Math 307A, Midterm 1 Name: ____________________

Spring 2013

Instructions.

- **DON'T PANIC!** If you get stuck, take a deep breath and go on to the next question. Come back to the question you left if you have time at the end.
- There are 4 questions on 6 pages. Make sure your exam is complete.
- You are allowed one double-sided sheet of notes in your own handwriting. You may not use someone else's note sheet.
- You may use a simple scientific calculator, but you don't need to. No fancy calculators or other electronic devices allowed. If you didn't bring a simple calculator, then just don't use a calculator.
- It's fine to leave your answers in exact form. If you use a calculator, approximate to two decimal places.
- **Show your work**, unless instructed otherwise. If you need more space, raise your hand and I'll give you some extra paper to staple onto the back of your test.
- Don't cheat. If I see that you aren't following the rules, I will report you to UW.

Question	Points	Score
1	17	
2	10	
3	10	
4	11	
Total:	48	

1. (a) (5 points) Solve the IVP (find an explicit formula for y), and find the interval on which your solution is valid.

$$\frac{dy}{dx} = (1-2x)y^2, \qquad y(0) = -\frac{1}{2}$$

Solution: Separate the variables, and integrate both sides:

$$\begin{aligned} \frac{dy}{y^2} &= (1-2x)dx \\ -\frac{1}{y} &= x - x^2 + C \\ y &= \frac{1}{x^2 - x - C}. \end{aligned}$$

Using the initial condition, we get the solution $y = 1/(x^2 - x - 2)$.
Now the denominator factors as $(x-2)(x+1)$, so our formula is not defined at $x = 2$, $x = -1$. The formula for y' is fine everywhere. 0 is in between the bad x-values, so the interval on which our solution is valid is $-1 < x < 2$.

(b) (5 points) Solve the differential equation. You may leave your answer in implicit form: no need to solve for y.

$$3x^2 + y + (x+2y)\frac{dy}{dx} = 0.$$

Solution: First, check for exactness: $M_y = 1 = N_x$, so the equation is exact. So let

$$\begin{aligned} \psi(x,y) &= \int M(x,y)dx \\ &= \int 3x^2 + y\,dx \\ &= x^3 + xy + f(y). \end{aligned}$$

Now $\frac{\partial \psi}{\partial y} = N$, so we have

$$x + f'(y) = x + 2y,$$

so $f'(y) = 2y$ and $f(y) = y^2$.
So $\psi(x,y) = x^3 + xy + y^2$, and the solutions to our DE are given implicitly by the formula

$$x^3 + xy + y^2 = C.$$

(c) (7 points) Consider the nonseparable, nonlinear differential equation

$$\frac{dy}{dt} + \frac{2}{t}y = \frac{y^3}{t^2}, \qquad t > 0.$$

It is called a *Bernoulli equation.*

Use the substitution $v = y^{-2}$ to solve the equation. Leave it in general form, with the constant C. Be sure your final answer has the variables y and t only, no v. You may leave your answer in implicit form: no need to solve for y.

Solution: Following the hint, set $v = y^{-2}$ and compute $v'(t)$:

$$\begin{aligned} \frac{dv}{dt} &= \frac{dv}{dy}\frac{dy}{dt} \\ &= -2y^{-3}\frac{dy}{dt}. \end{aligned}$$

Multiply the whole DE by $-2y^{-3}$, then we have

$$\begin{aligned} -2y^{-3}\frac{dy}{dt} - 4t^{-1}y^{-2} &= -2t^{-2}, \text{or} \\ \frac{dv}{dt} - 4t^{-1}v &= -2t^{-2}. \end{aligned}$$

Now the DE is linear, with integrating factor $\mu(t) = e^{-4\ln|t|} = e^{-4\ln t}$ (since $t > 0$). Simplify to get $\mu = t^{-4}$. Multiplying through by μ and rearranging, we have

$$\begin{aligned} \frac{d}{dt}\left(vt^{-4}\right) &= -2t^{-6} \\ vt^{-4} &= \frac{2}{5}t^{-5} + C \\ v &= \frac{2}{5}t^{-1} + Ct^4 \\ v &= \frac{2 + Ct^5}{5t} \\ y^{-2} &= \frac{2 + Ct^5}{5t} \\ y^2 &= \frac{5t}{2 + Ct^5} \\ y &= \pm\sqrt{\frac{5t}{2 + Ct^5}} \end{aligned}$$

2. Consider the differential equation

$$y' = 2y + 3e^t.$$

(a) (4 points) Find the general form of the solution $y(t)$.

