{-\frac{1}{2}t},$$

and so the general solution is given by

$$\begin{aligned}
y(t) &= \frac{1}{\mu(t)}\left(\int \mu(t)g(t)\,dt + C\right)\\
&= e^{\frac{1}{2}t}\left(-600\int e^{-\frac{1}{2}t}(1-e^{-t})\,dt + C\right)\\
&= e^{\frac{1}{2}t}\left(-600\int e^{-\frac{1}{2}t}\,dt + 600\int e^{-\frac{3}{2}t}\,dt + C\right)\\
&= e^{\frac{1}{2}t}\left(1200e^{-\frac{1}{2}t} - 400e^{-\frac{3}{2}t} + C\right)\\
&= 1200 - 400e^{-t} + Ce^{\frac{1}{2}t}.
\end{aligned}$$

Applying the initial condition $y(0) = 750$ yields

$$750 = 1200 - 400e^{0} + Ce^{0},$$

so $C = -50$. Hence the biomass in the lake at time t is given by the function

$$y(t) = 1200 - 400e^{-t} - 50e^{\frac{1}{2}t}.$$

(b) (3 points) Will efforts to completely remove the water hyacinth from the lake be successful? Justify your answer.

We see that the solution

$$y(t) = 1200 - 400e^{-t} - 50e^{\frac{1}{2}t}$$

has an exponentially growing term with a negative coefficient; this term will eventually dominate the other two terms and so $lim_{t\to\infty}y(t) = -\infty$. Thus there must be some point in the future where $y(t) = 0$, i.e. there is no more water hyacinth in the lake.

Thus yes, the removal effort under the assumptions above will ultimately be successful.

(c) (Bonus: 3 points) If the answer to the above question is yes, estimate how many weeks it will take for the water hyacinth to be removed completely from the lake. If the answer to the above question is no, estimate how many weeks it will take for the water hyacinth to reach 10000kg biomass. You may use decimal approximations in your final answer (but keep at least 4 digits precision at all points).

We seek the time where there is no more water hyacinth in the lake, i.e. the t for which $y(t) = 0$. Thus we must solve the equation

$$1200 - 400e^{-t} - 50e^{\frac{1}{2}t} = 0.$$

This equation can in fact be solved exactly through logs and cubic equations, but it's ugly; there's no way I'm expecting you to do this in an exam. The best simplifying observation we can make is that $50e^{\frac{1}{2}t}$ increases exponentially in size with time, while $400e^{-t}$ decreases exponentially. So if t is more than a few units in size we can expect the $-400e^{-t}$ term in the above expression to be negligible in size compared to the $-50e^{\frac{1}{2}t}$ term.

It thus makes sense to discard the $-400e^{-t}$ term and solve instead for t in the equation

$$1200 - 50e^{\frac{1}{2}t} = 0.$$

We thus get that $e^{\frac{t}{2}} = 24$ so $t = 2\ln(24) = 4\ln(6) = 6.3561\ldots$

Indeed, if we plug this t-value into $y(t) = 1200 - 400e^{-t} - 50e^{\frac{t}{2}}$ we get

$$1200 - 400e^{-4\ln 6} - 1200 = -\frac{25}{81} = -0.3086\ldots.$$

This is only a hair below zero (considering the starting value for y of 1200). We therefore expect the t-value for which $y(t) =$ to be just slightly less than our estimate

$$y = 4\ln(6) = 6.3561\ldots$$

(In fact, the true solution to the equation is

$$y = 6.3550\ldots,$$

so our estimate is accurate to 2 decimal places.)

In other words, the lake will be cleared of invasive water hyacinth in just under $6\,\frac{1}{2}$ weeks time.