Your Name

Student ID #

In these problems, you may use the following table of selected Laplace transform identities:

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$
$tf(t)$	$-F'(s)$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$

1. **Don't panic—it's not as bad as it looks.** In this problem, you are given the proof of a Laplace transform identity and are asked to describe what happened in some of the steps. The identity is

$$\mathcal{L}\left\{\frac{1}{t}f(t)\right\}(s) = \int_s^\infty F(\sigma)\,d\sigma.$$

The proof is

$$\begin{aligned}\int_s^\infty F(\sigma)\,d\sigma &\overset{①}{=} \int_s^\infty \mathcal{L}\{f(t)\}(\sigma)\,d\sigma \overset{②}{=} \int_s^\infty \mathcal{L}\left\{t\left(\frac{1}{t}f(t)\right)\right\}(\sigma)\,d\sigma \\ &\overset{③}{=} \int_s^\infty -\frac{d}{d\sigma}\mathcal{L}\left\{\frac{1}{t}f(t)\right\}(\sigma)\,d\sigma \overset{④}{=} -\mathcal{L}\left\{\frac{1}{t}f(t)\right\}(\sigma)\Big|_s^\infty \\ &= \mathcal{L}\left\{\frac{1}{t}f(t)\right\}(s) - \lim_{b\to\infty}\mathcal{L}\left\{\frac{1}{t}f(t)\right\}(b) = \mathcal{L}\left\{\frac{1}{t}f(t)\right\}(s).\end{aligned}$$

What happened in each of the labeled steps? That is, what justifies the labeled equalities?

①

Solution: By definition, $F(\sigma) := \mathcal{L}\{f\}(\sigma)$.

②

Solution: In general, $t\frac{1}{t} = 1$.

③

Solution: Use the identity $\mathcal{L}\{tf(t)\} = -F'(s)$ with $f(t)$ replaced with $\frac{1}{t}f(t)$ and s replaced with σ.

④

Solution: This is the fundamental theorem of calculus, since we're integrating a derivative.

(Note: assume that all terms—integrals, limits, Laplace transforms, etc.—exist and the identities in the table apply. Generally ignore technical conditions, like continuity or convergence.)

2. Solve the following initial value problem using the Laplace transform:

$$y''' = y, \qquad y''(0) = 0, y'(0) = 1, y(0) = 0.$$

If you use partial fractions, you **do not** need to solve for the constants.

Hint: $s^3 - 1 = (s-1)(s^2+s+1) = (s-1)((s+\frac{1}{2})^2 + \frac{3}{4})$.

Solution: We have

$$\begin{aligned}
\mathcal{L}\{y\} &= \mathcal{L}\{y'''\} \\
&= s\mathcal{L}\{y''\} - y''(0) \\
&= s^2\mathcal{L}\{y'\} - sy'(0) - y''(0) \\
&= s^3\mathcal{L}\{y\} - s^2y''(0) - sy'(0) - y''(0) \\
&= s^3\mathcal{L}\{y\} - s
\end{aligned}$$

so that

$$\begin{aligned}
\mathcal{L}\{y\} = \frac{s}{s^3-1} &= \frac{s}{(s-1)((s+\frac{1}{2})^2+\frac{3}{4}} \\
&= \frac{A}{s-1} + \frac{B(s+\frac{1}{2}) + C\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}
\end{aligned}$$

and hence

$$y = Ae^t + Be^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) + Ce^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

(As it turns out, $A = \frac{1}{3}, B = -\frac{1}{3}, C = \frac{\sqrt{3}}{3}$.)