Solution: This DE is linear, so we can use integrating factors with $\mu(t) = e^{-2t}$. Multiplying through by μ and rearranging, we have

$$\begin{aligned} \frac{d}{dt}\left(ye^{-2t}\right) &= 3e^{-t} \\ ye^{-2t} &= -3e^{-t} + C \\ y &= -3e^t + Ce^{2t}. \end{aligned}$$

(b) (6 points) Find the solution $y(t)$ that is tangent to the horizontal line $y = -1$.

Solution: If y is to be tangent to the line $y = -1$, then at some t-value we must have equations

$$y = -1, \qquad y' = 0.$$

So use the formulas for y and y' to get equations. Our goal is to find the t-value where the solution touches the line $y = -1$, and then use it to find C.

Setting $y' = 0$ and $y = -1$ in the differential equation, we have the equation

$$2 = 3e^t,$$

which means that $t = \ln\left(\frac{2}{3}\right)$. This is the t-value where the solution and the line are tangent. Now plug $t = \ln\left(\frac{2}{3}\right)$ and $y = -1$ into the general solution:

$$\begin{aligned} -1 &= -3e^{\ln\left(\frac{2}{3}\right)} + Ce^{2\ln\left(\frac{2}{3}\right)} \\ &= -3\left(\frac{2}{3}\right) + C\left(\frac{2}{3}\right)^2 \\ &= -2 + \frac{4}{9}C. \end{aligned}$$

Thus $C = \frac{9}{4}$, so our desired solution is $y = -3e^t + \frac{9}{4}e^{2t}$.

3. Read the instructions here carefully, so you avoid doing extra work!

(a) (5 points) A tank has 80 liters (L) of water with 15 grams (g) of salt dissolved in it. At time $t = 0$ water with 20 g/L of salt flows into the tank at a rate of $\frac{1}{4}$ L/s. At the same time, a drain opens in the bottom of the tank and the well-mixed solution drains out at 1 L/s.

Let $Q = Q(t)$ be the quantity of salt in the tank in grams. Write a differential equation relating Q, t (in seconds), and Q'. Be sure there are no other variables in your expression. **You don't have to solve the equation**; just set it up.

Solution: The volume V of water in the tank has the formula $V = 80 - \frac{3}{4}t$. The rate of salt in is $(20)\frac{1}{4} = 5$ g/s, and the rate out is $(1)Q/V$ g/s. So the differential equation is

$$\frac{dQ}{dt} = 5 - \frac{Q}{80 - \frac{3}{4}t}.$$

Note that the initial condition $Q(0) = 15$ doesn't matter for setting up the equation; only for solving it (which you don't have to do).

(b) (5 points) Let $P = P(t)$ be the number of bacteria in a certain area, where t is in days. The population is modeled by the differential equation

$$P' = rP$$

for some r. The population is dying off: every five days, the population is cut in half. What is r?

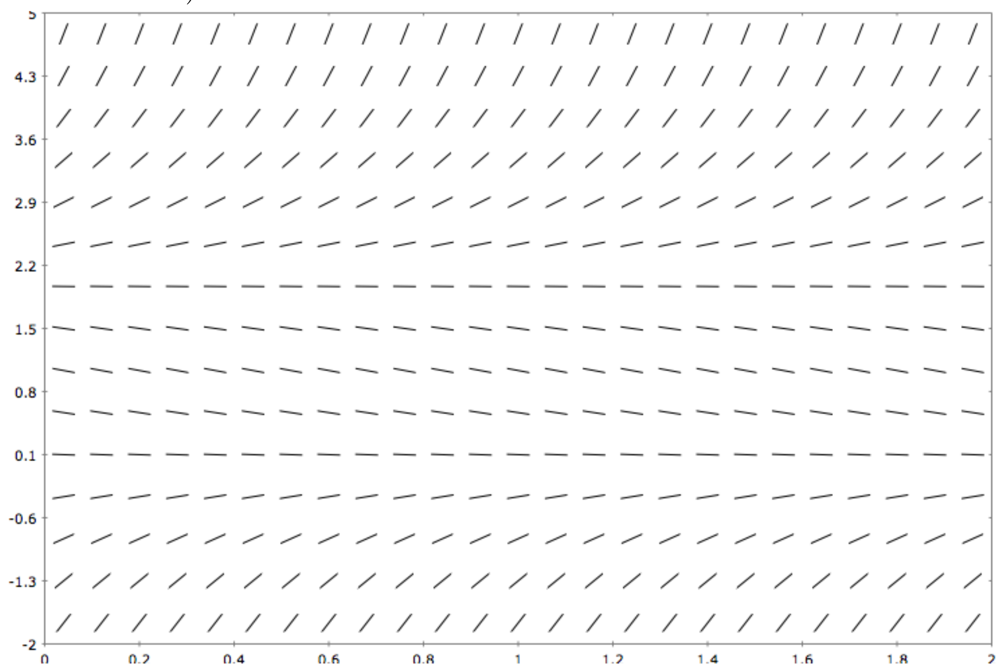
Solution: Solving this DE, you get $P = P_0 e^{rt}$, where $P_0 = P(0)$. Plugging in $t = 5$ and $P = P_0/2$, we get the equation

$$\frac{P_0}{2} = P_0 e^{5r},$$

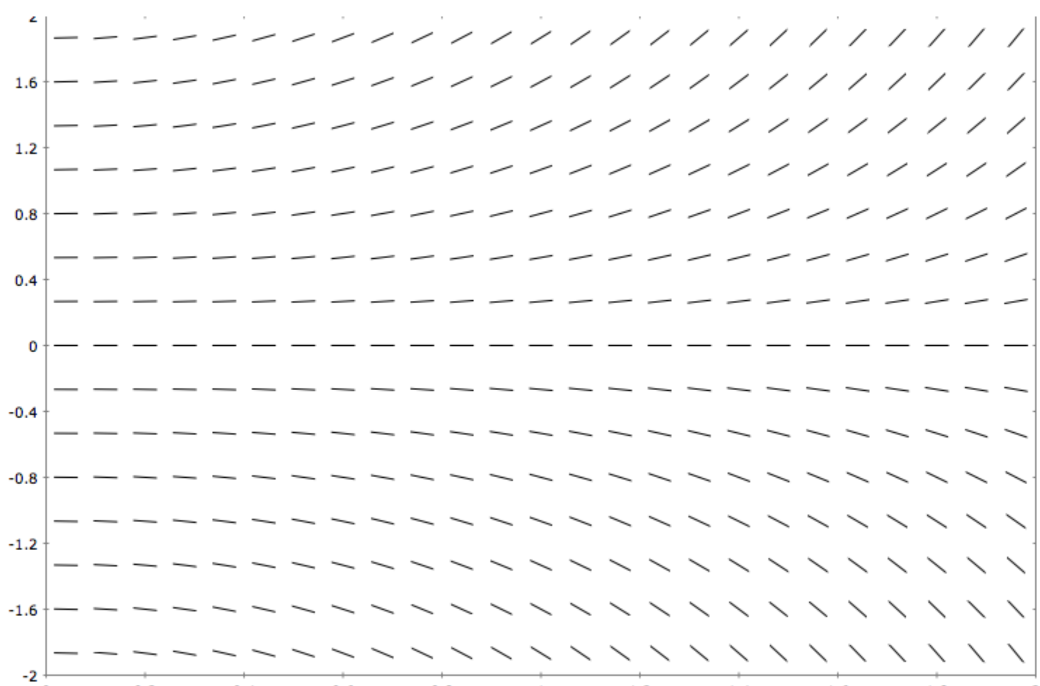
and we can solve for r. Final answer:

$$r = \frac{\ln\left(\frac{1}{2}\right)}{5}.$$

4. (a) (4 points) Each of the two slope fields below has a list of differential equations below it. Circle the DE that matches the slope field. (t is the horizontal axis; y is the vertical axis.)



$\boxed{y' = y(y-2)}$ $\qquad y' = -y(y-2) \qquad y' = y(y+2) \qquad y' = -y(y+2)$



$y' = y \qquad y' = -y \qquad \boxed{y' = ty} \qquad y' = -ty$

(b) (7 points) Consider the differential equation

$$\frac{dy}{dt} = (y^2 - 1)(y + 2)$$

Draw a coordinate plane below. Label the axes, and sketch at least ten solutions to the differential equation.

Read this carefully!: Include all equilibrium solutions. Make sure your solutions start at $t = 0$ (or before), and draw them for long enough so that their eventual behavior is clear to me. Include as many different behaviors as possible.

Solution: First, factor the right side: $y' = (y - 1)(y + 1)(y + 2)$. Then make the phase line, which shows that $y = 1, -2$ are unstable equilibrium solutions and $y = -1$ is stable. Then sketch solutions: