Intro Differential Equations

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Contents

Chapter 0

Review

In this chapter, we will review the material needed for the rest of the course. First, we review ordinary and partial differentiation, as well as the various integration techniques from calculus. Next, we review Taylor polynomials and Taylor series. Finally, we review complex numbers and Euler's formula.

It is normal that you will not be familiar with every topic in this chapter - That's okay! You should read this chapter as soon as possible, and ask me for help with anything you don't understand. As we need these techniques later in the course, we will quickly review them in class as well. At that time you can refer back to this chapter as needed.

1 Calculus Review

In this section, we review the two main concepts that you studied in calculus: differentiation and integration. We also will consider partial derivatives of functions of several variables.

1.1 Differentiation

Let $f(x)$ be a differentiable function of just one variable. This means that for every x in the real numbers $\mathbb{R}$, we have

$$f'(x) := \lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h}$$

exists, i.e. the above limit converges to a finite real number. We often use the "Leibniz notation" $\frac{df}{dx}$ for $f'(x)$.

This is the definition that you learned in your first calculus course - but it is not how you actually compute derivatives. The following properties allow us to compute derivatives in practice:

1.1 Theorem. *If $f(x)$ and $g(x)$ are differentiable functions, and $c \in \mathbb{R}$ a constant then*

i) $c' = 0$

ii) (Linearity): $[cf(x) + g(x)]' = cf'(x) + g'(x)$

iii) (Product Rule): $[f(x) \cdot g(x)]' = f'(x)g(x) + f(x)g'(x)$

iv) (Quotient Rule): $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$

v) (Chain Rule): $[f(g(x))]' = f'(g(x)) \cdot g'(x)$

vi) (Power Rule): $[x^c]' = cx^{c-1}$

1.2 Examples. Compute the derivative of $f(x) = 3\sin(\sqrt{x}) - \pi x^{3/2}$.

1.3 Solution. Using linearity, it will suffice to compute $[\sin(\sqrt{x})]'$ and $[x^{3/2}]'$, separately. We use the power rule for $x^{3/2}$, and get

$$[x^{3/2}]' = \frac{3}{2}x^{1/2}.$$

We use the chain rule to evaluate $[\sin(\sqrt{x})]'$, where $g(x) = \sqrt{x}$. We need to know $g'(x)$, which we find using the power rule:

$$[\sqrt{x}]' = [x^{1/2}]' = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}.$$

Then $[\sin(\sqrt{x})]' = \cos(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$, since $\sin'(x) = \cos(x)$. Finally, by linearity, we have

$$\begin{aligned} f'(x) &= 3[\sin(\sqrt{x})]' - \pi[x^{3/2}]' \\ &= 3\frac{\cos(\sqrt{x})}{2\sqrt{x}} - \pi\frac{3}{2}x^{1/2} \end{aligned}$$

1.2 Partial Derivatives

Now we consider the case when $f(x, y)$ is a function of two variables; this means that for every pair of real numbers (x, y) in $\mathbb{R}^2$, the xy-plane, we have associated a real number $f(x, y)$ in $\mathbb{R}$. Notice that if we fix a specific value of y, say $y = y_0$, then the function $f(x, y_0)$ is really only a function of x. We denote this function $f^{y_0}(x)$; it is called a section of f. Similarly, we could fix $x = x_0$ and consider the section $f^{x_0}(y)$.

We say that $f(x, y)$ is differentiable with respect to x if for every choice of y_0, the section $f^{y_0}(x)$ is differentiable. In other words, $f(x, y)$ is differentiable with respect to x if for each choice of y_0, the limit

$$f_x(x, y_0) := \lim_{h \longrightarrow 0} \frac{f(x + h, y_0) - f(x, y_0)}{h}$$

exists, i.e. converges to a finite number. The definition for the derivative of $f(x, y)$ with respect to y is similar, and is denoted $f_y(x_0, y)$.

The Leibniz notation for partial derivatives is convenient:

$$\frac{\partial}{\partial x}f(x, y) := f_x(x, y) \qquad \frac{\partial}{\partial y}f(x, y) = f_y(x, y).$$

Of course, if $f(x, y, z)$ is a function of 3 variables, or more generally if $f(x_1, \ldots, x_n)$ is a function of n variables we can similarly define a section by fixing $n - 1$ of the variables; then the partial derivative with respect to the remaining variable is just the usual derivative of this section.

1.4 Examples. Suppose $f(x, y) = \sin(xy) + 3x^2y$. Find f_x and f_y.

1.5 Solution. To find $f_x(x, y)$, we first find the section $f^{y_0}(x)$ at y_0:

$$f^{y_0}(x) = \sin(y_0x) + 3y_0x^2.$$

Then we take the derivative as usual, using linearity, the chain rule and power rule. We get:

$$[f^{y_0}(x)]' = y_0 \cos(y_0x) + 6y_0x.$$

Thus $f_x(x, y) = y \cos(xy) + 6yx$.

Observe that using the notation y_0 is pedagogical; it just reminds us which variable(s) are fixed and can be treated as constants. With practice, it is easy to just use y in place of y_0. The Leibniz notation is particularly helpful for this, since it indicates which variable is the ONLY variable which is not considered constant. Observe:

To find $f_y(x, y) = \frac{\partial}{\partial y} f(x, y)$, we compute

$$\begin{aligned} \frac{\partial}{\partial y}[\sin(xy) + 3x^2y] &= \frac{\partial}{\partial y}\sin(xy) + \frac{\partial}{\partial y}3x^2y \\ &= x\cos(xy) + 3x^2. \end{aligned}$$

Finally, it is often the case that we want to take multiple partial derivatives. The partial derivative $f_x(x, y)$ is again a function of x and y, so we could compute its partial derivative with respect to x or y, denoted $f_{xx} := (f_x)_x$ or $f_{xy} := (f_x)_y$.

1.6 Examples. Using $f(x, y) = \sin(xy) + 3x^2y$, compute $f_{xy}(x, y)$ and $f_{yx}(x, y)$.

1.7 Solution. Since $f_x(x, y) = y \cos(xy) + 6xy$, we get

$$f_{xy}(x, y) = \frac{\partial}{\partial y}\left[y \cos(xy) + 6xy\right] = \cos(xy) - yx \sin(xy) + 6x.$$

Since $f_y(x, y) = x \cos(xy) + 3x^2$, we get

$$f_{yx}(x, y) = \frac{\partial}{\partial x}\left[x \cos(xy) + 3x^2\right] = \cos(xy) - xy \sin(xy) + 6x.$$

Notice, that $f_{xy}(x, y) = f_{yx}(x, y)$.

1.8 Theorem (Equality of Mixed Partials)**.** *If $f(x, y)$ is twice differentiable, so that all of the partial derivatives f_{xx}, f_{yx}, f_{xy} and f_{yy} exist, and if all of these second partials are continuous functions, then*

$$f_{xy} = f_{yx}.$$

The previous theorem is an important fact that is proved rigorously in a standard undergraduate courses in real analysis.

1.3 Integration

In this section, we will review the major techniques for evaluating integrals. I take it for granted that the reader is aware of the definition of the definite integral as a limit of Riemann sum approximations to the "signed area under the curve," as well as the relationship between the definite and an indefinite integral. We first list some properties of the integral:

1.9 Theorem. *Suppose that $f(x)$ and $g(x)$ are differentiable functions, and c in $\mathbb{R}$ is a real number. Let $-\infty \leq a < b < +$*
$infty$.

$\int cdx = cx$

(Linearity): $\int cf(x) + g(x)dx = c\int f(x)dx + \int g(x)dx$

(By parts): $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$

(Substitution): $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$

(Power Rule): $\int x^c dx = \frac{x^{c+1}}{c+1}$ *if* $c \neq -1$*. If* $c = -1$*, then* $\int \frac{1}{x}dx = \ln(x)$*.*

1.10 Theorem (Fundamental theorem of calculus)**.** *If $f(x)$ is a continuous function of x, then then the function $F(x)$ defined by the definite integral*

$$F(x) = \int_0^x f(t)dt$$

is a differentiable function, and we have the equality $F'(x) = f(x)$. Conversely, if $G(x)$ is any function such that the derivative $G'(x) = f(x)$, then $G(x) = F(x) + c$ for some constant $c \in \mathbb{R}$.

Remember that the choice of 0 as the lower limit of integration for $F(x)$ is arbitrary. If we choose a different constant for the lower limit, say $G(x) = \int_{-1}^x f(t)dt$, then we effectively have added $c = \int_{-1}^0 f(t)dt$ to $F(x)$, since

$$G(x) = \int_{-1}^x f(t)dt = \int_{-1}^0 f(t)dt + \int_0^x f(t)dt = c + F(x).$$

Substitution

The first technique we learn for evaluating indefinite integrals is direct substitution. Basically, if we are given an integral of the form

$$\int f(g(x))g'(x)dx$$

then we can substitute $u = g(x)$. Then $du = g'(x)dx$, so that

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

If the anti-derivative of f is known, then we have solved the problem!

1.11 Examples. Evaluate $\int xe^{-x^2}dx$.

1.12 Solution. We let $u = x^2$, so that $du = 2xdx$. Then

$$\int xe^{-x^2}dx = \int e^{-u}\frac{du}{2} = -e^u + c = -e^{-x^2} + c,$$

where c is an arbitrary constant of integration.

Heuristically, you should identify constituent functions which are a multiple of the derivative of another constituent function of the integrand. In the above example, we identified that x was a multiple of the derivative of x^2.

1.13 Examples. Evaluate $\int \frac{\sin(\sqrt{x})}{\sqrt{x}}dx$.

1.14 Solution. We know that $\frac{d\sqrt{x}}{dx} = \frac{1}{2\sqrt{x}}$, which is a multiple of the function $\frac{1}{\sqrt{x}}$. So we try $u = \sqrt{x}$, $du = \frac{dx}{2\sqrt{x}}$, and

$$\int \frac{\sin(\sqrt{x})}{\sqrt{x}}dx = \int \sin(u)2du = 2\int \sin(u)du = -2\cos(u) + c = -2\cos(\sqrt{x}) + c,$$

where c is an arbitrary constant of integration.

Instead of a direct substitution, we sometimes use an implicit substitution. For example, $\cos(\theta) = \frac{1}{\sqrt{1+x^2}}$ is an example of an implicit substitution. This implies that $\tan(\theta) = x$, because

$$\sin^2(\theta) = 1 - \cos^2(\theta) = 1 - \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \frac{x^2}{1+x^2}$$

and so

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{x/\sqrt{1+x^2}}{1/\sqrt{1+x^2}} = x.$$

1.15 Examples. Evaluate $\int \frac{1}{\sqrt{1+x^2}}dx$.

1.16 Solution. We make the implicit (or trigonometric) substitution $\cos(\theta) = \frac{1}{\sqrt{1+x^2}}$. By the previous reasoning, $\tan(\theta) = x$; by differentiating implicitly we have $\sec^2(\theta)d\theta = dx$. Then

$$\int \frac{1}{\sqrt{1+x^2}}dx = \int \cos(\theta)\sec^2(\theta)d\theta = \int \sec(\theta)d\theta.$$

The strategy is now far from obvious, but we multiply and divide by $\sec(\theta) + \tan(\theta)$ to get

$$\int \sec(\theta)\frac{\sec(\theta)+\tan(\theta)}{\sec(\theta)+\tan(\theta)}d\theta = \int \frac{\sec(\theta)\tan(\theta)+\sec^2(\theta)}{\sec(\theta)+\tan(\theta)}d\theta.$$

Now we try the direct substitution $u = \sec(\theta) + \tan(\theta)$ to get $du = [\sec(\theta)\tan(\theta) + \sec^2(\theta)]d\theta$, and so

$$\int \frac{\sec(\theta)\tan(\theta)+\sec^2(\theta)}{\sec(\theta)+\tan(\theta)}d\theta = \int \frac{du}{u} = \ln|u| = \ln|\sec(\theta)+\tan(\theta)|.$$

Finally, we want to substitute back for x; since $\sec(\theta) = \frac{1}{\cos(\theta)} = \sqrt{1+x^2}$, and $\tan(\theta) = x$, we get

$$\ln|\sec(\theta)+\tan(\theta)| = \ln\left|x + \sqrt{1+x^2}\right|.$$

Partial Fractions

Consider the rational function $f(x) = \frac{1}{x(x+1)}$. We would like to compute

$$\int f(x)dx = \int \frac{1}{x(x+1)}dx$$

The idea is that $\frac{1}{x(x+1)}$ is a really a linear combination of simpler functions, namely $\frac{1}{x}$ and $\frac{1}{x+1}$. Indeed, we have

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}.$$

Then the integral is easy:

$$\int \frac{1}{x(x+1)}dx = \int \frac{1}{x}dx - \int \frac{1}{x+1}dx = \ln|x| - \ln|x+1|.$$

The technique of expanding a rational function in the previous way as a linear combination of simpler rational functions is called *partial fractions*.

There are three important steps to using partial fractions to expand a rational function. The first step is to factor the denominator into linear factors and irreducible quadratic factors. For example,

$$\frac{1}{x^3 + x^2 + x + 1} = \frac{1}{(x+1)(x^2+1)}.$$

The factor x^2+1 is an irreducible quadratic because it does not have any real roots. The second step is to "guess" the correct form of the expansion. In this example, the correct form is

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}.$$

Notice that the irreducible quadratic terms must contain a generic linear function of x in the numerator. Finally, the third step is to solve for the unknown constants in some way. The most obvious way is to find a common denominator on the right hand side, and by comparing coefficients obtain a system of equations. This is time-consuming and difficult; it is often easier to multiply the equation by one of the denominators, and evaluate both sides at whatever makes that denominator zero. For example, if we multiply by $x+1$ and evaluate at $x = -1$ we obtain

$$\begin{aligned} \left.\frac{1}{x^2+1}\right|_{x=-1} &= A + (x+1)\left.\frac{Bx+C}{x^2+1}\right|_{x=-1} \\ \frac{1}{1+1} &= A \end{aligned}$$

so that $A = 1/2$. We can do exactly the same thing for the denominator x^2+1, though we have to evaluate at the complex number $i = \sqrt{-1}$, because this is what makes $x^2+1 = 0$. We obtain:

$$\begin{aligned} \left.\frac{1}{x+1}\right|_{x=1} &= (x^2+1)\frac{A}{x+1} + (Bx+C)\bigg|_{x=i} \\ \frac{1}{i+1} &= Bi + C. \end{aligned}$$

Observe that

$$\frac{1}{i+1} = \frac{1}{i+1}\frac{1-i}{1-i} = \frac{1-i}{2} = \frac{1}{2} - \frac{1}{2}i,$$

from which it follows that $C = \frac{1}{2}$, and $B = -\frac{1}{2}$. You may wish to skim the section on complex numbers if this is new.

Sometimes when you factor the denominator of a rational function you end up with repeating linear or quadratic factors. For example, you might obtain

$$\frac{1}{(x^2+1)^2(x-1)^3}.$$

In this case, what do we guess for the partial fractions expansion? We guess

$$\frac{1}{(x^2+1)^2(x-1)^3} = \frac{A_1x+B_1}{x^2+1} + \frac{A_2x+B_2}{(x^2+1)^2} + \frac{C_1}{x-1} + \frac{C_2}{(x-1)^2} + \frac{C_3}{(x-1)^3}.$$

In general, if we have a factor raised to the nth power, then we have to have repeated terms in the guess for all powers from 1 up until n.

2 Taylor Series

Taylor series are limits of things called Taylor polynomials. So it is first necessary to understand these.

2.1 Taylor Polynomials

Consider the function $f(x) = (x-3)^2 - 3(x+1) + 4$. We know that this is a polynomial of degree 2, so that we can re-write it in the form

$$f(x) = a_0 + a_1x + a_2x^2,$$

for some constants a_0, a_1, a_2 which we need to determine. To do this, we observe the following:

$$\begin{aligned} f(0) &= (0-3)^2 - 3(0+1) + 4 = 10 \\ f'(0) &= 2(0-3) - 3 = -9 \\ f''(0) &= 2. \end{aligned}$$

We can also compute these values for $f(x) = a_0 + a_1x + a_2x^2$:

$$\begin{aligned} f(0) &= a_0 \\ f'(0) &= a_1 \\ f''(0) &= 2a_2. \end{aligned}$$

This gives us the equations:

$$\begin{aligned} 10 &= a_0 \\ -9 &= a_1 \\ 2 &= 2a_2, \end{aligned}$$

i.e. $f(x) = 10 - 9x + x^2$. We have re-written the function $f(x)$ in a "standard form" using only the data of its derivatives at $x = 0$.

More generally, if we have a polynomial of degree n, $f(x) = a_0 + a_x + \cdots + a_n x^n$, we can compute its derivatives at $x = 0$:

$$\begin{aligned} f(0) &= a_0 \\ f'(0) &= a_1 \\ &\vdots \\ f^{(n)}(0) &= n! a_n. \end{aligned}$$

Now we solve these equations for the values a_i, $0 \leq j \leq n$. We get $a_j = \frac{f^{(j)}(0)}{j!}$. Then our function

$$f(x) = f(0) + f'(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \cdots \frac{f^{(n)}(0)}{n!}x^n = \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!}x^j$$

clearly depends only on the data of the derivatives of $f(x)$ at $x = 0$. We call the sum on the right the *nth Taylor polynomial of $f(x)$ at $x = 0$*, denoted $T_{f,n}(x)$ or $T_n(x)$ if $f(x)$ is understood:

$$T_{f,n}(x) := \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!}x^j.$$

Since $f(x)$ is itself a polynomial of degree n, we have that $f(x) = T_n(x)$ just like in the preceding example.

Now suppose that $f(x) = (x+1)^2$ is a polynomial of degree 2. What is $T_1(x)$? By our formula, it should be

$$\begin{aligned} T_1(x) = \sum_{j=0}^{1} \frac{f^{(j)}(0)}{j!}x^j &= \frac{f^{(0)}(0)}{0!} + \frac{f^{(1)}(0)}{1!}x \\ &= f(0) + f'(0)x = 1 + 2x. \end{aligned}$$

We notice that this is exactly the tangent line to the parabola at $x = 0$, which is the *best* degree 1 approximation of $f(x)$ for small values of x. More generally, $T_k(x)$ is the best kth degree polynomial approximation to $f(x)$ for values of x near $x = 0$.

But what if $f(x)$ is not a polynomial at all? For example, what if $f(x) = e^x$? We still want to write it in a standard form based on the data of its derivatives at $x = 0$. The idea is to use our formula to approximate e^x by these "standard polynomials" near $x = 0$; by letting the degree get larger and larger, our approximation becomes more and more accurate. In this case,

$$\begin{aligned} T_{f,n}(x) = \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!}x^j = \sum_{j=0}^{n} \frac{e^0}{j!}x^j &= \sum_{j=0}^{n} \frac{1}{j!}x^j \\ &= 1 + x + \frac{1}{2!}x^2 + \cdots \frac{1}{n!}x^n; \end{aligned}$$

here we have used that the derivative of e^x is e^x, so $f^{(j)}(x) = e^x$ for all $j \geq 0$. Then, for values of x near 0, we have

$$e^x = \lim_{n\longrightarrow\infty} T_{f,n}(x) = \sum_{j=0}^{\infty} \frac{1}{j!}x^j = 1 + x + \frac{x^2}{2} + \cdots \frac{x^n}{n!} + \cdots .$$

This is called the *Taylor series* of e^x based at $x = 0$. For certain nice functions, like e^x, this equality is actually true for *ALL* values of x, not just values close to $x = 0$.

2.2 Taylor Series

2.1 Definition. If $f(x)$ is infinitely differentiable at $x = 0$, then the *Taylor series of $f(x)$ at $x = 0$* is

$$T_f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!}x^j = \lim_{n\longrightarrow\infty} T_{n,f}(x),$$

which may be infinite. When $f(x) = T_f(x) < \infty$ for all values of x in some interval, we say that f is *real-analytic* in that interval.

If this is your first time seeing Taylor series, you probably have some questions. For example:

i) For which values of x is the limit $T_f(x) = \lim_{n\longrightarrow\infty} T_{f,n}(x)$ finite?

ii) Which functions $f(x)$ are real-analytic? In other words, when can we replace a function with its Taylor series?

iii) What is so special about $x = 0$? What if I have the data of all derivatives of $f(x)$ at $x = 1$?

Think about these questions, and consult your calculus books if necessary. If you'd rather ask me, come to my office hours! For this class, all of the functions we consider are real-analytic. There is nothing special about basing the series at $x = 0$, but we will stick to this case. All of the bounded trig functions, like $\sin(x)$, $\cos(x)$, and exponential functions will equal their Taylor series for all values of x, not just a finite interval.

Look at the last page for a standard table of Taylor series. The functions $\cosh(x)$ and $\sinh(x)$ might be new to you:

$$\cosh(x) := \frac{e^x + e^{-x}}{2} \qquad \sinh(x) := \frac{e^x - e^{-x}}{2};$$

observe that from both the definitions above, as well as from the Taylor series in the table it is clear that

$$\cosh(x) + \sinh(x) = e^x.$$

Now, let's review some tricks for finding Taylor series. For example, if $\tan^{-1}(x)$ were not in the table, how would we compute it? Not from using the definition! That would be difficult. Instead, remember that $\tan^{-1}(x) = \int_0^x \frac{1}{1+s^2}ds$, or in other words

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}.$$

Now using a different entry on the table, namely

$$\frac{1}{1-x} = \sum_{j=0}^{\infty} x^j,$$

we can deduce that

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{j=0}^{\infty} (-x^2)^j = \sum_{j=0}^{\infty} (-1)^j x^{2j}.$$

Integrating both sides with respect to x, we obtain

$$\tan^{-1}(x) = \sum_{j=0}^{\infty} (-1)^j \int x^{2j} dx = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{2j+1},$$

which agrees with the entry in the table.

Another technique you may need to employ is multiplying two Taylor series together. This is often called the *Cauchy product* of the sequences:

$$\left(\sum_{j=0}^{\infty} a_j x^j\right)\left(\sum_{j=0}^{\infty} b_j x^j\right) = \sum_{j=0}^{\infty} c_j x^j, \qquad c_j = \sum_{k=0}^{j} a_k b_{j-k}.$$

Consider what is going on: we are just distributing the product; each term must be multiplied by every other term. If we fix a term in the product, like $c_j x^j$, it must be the sum of products of terms $(a_s x^s)(b_t x^t)$, where $s, t \leq j$. Importantly, $s + t = j$ since $x^s x^t = x^j$. Thus if we know that $s = k$, it must be that $t = j - k$, c.f. the formula above. Finally, we sum over all possibilities, namely $k = 0$ to $k = j$, since this exhausts the terms of degree less than j.

As a silly example, if $y(x) = \frac{1}{1-x} = \sum_{j=0}^{\infty} x^j$, then

$$y^2 = \sum_{j=0}^{\infty} (\sum_{k=0}^{j} 1) x^j = \sum_{j=0}^{\infty} (j+1) x^j.$$

2.2 Excercise. Verify this Taylor series in a different way by differentiating both sides of the equation $\frac{1}{1-x} = \sum_{j=0}^{\infty} x^j$.

2.3 Solving First-Order Equations

Consider the equation

$$\frac{dy}{dx} = y.$$

Here's the new idea: we assume that $y(x)$ is real analytic, which allows us to replace $y(x)$ by its Taylor series. Indeed, let $y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{j=0}^{\infty} a_j x^j$. Then

$$\frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots = \sum_{j=1}^{\infty} j a_j x^{j-1}.$$

using the power rule. Observe that we can also write

$$\frac{dy}{dx} = \sum_{j=0}^{\infty}(j+1)a_{j+1}x^j$$

by re-indexing the sum above. This is an important technique for this method.

Since $\frac{dy}{dx} = y$, we must have

$$a_1 + 2a_2x + 3a_3x^2 + \cdots = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots .$$

This is an equality of functions in some neighborhood of $x = 0$. In particular, choosing $x = 0$ gives us that $a_1 = a_0$. Now taking the derivatives of both sides with respect to x, and evaluating at $x = 0$, we get that $2a_2 = a_1$. Likewise, these two series are equal if and only if they agree on every coefficient of x^j for $j \geq 0$. The resulting system of equations is this:

$$\begin{aligned}
& & a_1 &= a_0 \\
2a_2 &= a_1 = a_0 & a_2 &= \frac{a_0}{2} \\
3a_3 &= a_2 = \frac{a_0}{2} & a_3 &= \frac{a_0}{2\cdot 3} \\
&\vdots & & \\
na_n &= a_{n-1} = \frac{a_0}{(n-1)!} & a_n &= \frac{a_0}{n!}.
\end{aligned}$$

It follows that

$$\begin{aligned}
y(x) &= a_0 + a_0x + \frac{a_0}{2}x^2 + \cdots + \frac{a_0}{n!}x^n + \cdots \\
&= a_0\left(1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots\right).
\end{aligned}$$

The final step is to try to identify the Taylor series that we have above: of course it is e^x, so that $y(x) = a_0e^x$. Anyway, we already knew this was the answer because $\frac{dy}{dx} = y$ is separable and has solution $y = y_0e^x$. It is good to first see the method work on an easy example to get the idea of what's going on.

In summary, we guess that the solution has a particular form (the form of a Taylor series based at $x = 0$), and then we solve for the parameters in that form (all of the coefficients a_j). Finally, we re-write in terms of elementary functions if it's convenient. Now let's do a more interesting

2.3 Examples. Solve the differential equation

$$\frac{dy}{dx} = y + e^{-x}.$$

2.4 Solution. We replace $y(x)$ with $\sum_{j=0}^{\infty} a_jx^j$, its Taylor series. Then $\frac{dy}{dx} = \sum_{j=0}^{\infty}(j+1)a_{j+1}x^j$, as before. Substituting $-x$ for x in the standard formula gives

$$e^{-x} = \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}x^j.$$

Thus, $\frac{dy}{dx} = y + e^{-x}$ becomes

$$\begin{aligned}\sum_{j=0}^{\infty}(j+1)a_{j+1}x^j &= \sum_{j=0}^{\infty} a_j x^j + \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}x^j \\ &= \sum_{j=0}^{\infty}(a_j + \frac{(-1)^j}{j!})x^j.\end{aligned}$$

Comparing coefficients gives us a system of equations. We get:

$$\begin{array}{rrr}
j=0 & & a_1 = a_0 + 1 \\
j=1 & 2a_2 = a_1 - 1 = a_0 & a_2 = \dfrac{a_0}{2} \\
j=2 & 3a_3 = a_2 + \dfrac{1}{2} = \dfrac{a_0+1}{2} & a_3 = \dfrac{a_0+1}{3!} \\
j=3 & 4a_4 = a_3 - \dfrac{1}{3!} = \dfrac{a_0}{3!} & a_4 = \dfrac{a_0}{4!} \\
 & \vdots & \\
j=2n-1 & 2n\,a_{2n} = a_{2n-1} - \dfrac{1}{(2n-1)!} = \dfrac{a_0}{(2n-1)!} & a_{2n} = \dfrac{a_0}{(2n)!} \\
j=2n & (2n+1)\,a_{2n+1} = a_{2n} + \dfrac{1}{(2n)!} = \dfrac{a_0+1}{(2n)!} & a_{2n+1} = \dfrac{a_0+1}{(2n)!}.
\end{array}$$

Finding the general pattern isn't too bad in this example, which is lucky. In this case, there's a difference between the odd and even coefficients.

To really be sure that we get the right formulas, we would use a technique called "mathematical induction." We first guess the answer (so every even and odd term as above). Then we check that if we are correct up until $j = 2n$, then we are also correct up until $j = 2(n+1) = 2n+2$. I will leave that last step to the you. The logic goes like this: since we are clearly correct for $j = 2$, we must be correct for $j = 4$; since we are correct until $j = 4$, we must be correct until $j = 6$, etc. Don't worry about this too much if you haven't yet taken a class in mathematical proofs.

To recognize the solution, we write

$$\begin{aligned}y(x) &= \sum_{j=0}^{\infty} a_j x^j = a_0 \sum_{j=0}^{\infty}\frac{1}{j!}x^j + \sum_{j=0}^{\infty}\frac{1}{(2j+1)!}x^{2j+1} \\ &= a_0 e^x + \sinh(x).\end{aligned}$$

The Standard Examples.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{j=0}^{\infty} \frac{x^j}{j!} \qquad \text{all } x$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} x^{2j} \qquad \text{all } x$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} x^{2j+1} \qquad \text{all } x$$

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{j=0}^{\infty} \frac{x^{2j}}{(2j)!} \qquad \text{all } x$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{j=0}^{\infty} \frac{x^{2j+1}}{(2j+1)!} \qquad \text{all } x$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{j=0}^{\infty} x^j \qquad |x| < 1$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!} x^j \qquad -1 < x \leq 1$$

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{j=0}^{\infty} \frac{(-1)^{2j+1}}{2j+1} x^{2j+1} \qquad |x| \leq 1$$

$$(1+x)^r = 1 + rx + \frac{r(r-1)}{2} x^2 + \frac{r(r-1)(r-2)}{3!} x^3 + \cdots = \sum_{j=0}^{\infty} \binom{r}{j} x^j \qquad |x| < 1$$

3 Complex Numbers

Consider the following quadratic equation:

$$x^2 + 1 = 0.$$

In high school, we learn that solving equations like this one corresponds to finding the values of x for which the function $f(x) = x^2 + 1$ crosses the x-axis. Since the graph of this function $f(x)$ is a parabola sitting “above” the x-axis, it never crosses the x-axis and so there are no solutions to this equation. Indeed, $x^2 = -1$ has no real solutions since $x^2 \geq 0$ for every real number x.

But now we extend our set of numbers to allow for solutions to this equation! This may seem like a radical idea, but it is not the first time you have done this. Probably the easiest numbers to understand are positive integers. As children, we first learn this concept by counting objects. Next, we learn negative integers – perhaps by learning to accept solutions to equations like $x + 5 = 2$; $x = -3$, the “opposite of 3.” The next conceptual leap is to accept solutions to equations like $2x - 4 = 1$ as fractions: $x = \frac{5}{2}$. Finally, we accept solutions to certain algebraic equations like $x^2 + x - 1 = 0$, $x = \frac{-1 \pm \sqrt{5}}{2}$.

The picture of the *real number line* is probably what convinces us intuitively that this "makes sense," since we can point to solutions on the line and see how different numbers have different "sizes" and negatives are on the "other side" of zero, etc.

3.1 Rectangular Coordinates

Now, we introduce a picture for our new set of numbers. Instead of a line, we have the plane. Each point $(a, b) \in \mathbb{R}^2$ of the plane represents the *complex number* $a + bi$. We add (subtract) and multiply these numbers exactly like we do real numbers, with the caveat that $i^2 = -1$. For example:

$$\begin{aligned}(1 - i) + (2 + 3i) &= 1 + 2 - i + 3i \\ &= 3 + 2i. \\ (1 - i) \cdot (2 + 3i) &= 1 \cdot 2 + 1 \cdot 3i - i \cdot 2 - i \cdot 3i \\ &= 2 + i - 3i^2 = 2 + i + 3 \\ &= 5 + i.\end{aligned}$$

The entire set of complex numbers is denoted by $\mathbb{C}$, just like the set of real numbers is denoted by $\mathbb{R}$. Notice that $\mathbb{C}$ contains the set of real numbers, since $a + 0 \cdot i = a \in \mathbb{R}$. More generally, if we have a complex number $a + bi$ in $\mathbb{C}$, then we call a the *real part* of $a + bi$. We call b the *imaginary part* of $a + bi$. Graphically, these parts correspond to the x and y-components of the point $(a, b) \in \mathbb{R}^2$.

If $a + bi$ is a complex number, and $a = 0$, we say that the number is *purely imaginary.* For example, the number $5i$ is purely imaginary. Also, notice that the imaginary part of a complex number $a + bi$ is always a *real* number, $b \in \mathbb{R}$. Do not be confused by this; for example, the imaginary part of $5i$ is 5, and 5 is a real number.

What about division of complex numbers? How do we compute $\frac{1}{1-i}$, for example? Remember that division (fractions) is really just a way to find solutions to equations. In this case, the relevant equation is

$$1 = (a + bi) \cdot (1 - i).$$

So we may solve the equation

$$1 = a + bi - ai + b = (a + b) + (b - a)i$$

for a and b. At first this seems to be one equation with two unknowns; actually, there are two equations here! The real parts of both sides must be equal, as well as their imaginary parts. So we get

$$\begin{cases} 1 = a + b \\ 0 = b - a. \end{cases}$$

This system of equations has the solution $b = 1/2 = a$, so that $\frac{1}{1-i} = \frac{1}{2} + \frac{1}{2}i$.

There is a trick to computing fractions much more easily. First, we consider a complex number $a + bi$. We define the *complex conjugate* of $a + bi$ to be $a - bi$. This is the original complex number with the imaginary part multiplied by -1. Here's the point:

$$(a + bi) \cdot (a - bi) = a^2 + ab\,i - ab\,i - b^2 i^2 = a^2 + b^2 \in \mathbb{R}.$$

In words, a complex number multiplied by its conjugate is a non-negative real number. Moreover, if the original complex number is not zero, then the product is positive. Assume that $a+bi \neq 0$, so we may invert both sides of the previous equation. We get

$$\frac{1}{(a+bi)(a-bi)} = \frac{1}{a^2+b^2},$$

so that

$$\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i.$$

This shows that every non-zero complex number has a multiplicative inverse in $\mathbb{C}$.

Remarks.

- Thinking of the "picture" of $\mathbb{C}$ as points $(a,b) \in \mathbb{R}^2$, then $a^2+b^2 = |(a,b)|^2$ is just the square of the length of the line segment from $(0,0)$ to (a,b). We call the length $|(a,b)|$ of that line segment the *modulus* of $a+bi$.
- The procedure above is exactly "rationalizing the denominator." For example,

$$\begin{aligned}\frac{1}{1+\sqrt{2}} &= \frac{1}{1+\sqrt{2}} \cdot \frac{1-\sqrt{2}}{1-\sqrt{2}} \\ &= \frac{1-\sqrt{2}}{-1} = -1+\sqrt{2}.\end{aligned}$$

  We just have -1 under the radical instead of 2, since $i=\sqrt{-1}$.

Example. Simplify (compute) $\frac{2-3i}{1+i}$.
We first compute $\frac{1}{1+i} = \frac{1}{1+i} \cdot \frac{1-i}{1-i} = \frac{1-i}{2} = \frac{1}{2} - \frac{1}{2}i$. Then we multiply

$$\begin{aligned}\frac{2-3i}{1+i} &= \frac{1}{1+i}(2-3i) = (\frac{1}{2}-\frac{1}{2}i)\cdot(2-3i) \\ &= 1 - \frac{3}{2}i - i + \frac{3}{2}i^2 = (1-\frac{3}{2}) + (-\frac{3}{2}-1)i \\ &= -\frac{1}{2} - \frac{5}{2}i.\end{aligned}$$

Exercises. Show the following by direct computation.

i) $(\sqrt{2}-i) - i(1-\sqrt{2}i) = -2i$

ii) $\frac{1+2i}{3-4i} + \frac{2-i}{5i} = -\frac{2}{5}$

iii) $\frac{5}{(1-i)(2-i)(3-i)} = \frac{1}{2}i$

iv) $-1 = \frac{4}{(1-i)^4}$.

v) Show that $(a+bi)^2 = 1+i$ admits only the two solutions $\pm\left[\sqrt{\sqrt{2}+1/2} + i\sqrt{\sqrt{2}-1/2}\right]$.

3.2 Exp of a complex number.

Perhaps e^x is the most important function in calculus; so we should figure out how to compute e^{a+bi}. This means we should figure out how to write it as a complex number $a' + b'i$. We expect that it satisfies the usual rule of summing in the exponent, so that

$$e^{a+bi} = e^a(e^{bi}).$$

Since e^a is a real number, it is sufficient to compute e^{bi}. We do this using Taylor series. Indeed, we know that

$$e^x = 1 + x + x^2/2 + x^3/3! + \cdots$$

so that

$$\begin{aligned} e^{bi} &= 1 + bi + (bi)^2/2 + (bi)^3/3! + (bi)^4/4! + (bi)^5/5! + \cdots \\ &= 1 + bi - b^2/2! - b^3/3!\, i + b^4/4! + b^5/5!\, i + \cdots \\ &= (1 - b^2/2! + b^4/4! + \cdots) + (b - b^3/3! + b^5/5! + \cdots)i \\ &= \cos(b) + \sin(b)i. \end{aligned}$$

The previous formula is called *Euler's formula*; since it is very important, I will reproduce it here:

$$\boxed{e^{bi} = \cos(b) + \sin(b)\, i.}$$

It follows that $e^{a+bi} = e^a \cos(b) + e^a \sin(b)i$.

Example. Compute 2^{5i}.

We first rewrite 2^{5i} as $(e^{\ln(2)})^{5i} = e^{5\ln(2)i}$. Now we may use Euler's formula with $b = 5\ln(2)$. We get:

$$2^{5i} = e^{5\ln(2)i} = \cos(5\ln(2)) + \sin(5\ln(2))\, i.$$

Remark. This trick works perfectly well when the base, in this case 2, is a positive real number. We will stick to that case in this class.

3.3 Polar coordinates

Let us recall the polar representation of the plane $\mathbb{R}^2$. For each point $p = (a, b) \in \mathbb{R}^2$, we can associate $r = \sqrt{a^2 + b^2}$. This r is the length of the line segment $\vec{p}$ from 0 to (a, b) (or what we called the *modulus* of the complex number $a + bi$). If one draws a circle centered at the origin of radius r, then the point (a, b) will lie somewhere on that circle. But where? We measure counter-clockwise the angle that $\vec{p}$ makes with the positive x-axis, and denote that angle by θ. Basic trigonometry tells us that

$$\begin{aligned} a &= r\cos(\theta) \\ b &= r\sin(\theta). \end{aligned}$$

So the point $(a, b) \in \mathbb{R}^2$ can be specified either by its x and y-coordinates (which is how we have just specified it), or it can be specified by its modulus r and its *argument* (angle) θ. If we have a, b and want to know (r, θ), we may use the identities

$$r = \sqrt{a^2 + b^2}$$
$$\theta = \tan^{-1}(b/a).$$

Now we want to apply this polar representation of $\mathbb{R}^2$ to our set of complex numbers $\mathbb{C}$. We know that every complex number is written as $a + bi$. So in polar form, we would have

$$\begin{aligned} a + bi &= r\cos(\theta) + r\sin(\theta)i = r(\cos(\theta) + i\sin(\theta)) \\ &= re^{\theta\, i} \end{aligned}$$

using Euler's formula. Polar coordinates allows one to multiply complex numbers much more quickly than when using rectangular coordinates. For example:

$$2e^{\frac{\pi}{4}i} \cdot \frac{2}{3}e^{\frac{\pi}{4}i} = \frac{4}{3}e^{\frac{\pi}{2}i}.$$

Compare this with the same computation in rectangular coordiantes:

$$(\sqrt{2} + \sqrt{2}i) \cdot (\frac{\sqrt{2}}{3} + \frac{\sqrt{2}}{3}i) = 2/3 + 2/3i + 2/3i - 2/3 = \frac{4}{3}i.$$

More importantly, now we can conceptually understand what complex multiplication means in terms of our "picture" of $\mathbb{C}$ as the plane. Consider two complex numbers, $r_1e^{\theta_1\, i}$ and $r_2e^{\theta_2\, i}$. Then the product is

$$r_1e^{\theta_1\, i} \cdot r_2e^{\theta_2\, i} = r_1r_2e^{(\theta_1+\theta_2)i}.$$

So the argument of the product, that is the angle that the line segment from 0 to the product makes with the positive x-axis, is exactly the sum of the arguments of each factor. Also, the modulus of the product is just the product of the moduli of the factors. If we think a little bit geometrically, multiplication by a fixed complex number has the effect of scaling (by the modulus of that number) and rotating (by the argument of that number).

When we do not want to specify the coordinate system for the complex number, we will usually write z instead of $a + bi$ or $re^{i\theta}$. In all cases, z will mean a complex number, so an element in the set $\mathbb{C}$.

Exercise.

i) Find all complex numbers such that $z^3 = -1$.

ii) Find all complex z such that $z^2 = \sqrt{2}e^{i\pi/4}$. Compare with the last of the previous exercises.

Solutions.

i) $z = e^{(\pi/3)\, i}, e^{\pi\, i}, e^{(5\pi/3)\, i}$

ii) $z = 2^{1/4}e^{(\pi/8)\, i}, 2^{1/4}e^{(9\pi/8)\, i}$

Chapter 1

1^{st}-Order Differential Equations

A first order differential equation is an equation of the form

$$\frac{dy}{dt} = f(t, y), \qquad (\star)$$

where $f(t, y)$ is some function of two variables. The special case when $f(t, y)$ is a function of only t is already well known to you! In this case, the differential equation $\frac{dy}{dt} = f(t)$ is solved by integrating: $y = \int f(t)dt$. Indeed, every technique that we will learn for solving differential equations will involve integration in some way.

A *particular solution* to the differential equation $\frac{dy}{dt} = f(t, y)$ is some particular function $y_p(t)$ such that

$$y_p'(t) = f(t, y_p(t)) \qquad \text{for all } t \in I,$$

where $I = (t_0, t_1)$ is some interval. Remember that $t \in I$ means that $t_0 < t < t_1$. In other words, $y_p(t)$ satisfies ($\star$).

0.1 Examples. i) $\frac{dy}{dt} = 1 + t - y$

ii) $\frac{dy}{dt} = 1 - \frac{t}{2}y$

iii) $\frac{dy}{dt} = \frac{-t}{y}$.

Solutions.

i) We can guess that $y_p(t) = t$ is a solution for all values of t.

ii) We will be able to solve this soon; a solution may be difficult to guess at this stage.

iii) We guess that $y_p(t) = \sqrt{1 - t^2}$ works! Just check that

$$y_p'(t) = \frac{-2t}{2\sqrt{1 - t^2}} = \frac{-t}{y_p} \qquad \text{for } t \in (-1, 1).$$

This is the "algebraic" way of understanding solutions to ($\star$). Basically, we write down a possible solution and check that it works. Most of this class is dedicated to methods for finding these "formulas" for solutions. However, in general there is no closed formula to find! This leads us to a "geometric view" of solving first-order differential equations and related numerical techniques, which we pursue at the end of the chapter.

1 Separation of Variables

We are trying to solve special kinds of first order differential equations:

$$\frac{dy}{dt} = f(t, y).$$

In today's lecture, we focus on the special case when $f(t, y) = g(t) \cdot h(y)$. In words, we will assume that $f(t, y)$ can be written as the product of two functions, one of which depends only on t and the other depends only on y. In this case, the solution is as follows:

$$\begin{aligned} \frac{dy}{dt} &= g(t)h(y) \\ \text{rewriting:} \quad \frac{dy}{h(y)} &= g(t)dt \\ \text{integrating:} \quad \int \frac{dy}{h(y)} &= \int g(t)dt + C. \end{aligned}$$

Normally (i.e. for the examples chosen in class), these integrals will be possible to compute using the elementary methods that we learn in calculus. Sometimes it is possible to solve the resulting equation for y *explicitly*; when this is possible, you should do it!

1.1 Examples. Solve the initial value problem (IVP):

$$\begin{cases} \frac{dy}{dx} = 2xy^2 \\ y_1(2) = 1. \end{cases}$$

1.2 Solution.

Observe that the right hand side (RHS) can be written as $(2x)(y^2)$. This is a product of a function depending only on x, with a function depending only on y. Thus, we can separate the variables:

$$\begin{aligned} \frac{dy}{dx} &= (2x)(y^2) \\ \text{rewriting:} \quad \frac{dy}{y^2} &= 2xdx \\ \text{integrating:} \quad -\frac{1}{y} &= x^2 + C. \end{aligned}$$

We can solve for C using the initial value: $y = 1$ when $x = 2$. Thus

$$-\frac{1}{1} = 2^2 + C \qquad \text{so that} \qquad C = -5.$$

Solving for y, we have

$$y(x) = \frac{1}{5 - x^2} \qquad \text{for } -\sqrt{5} < x < \sqrt{5}.$$

Note. The restrictions on the domain of our solution were evident only *after* we found the explicit form of the solution. Be prepared to specify the domain of the function - it is important to know for which values of x the solution makes sense. **Solution.**

1.3 Examples. Solve the differential equation

$$\cos(y) \cdot y' = 1.$$

1.4 Solution.

$$\begin{aligned} & \cos(y)\frac{dy}{dt} = 1 \\ \text{rewriting:} \quad & \cos(y)dy = dt \\ \text{integrating:} \quad & \sin(y) = t + C. \end{aligned}$$

Since there is no initial value, we cannot choose C. However, we can solve for y:

$$y(t) = \sin^{-1}(t + C).$$

Check. Do you remember how to find the derivative of an inverse function?

1.5 Proposition. If $y = f^{-1}(x)$, then

$$y'(x) = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}.$$

Proof. Since $y = f^{-1}(x)$, we have $f(y) = f \circ f^{-1}(x) = x$. Now differentiate implicitly:

$$f'(y)\frac{dy}{dx} = 1,$$

so that $\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$. □

Now we compute $y'(t)$ from the problem above. We have:

$$y'(t) = \frac{1}{\cos(y)},$$

so that $\cos(y) \cdot y'(t) = 1$. This checks that $y(t)$ is a solution!

Exercises.

i) $2(y - 1)y' = e^x$

ii) $3y^2y' = (1 + y^3)\cos(x)$

iii) $\cos^2(x)y' = y^2(y - 1)\sin(x)$

Hints/Solutions.

i) Separate, and get that

$$\int (y - 1)dy = \frac{1}{2}\int e^x dx.$$

ii) Separate, and get that

$$3\int \frac{y^2}{1+y^3}dy = \int \cos(x)dx.$$

iii) Separate, and get that

$$\int \frac{dy}{y^2(y-1)} = \int \frac{\sin(x)}{\cos^2(x)}dx.$$

Use partial fractions for the integral on the left. Substitute $u(x) = \cos(x)$ for the integral on the right.

1.1 Substitution.

Consider the following differential equation:

$$\frac{dy}{dx} = \frac{y}{x} - e^{\frac{y}{x}}.$$

It is impossible to write the RHS as a product of $g(x)h(y)$. So it seems that our method of separation of variables will not work! This is a problem.

However, we notice that the quantity $\frac{y}{x}$ appears twice on the RHS. It might seem intuitive to try to substitute $u = \frac{y}{x}$, and see what happens. Equivalently,

$$\begin{aligned} \text{Substitution:} \quad & x \cdot u = y \\ \text{Implicit Diff:} \quad & u + x\frac{du}{dx} = \frac{dy}{dx}. \end{aligned}$$

Thus:

$$u + x\frac{du}{dx} = \frac{dy}{dx} = \frac{y}{x} - e^{\frac{y}{x}} = u - e^u.$$

This shows that we have simplified our problem! We only need to solve now the differential equation

$$x\frac{du}{dx} = -e^u,$$

which is separable! Indeed,

$$\begin{aligned} \text{rewriting:} \quad & e^{-u}du = \frac{-1}{x}dx \\ \text{integrating:} \quad & -e^{-u} = -\ln|x| + C. \end{aligned}$$

Now we solve this equation for u:

$$u = -\ln(\ln|x| - C),$$

and since $u = y/x$,

$$y(x) = -x\ln(\ln|x| - C).$$

When the substitution $u = y/x$ works, we say that the differential equation is *homogeneous*. To check this, we can just check that multiplying x and y (both) by any number α does not change the RHS.

1.6 Examples. Solve the differential equation

$$\frac{dy}{dx} = 2x - 5y.$$

1.7 Solution. This is not separable, nor is it homogeneous. But let's try a substitution, say

$$v = 2x - 5y.$$

It follows that $\frac{dv}{dx} = 2 - 5\frac{dy}{dx}$; thus

$$\frac{dv}{dx} = 2 - 5\,(2x - 5y) = 2 - 5v.$$

This is separable!

$$\begin{aligned} \text{rewriting:} \quad & \frac{dv}{2-5v} = dx \\ \text{integrating:} \quad & \frac{-1}{5}\ln|2-5v| = x + C. \end{aligned}$$

Solving:

$$\begin{aligned} \ln|2-5v| &= C' - 5x \\ \ln|2-5(2x-5y)| &= C' - 5x \\ 2 - 10x + 25y &= C'' e^{-5x} \end{aligned}$$

Solving for y:

$$y(x) = \frac{C''}{25}e^{-5x} - \frac{2}{25} + \frac{2x}{5}.$$

Exercise. Solve the differential equations using a substitution.

i) $x^2y' = y^2 + 2xy$

ii) $xy' = y + (x^2 + y^2)^{1/2}$

iii) $y' = e^{2y-x}$.

Hints/Solutions.

i) Normalize. Get $\frac{dy}{dx} = (y/x)^2 + 2(y/x)$. Substitute $u = y/x$. Obtain

$$\begin{aligned} u + x\frac{du}{dx} &= u^2 + 2u \\ \frac{du}{dx} &= \frac{1}{x}\left(u(u+1)\right). \end{aligned}$$

This is separable, and the u-integral can be done using partial fractions.

ii) Normalize. Get $\frac{dy}{dx} = (y/x) + (1 + (y/x)^2)^{1/2}$. Substitute $u = y/x$. Obtain

$$\begin{aligned} u + x\frac{du}{dx} &= (y/x) + \left(1 + (y/x)^2\right)^{1/2} = u + (1+u^2)^{1/2} \\ \frac{du}{dx} &= \frac{1}{x}(1+u^2)^{1/2}. \end{aligned}$$

This is separable, and the u-integral can be done by trig substitution. You may use the following fact:

$$\int \sec^3\theta = \frac{1}{2}\left[\sec(\theta)\tan(\theta) + \ln|\sec(\theta) + \tan(\theta)|\right].$$

iii) Substitute $u = 2y - x$. Then $\frac{du}{dx} = 2\frac{dy}{dx} - 1$. So

$$\frac{du}{dx} = 2\left(e^u\right) - 1.$$

This is separable. Multiply the numerator and denominator of the u-integral by e^{-u}, so that integral can be done by substitution.

1.2 Advanced.

We will walk through an example where the substitution is not very obvious.

1.8 Examples.

$$xy' = \sqrt{x}y^2 + \frac{y}{2}.$$

First, we want to normalize our differential equation, so we divide both sides by x.

$$\frac{dy}{dx} = \frac{y^2}{\sqrt{x}} + \frac{y}{2x}.$$

Go ahead and check that the RHS cannot be factored as $g(x)h(y)$. First, we might try the substitution, $u = y/x$. We obtain, as before:

$$u + x\frac{du}{dx} = \frac{dy}{dx} = \frac{y^2}{\sqrt{x}} + \frac{y}{2x} = x^{3/2}u^2 + \frac{u}{2}.$$

Simplifying, we have the differential equation

$$\frac{du}{dx} = \sqrt{x}u^2 - \frac{u}{2x}.$$

But look - the RHS still does not separate!

1.9 Solution. Now, let's try the substitution $v = y^2/x$. Then $xv = y^2$, and differentiating implicitly,

$$v + x\frac{dv}{dx} = 2y\frac{dy}{dx}.$$

So we have that

$$\begin{aligned} v + x\frac{dv}{dx} &= 2y\frac{dy}{dx} = 2y\left(\frac{y^2}{\sqrt{x}} + \frac{y}{2x}\right) \\ &= \frac{2y^3}{\sqrt{x}} + \frac{y^2}{x} = \frac{2y^3}{\sqrt{x}} + v. \end{aligned}$$

Simplifying, we have the differential equation

$$\frac{dv}{dx} = 2\frac{y^3}{x^{3/2}} = 2\frac{(xv)^{3/2}}{x^{3/2}} = 2v^{3/2}.$$

Finally, the RHS separates. So we have that

$$v^{-3/2}dv = 2dx,$$

and integrating we find that

$$\frac{1}{\sqrt{v}} = C - x,$$

so that $y(x) = \frac{\sqrt{x}}{C-x}$.

Homework Problems

These problems are taken from section §2.2 of the textbook.

1. Solve the given differential equations.

B4) $y' = (3x^2 - 1)/(3 + 2y)$

B5) $y' = (\cos^2 x)(\cos^2 2y)$

B6) $xy' = (1 - y^2)^{1/2}$

2 (B12). Consider the differential equation $\frac{dr}{d\theta} = r^2/\theta$, where $r(1) = 2$.

a) Find the solution of the given initial value problem explicitly.

b) Plot the graph of the solution.

c) Determine (at least approximately) the interval in which the solution is defined.

3 (B30). Consider the equation

$$\frac{dy}{dx} = \frac{y - 4x}{x - y}.$$

a) Show that this equation can be rewritten as

$$\frac{dy}{dx} = \frac{(y/x) - 4}{1 - (y/x)}.$$

b) Introduce a new dependent variable v so that $v = y/x$, or $y = xv$. Express $\frac{dy}{dx}$ in terms of x, v, and $\frac{dv}{dx}$.

c) Replace y and $\frac{dy}{dx}$ in part (a) with the expressions in part (b) that involve v and $\frac{dv}{dx}$. Show that the resulting diff-eq is

$$v + x\frac{dv}{dx} = \frac{v-4}{1-v},$$

or

$$x\frac{dv}{dx} = \frac{v^2-4}{1-v}.$$

Observe that this is separable.

d) Solve this separable equation, obtaining v implicitly in terms of x.

e) Find the solution of the original equation by replacing v with y/x in the previous solution.

2 Integrating Factors

Consider a generic first order differential equation:

$$\frac{dy}{dt} = f(t, y).$$

In today's lecture, we focus on the case when $f(t, y) = m(t) \cdot y + b(t)$. Since we call $y(x) = mx + b$ a linear function of x, we call these differential equations *linear* equations. Notice that the coefficients m and b can depend on the independent variable, t.

Before reading on you should compare this form with the form from the last lecture. In that lecture, the equations had the form

$$\frac{dy}{dt} = g(t)h(y).$$

Remember what we did: we divide by $h(y)$ and integrate both sides with respect to t. This gives us an implicit equation for the function $y(t)$, in general.

Now, consider the following linear differential equation:

$$x^2\frac{dy}{dx} = -2xy + \cos(x).$$

This is not a separable equation, but we can rewrite it as

$$x^2\frac{dy}{dx} + 2xy = \cos(x).$$

Here's the **key observation** for this entire section: the left hand side is exactly the derivative of $x^2 \cdot y$. Indeed,

$$\frac{d}{dx}\left(x^2y\right) = \frac{d(x^2)}{dx}y + \frac{dy}{dx}x^2 = x^2\frac{dy}{dx} + 2xy.$$

This allows us to solve the differential equation, again by integrating:

$$\begin{aligned} \text{original:} \quad & x^2\frac{dy}{dx} = -2xy + \cos(x) \\ \text{rewrite:} \quad & x^2\frac{dy}{dx} + 2xy = \cos(x). \\ \text{simplify:} \quad & \frac{d}{dx}\left(x^2 y\right) = \cos(x) \\ \text{integrate:} \quad & x^2 y = \sin(x). \end{aligned}$$

So we have a solution: $y = \frac{\sin(x)}{x^2}$.

Before moving on, compare this method to separating variables. In both cases, we reduce the problem to integration.

2.1 Examples. Solve the differential equation

$$\frac{dy}{dx} = 2xy + e^{x^2}.$$

2.2 Solution. First of all, notice that the right hand side (RHS) of the differential equation does not separate. Secondly, the RHS is not homogeneous, and no other substitution seems likely to work. Let's try the trick of the motivating example.

$$\begin{aligned} \text{original:} \quad & \frac{dy}{dx} = 2xy + e^{x^2} \\ \text{rewrite:} \quad & \frac{dy}{dx} - 2xy = e^{x^2} \end{aligned}$$

But now we are stuck, since $\frac{dy}{dx} - 2xy$ is not the derivative of a function times y, like before. But let's do something ingenious: multiply the equation by $\mu(x) = e^{-x^2}$: Let's see what happens:

$$\begin{aligned} \text{multiply:} \quad & e^{-x^2}\frac{dy}{dx} = (2xe^{-x^2})y + e^{-x^2}e^{x^2} \\ \text{rewrite:} \quad & e^{-x^2}\frac{dy}{dx} - 2xe^{-x^2}y = 1. \\ \text{simplify:} \quad & \frac{d}{dx}\left(e^{-x^2}y\right) = 1 \\ \text{integrate:} \quad & e^{-x^2}y = x + C. \end{aligned}$$

So that $y(x) = xe^{x^2} + Ce^{x^2} = (x + C)e^{x^2}$.

2.3 Definition. An *integrating factor* is a function $\mu(x)$ which, when multiplied by a differential equation, allows us to write the left hand side as the derivative of $\mu(x) \cdot y$.

Given a linear differential equation, we have just seen how to find the solution if we can find an integrating factor. In the next example, we will see how to find one.

2.4 Examples. Solve the differential equation

$$\frac{dy}{dx} = y + \sin(x).$$

2.5 Solution. Let's first rewrite the differential equation:

$$\frac{dy}{dx} - y = \sin(x).$$

We want to multiply by some $\mu(x)$, an integrating factor, so that

$$\begin{aligned} \mu(x) \cdot LHS = \mu(x)\frac{dy}{dx} - \mu(x)y &= \frac{d}{dx}\left(\mu(x) \cdot y\right) \\ &= \mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y. \end{aligned}$$

Simplifying, we see the separable differential equation

$$\frac{d\mu}{dx}y = -\mu(x)y.$$

We solve this and find that $\mu(x) = e^{-x}$. Thus we have found our *integrating factor*! We use the same process as before:

$$\begin{aligned} \text{Original:} \quad & \frac{dy}{dx} = y + \sin(x) \\ \text{multiply:} \quad & e^{-x}\frac{dy}{dx} = e^{-x}y + e^{-x}\sin(x) \\ \text{rewrite:} \quad & e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}\sin(x) \\ \text{simplify:} \quad & \frac{d}{dx}\left(e^{-x}y\right) = e^{-x}\sin(x) \\ \text{integrate:} \quad & e^{-x}y = \int e^{-x}\sin(x)dx + C. \end{aligned}$$

To finish this problem, we need to calculate $\int e^{-x}\sin(x)dx$. We do integration by parts once with $u = e^{-x}$, $dv = \sin(x)dx$, and again with $u = e^{-x}$, $dv = \cos(x)dx$ to get

$$\begin{aligned} \int e^{-x}\sin(x)dx &= -e^{-x}\cos(x) - \int(-\cos(x))(-e^{-x})dx \\ &= -e^{-x}\cos(x) - \int e^{-x}\cos(x)dx \\ &= -e^{-x}\cos(x) - \left[e^{-x}\sin(x) - \int(\sin(x))(-e^{-x})dx\right] \\ &= -e^{-x}(\cos(x) + \sin(x)) - \int e^{-x}\sin(x)dx. \end{aligned}$$

If you haven't seen this before, it might seem like this was for nothing. But there's a trick! We add the integral $\int e^{-x}\sin(x)dx$ to both sides, and divide by 2. Then we have

$$\int e^{-x}\sin(x)dx = -\frac{e^{-x}}{2}(\cos(x) + \sin(x)).$$

Finally,

$$\begin{aligned} y(x) &= e^x \int e^{-x} \sin(x) dx + Ce^x \\ &= -\frac{\cos(x) + \sin(x)}{2} + Ce^x. \end{aligned}$$

General Solution

The general linear first order equation has the form

$$\frac{dy}{dx} = m(x)y + b(x).$$

We will rewrite this as:

$$\frac{dy}{dx} + p(x)y = g(x),$$

where $p(x) = -m(x)$ and $g(x) = b(x)$. We go through the same reasoning as we did in the previous example. First,

$$\begin{aligned} \text{multiply:} \quad & \mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)g(x) \\ \text{rewrite:} \quad & \mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)g(x). \end{aligned}$$

Then we want the LHS to be the derivative of $(\mu(x) \cdot y)$:

$$\mu(x)\frac{dy}{dx} + y\frac{d\mu}{dx} = \mu(x)\frac{dy}{dx} + \mu(x)p(x)y.$$

This is the case provided that

$$\frac{d\mu}{dx} = \mu(x)p(x).$$

This differential equation is separable, and we find that

$$\mu(x) = e^{\int p(x)dx}.$$

Finally:

$$\begin{aligned} \text{Original:} \quad & \frac{dy}{dx} = -p(x)y + g(x) \\ \text{multiply:} \quad & \mu(x)\frac{dy}{dx} = -\mu(x)p(x)y + \mu(x)g(x) \\ \text{rewrite:} \quad & \mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)g(x) \\ \text{simplify:} \quad & \frac{d}{dx}\left(\mu(x)y\right) = \mu(x)g(x) \\ \text{integrate:} \quad & \mu(x)y = \int \mu(x)g(x)dx + C. \end{aligned}$$

Thus, we get a *formula* for the solutions of this class of problems:

$$y(x) = \frac{1}{\mu(x)}\left(\int \mu(x)g(x)dx + C\right)$$
$$\text{when } \mu(x) = e^{\int p(x)dx}.$$

2.6 Examples. Use the formula above to solve the differential equation

$$\frac{dy}{dt} = 0.5 - t + 2y.$$

*2.7 Solution*Step 0 Normalize! We have

$$\frac{dy}{dt} = 2y + (0.5 - t),$$

so that $p(t) = -2$, and $g(t) = 0.5 - t$.

Step 1 Find an integrating factor! $\mu(t) = e^{\int p(t)dt}$: we have that $\int -2dt = -2t + C$, so we may choose

$$\mu(t) = e^{-2t}.$$

Step 2 Find $\int \mu(t)g(t)dt$. We compute $\int e^{-2t}(\frac{1}{2} - t)dt$ by parts. Let $u = \frac{1}{2} - t$, and $dv = e^{-2t}dt$. Then

$$\begin{aligned}\int e^{-2t}\left(\frac{1}{2} - t\right)dt &= \left(\frac{1}{2} - t\right)\frac{e^{-2t}}{-2} - \int \frac{e^{-2t}}{-2}(-1)dt \\ &= e^{-2t}\left(\frac{t}{2} - \frac{1}{4} + \frac{1}{4}\right) = t/2e^{-2t}.\end{aligned}$$

Step 3 Solve! We have that

$$\begin{aligned}y(t) &= \frac{1}{\mu(t)}\left(\int \mu(t)g(t)dt + C\right) \\ &= \frac{1}{e^{-2t}}\left(t/2e^{-2t} + C\right) \\ &= t/2 + Ce^{2t}.\end{aligned}$$

Exercises. Use the formula for the general solution to solve the following equations.

i) $xy' - 4y = x^3$

ii) $y' = y + 4x$

iii) $y' + 2xy = 2x$

Hints/Solutions.

i) Normalize, and find that

$$\frac{dy}{dx} = \frac{4}{x}y + x^2.$$

So $\mu(x) = e^{\int p(x)dx} = e^{-4\ln|x|} = \frac{1}{x^4}$. Then

$$\begin{aligned}y(x) &= x^4\left(\int x^{-2}dx + C\right) \\ &= -x^3 + Cx^4.\end{aligned}$$

ii) We have that $\mu(x) = e^{\int -1 dx} = e^{-x}$. Then

$$\begin{aligned} y(x) &= e^{x}\left(\int e^{-x}(4x)dx + C\right) \\ &= e^{x}\left(-4xe^{-x} + 4\int e^{-x}dx + C\right) \\ &= e^{x}\left(-4(x+1)e^{-x} + C\right) \\ &= -4(x+1) + Ce^{x}. \end{aligned}$$

iii) Since $p(x) = 2x$, we have $\mu(x) = e^{x^2}$. Then

$$\begin{aligned} y(x) &= e^{-x^2}\left(\int e^{x^2}(2x)dx + C\right) \\ &= e^{-x^2}\left(e^{x^2} + C\right) \\ &= 1 + Ce^{-x^2}. \end{aligned}$$

(Also, note that this equation is separable.)

Substitution

Sometimes, equations which are not linear can become linear after a substitution. Let's look at an example:

2.8 Examples. Solve the differential equation

$$xe^{y}y' - e^{y} = 3x^2.$$

2.9 Solution. We guess that $u = e^{y}$ is a good substitution. Observe then that

$$\frac{du}{dx} = e^{y}\frac{dy}{dx} = u\frac{dy}{dx}.$$

Then the differential equation becomes

$$\begin{aligned} xu\frac{dy}{dx} - u &= 3x^2 \\ x\frac{du}{dx} - u &= 3x^2. \\ \frac{du}{dx} &= \frac{1}{x}u + 3x. \end{aligned}$$

This equation is now linear! We have $p(x) = -\frac{1}{x}$. Then $\mu(x) = e^{\int p(x)} = \frac{1}{x}$. Thus,

$$\begin{aligned} u(x) &= x\left(\int \frac{1}{x}(3x)dx + C\right) \\ &= x\left(3x + C\right). \end{aligned}$$

Finally, we substitute back for y, and get

$$y = \ln(x(3x + C)).$$

Exercises. Find a substitution for the following equations.

i) $\frac{1}{y^2+1}y' + \frac{2}{t}\tan^{-1} y = \frac{2}{t}$.

ii) $y' - \frac{1}{x+1}y \cdot \log(y) = (x+1)y$.

iii) $y' = -p(x)y + q(x)y^5$. Examples of this type are *Bernoulli equations*.

Solutions.

i) Let $u = \tan^{-1}(y)$. Then $\frac{du}{dx} = \frac{1}{1+y^2}\frac{dy}{dx}$, so the differential equation is

$$\frac{du}{dt} + \frac{2}{t}u = \frac{2}{t}.$$

This is both linear *and* separable.

ii) Let $u = \log(y)$. Then $\frac{du}{dx} = \frac{1}{y}\frac{dy}{dx}$. Then the differential equation is

$$y\frac{du}{dx} - \frac{1}{x+1}y \cdot u = (x+1)y.$$

Divide by y, and obtain a linear equation in u.

iii) Let $u = y^{1-5} = \frac{1}{y^4}$. Then $\frac{du}{dx} = \frac{-4}{y^5}\frac{dy}{dx}$. So we divide the original equation by y^5 and multiply by -4, to get

$$\frac{-4}{y^5}\frac{dy}{dx} = \frac{4p(x)}{y^4} - 4q(x)$$
$$\frac{du}{dx} = 4p(x)u - 4q(x).$$

This is a linear equation in u.

Homework Problems

These problems are taken from section §2.1 of the textbook.

4. Solve the following problems using the method of integrating factors.

B4) $y' + (1/t)y = 3\cos 2t$

B7) $y' + 2ty = 2te^{-t^2}$

B10) $ty' - y = t^2e^{-t}$

5 (B16). Find the solution to the initial value problem

$$y' + (2/t)y = (\cos t)/t^2, \qquad y(\pi) = 0, \qquad t > 0$$

6 (23). Consider the initial value problem

$$3y' - 2y = e^{-\pi t/2}, \qquad y(0) = a.$$

How does the behavior of the solution, as t becomes large, depend on the choice of the initial value a? Let a_0 be the value of a for which the solution transitions from one type of behavior to the other. Determine a_0.

3 Exact Equations

As usual, we are trying to solve special kinds of first order differential equations:

$$\frac{dy}{dt} = f(t, y).$$

In today's lecture, we focus on the case when $f(t, y) = -\frac{M(t,y)}{N(t,y)}$. Observe that we can rewrite this equation:

$$\begin{aligned} \frac{dy}{dt} &= -\frac{M(t, y)}{N(t, y)} \\ N(t, y)\frac{dy}{dt} &= -M(t, y) \\ M(t, y) + N(t, y)\frac{dy}{dt} &= 0. \end{aligned}$$

We will work with this last form in today's lecture, because it is most convenient.
Now consider the following differential equation:

$$\frac{dy}{dx} = -\frac{1}{3}\cot(x) \cdot y + \frac{1}{3}\csc(x) \cdot y^{-2}.$$

This is not a separable equation; it is also not linear because of the appearance of y^{-2}; but we can rewrite it as

$$\begin{aligned} \frac{dy}{dx} &= -\frac{1}{3}\cot(x) \cdot y + \frac{1}{3}\csc(x) \cdot y^{-2} \\ \frac{dy}{dx} &= \frac{1 - \cos(x) \cdot y^3}{3\sin(x) \cdot y^2} \\ 3\sin(x) \cdot y^2\frac{dy}{dx} &= 1 - \cos(x) \cdot y^3 \\ \cos(x)y^3 + 3\sin(x) \cdot y^2\frac{dy}{dx} &= 1. \end{aligned}$$

Why would we rewrite it like this? Here's the **key observation** for this section: the left hand side is exactly the derivative of $\sin(x)y^3$. Indeed,

$$\frac{d}{dx}\left(\sin(x)y^3\right) = \frac{d(\sin(x))}{dx}y^3 + \sin(x)\frac{d(y^3)}{dx} = \cos(x)y^3 + 3\sin(x) \cdot y^2\frac{dy}{dx}.$$

This allows us to solve the differential equation, again by integrating:

$$\begin{aligned} \text{original:} \quad & \frac{dy}{dx} = -\frac{1}{3}\cot(x) \cdot y - \frac{1}{3}\sec(x) \cdot y^{-2} \\ \text{rewrite:} \quad & \cos(x)y^3 + 3\sin(x) \cdot y^2\frac{dy}{dx} = 1 \\ \text{simplify:} \quad & \frac{d}{dx}\left(\sin(x)y^3\right) = 1 \\ \text{integrate:} \quad & \sin(x)y^3 = x + C \end{aligned}$$

So we have a solution: $y = (\csc(x)(x + C))^{1/3}$.

Before moving on, compare this method to linear equations. In both cases, we want to reduce the LHS to the situation where it is $\frac{d}{dx}(\phi(x, y))$, where $\phi(x, y)$ is some function of x and y. In the case of linear equations, we always have that $\phi(x, y) = \mu(x)y$. In the previous example, we found $\phi(x, y) = \sin(x)y^3$, which allowed us to solve the problem by integrating.

Also, observe that the most useful form for us to work with in this problem was

$$\cos(x)y^3 + 3\sin(x) \cdot y^2 \frac{dy}{dx} = 1$$

which is in the "standard form":

$$M(x, y) + N(x, y)\frac{dy}{dt} = 0$$

for $M(x, y) = \cos(x)y^3 - 1$, and $N(x, y) = 3\sin(x)y^2$. The **key property** of M and N is the following:

$$\frac{\partial M}{\partial y} = 3\cos(x)y^2 = \frac{\partial N}{\partial x}.$$

The key observation suggests the following

3.1 Definition. An *exact* differential equation is one of the form

$$\frac{d\phi(x, y)}{dx} = 0.$$

In such a case, the function $\phi(x, y)$ is called a *potential function*.

3.2 Note. Potential functions are only determined up to a constant.

By implicitly differentiating, we can always write

$$\frac{d\phi(x, y)}{dx} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = M(x, y) + N(x, y)\frac{dy}{dx} = 0.$$

This tells us precisely how M and N depend on ϕ, the potential function. Indeed, we see that

$$\begin{cases} M(x, y) = \frac{\partial \phi}{\partial x} \\ N(x, y) = \frac{\partial \phi}{\partial y} \end{cases}$$

This is how we can derive the key property. Since $\frac{\partial \phi}{\partial y \partial x} = \frac{\partial \phi}{\partial x \partial y}$, we get

$$\frac{\partial M}{\partial y} = \frac{\partial \phi}{\partial y \partial x} = \frac{\partial \phi}{\partial x \partial y} = \frac{\partial N}{\partial x};$$

this was the key property of M and N in the example above. What we have just shown is that given any first-order differential equation $M(x, y) + N(x, y)\frac{dy}{dx} = 0$, if that equation is exact then it must be true that

$$M_y = N_x.$$

The amazing fact is that the *converse* of this statement is true:

3.3 Theorem (Exactness Test)**.** *Suppose that* $M(x,y)$*,* $N(x,y)$ $M_y(x,y)$ *and* $N_x(x,y)$ *are all continuous in some rectangle* R *in the xy-plane. Then*

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is exact, i.e. has a potential function $\phi(x,y)$ *such that*

$$\frac{d\phi(x,y)}{dx} = M(x,y) + N(x,y)\frac{dy}{dx},$$

if and only if

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} \text{ for all points } (x,y) \in R.$$

3.4 Examples. Solve the differential equation

$$y + (x+2y)\frac{dy}{dx} = 0.$$

3.5 Solution. First, we check to see if it is separable or linear, but it is not. Then we check to see if it is exact using the exactness test. Here,

$$M(x,y) = y \qquad N(x,y) = x + 2y,$$

so that we have

$$M_y = 1 = N_x.$$

So the equation is exact! But how do we find the potential function, $\phi(x,y)$? We know that

$$\begin{cases} \frac{\partial\phi}{\partial x} = M(x,y) = y \\ \frac{\partial\phi}{\partial y} = N(x,y) = x + 2y. \end{cases}$$

If we integrate the first equation with respect to x, we have that

$$\phi(x,y) = xy + h(y),$$

where $h(y)$ is just some function depending only on y. To determine what $h(y)$ is explicitly, we take the partial derivative with respect to y and compare with the second equation. Indeed,

$$\frac{\partial\phi}{\partial y} = x + \frac{dh}{dy} = x + 2y = N(x,y).$$

This tells us that $\frac{dh}{dy} = 2y$, so that $h(y) = y^2 + C$. To be concrete, we choose $C = 0$. With the potential in hand, we can solve:

$$\begin{aligned} \text{original:} \quad & y + (x+2y)\frac{dy}{dx} = 0 \\ \text{simplify:} \quad & \frac{d(xy+y^2)}{dx} = 0 \\ \text{integrate:} \quad & xy + y^2 = C. \end{aligned}$$

In this case, we can solve for y explicitly: $y = -x/2 \pm \frac{1}{2}\sqrt{x^2 + 4C}$.

Alternate Solution. There is not only one way to solve a differential equation. In this case, we can do the following:

$$\begin{aligned}
\text{original:} \quad & y + (x+2y)\frac{dy}{dx} = 0 \\
\text{simplify:} \quad & \frac{dy}{dx} = \frac{-y}{x+2y} = \frac{-\left(\frac{y}{x}\right)}{1+2\left(\frac{y}{x}\right)}
\end{aligned}$$

and now we try the substition $u = \frac{y}{x}$, since the RHS above is **homogeneous**. We get:

$$\begin{aligned}
\text{substitute:} \quad & u + x\frac{du}{dx} = \frac{-u}{1+2u} \\
\text{rewrite:} \quad & x\frac{du}{dx} = \frac{-u}{1+2u} - u = -\frac{2u(u+1)}{2u+1}. \\
\text{separate:} \quad & \frac{2u+1}{2u(u+1)}du = -\frac{1}{x}dx \\
\text{part. fract.:} \quad & \left(\frac{1}{2u} + \frac{1/2}{u+1}\right) du = -\ln|x|dx \\
\text{integrate:} \quad & \frac{1}{2}\left(\ln|u(u+1)|\right) = -\ln|x| + C_1. \\
\text{mult by 2 and exp:} \quad & |u^2+u| = C_2|x|^{-2} \\
\text{sub back:} \quad & |u^2+u| = \left|\frac{y^2+xy}{x^2}\right| = C_2|x|^{-2} \\
\text{multiply by } x^2\text{:} \quad & |y^2+xy| = C_2.
\end{aligned}$$

A lot more work!!

An Algorithm

If $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ is an exact equation, then we can find its potential function $\phi(x,y)$ as follows:

Step 0 Be sure that the equation is exact! Check this! If it isn't exact, this won't work.

Step 1 Let $\phi(x,y) = \int M(x,y)dx + h(y)$. (Compute the integral, leave $h(y)$ alone).

Step 2 Differentiate this expression with respect to y, and compare with $N(x,y)$:

$$N(x,y) = \frac{\partial \phi(x,y)}{\partial y} = \frac{\partial}{\partial y}\left(\int M(x,y)dx\right) + \frac{dh}{dy}.$$

This yields a separable differential equation in h, which can be solved by integrating. Once you have found $h(y)$, write down

$$\phi(x,y) = \int M(x,y)dx + h(y)$$

explicitly.

Step 3 The original differential equation is now:

$$\frac{d}{dx}\phi(x,y) = 0,$$

so the general solution is given implicitly: $\phi(x,y) = C$. If possible, solve for y explicitly.

Exercises.

i) $e^y - x + (xe^y - e^{2y})\frac{dy}{dx} = 0$.

ii) $xy^2 + (x^2y - y^3)\frac{dy}{dx} = 0$.

iii) $2x^2 + y + (x + y^2)\frac{dy}{dx} = 0$.

Hints/Solutions.

i) We first check that it is exact:

$$\frac{\partial}{\partial x}\left(xe^y - e^{2y}\right) = e^y = \frac{\partial}{\partial y}\left(e^y - x\right).$$

Then we integrate $\int (e^y - x)dx = xe^y - \frac{x^2}{2} + h(y)$. Then we compute

$$\frac{\partial}{\partial y}\left(xe^y - \frac{x^2}{2} + h(y)\right) = xe^y + \frac{dh}{dy} = xe^y - e^{2y}.$$

It follows that $h(y) = -\frac{1}{2}e^{2y}$, so that

$$\phi(x,y) = xe^y - \frac{x^2}{2} - \frac{1}{2}e^{2y},$$

and we have

$$\frac{d\phi}{dx} = 0$$

so $xe^y - \frac{x^2}{2} - \frac{1}{2}e^{2y} = C$. Solve for e^y using the quadratic formula, and then solve for y by taking logs.

ii) We first check that it is exact:

$$\frac{\partial}{\partial x}\left(x^2y - y^3\right) = 2xy = \frac{\partial}{\partial y}\left(xy^2\right).$$

Then we integrate $\int xy^2 dx = \frac{1}{2}x^2y^2 + h(y)$. Differentiating:

$$\frac{\partial}{\partial y}\left(\frac{1}{2}x^2y^2 + h(y)\right) = x^2y + \frac{dh}{dy} = x^2y - y^3,$$

so that $\frac{dh}{dy} = -y^3$. It follows that $h(y) = -\frac{1}{4}y^4$. Thus,

$$\phi(x,y) = \frac{1}{2}x^2y^2 - \frac{1}{4}y^4,$$

and the general solution is given by $\phi(x,y) = C$. One can use the quadratic formula to solve for y^2, and then solve for y by taking square roots.

iii) We first check that it is exact:

$$\frac{\partial}{\partial x}\left(x+y^2\right)=1=\frac{\partial}{\partial y}\left(2x^2+y\right).$$

So we integrate $\int(2x^2+y)dx=\frac{2}{3}x^3+xy+h(y)$, and differentiating:

$$\frac{\partial}{\partial y}\left(\frac{2}{3}x^3+xy+h(y)\right)=x+\frac{dh}{dy}=x+y^2.$$

So $h(y)=\frac{1}{3}y^3$, and $\phi(x,y)=\frac{2}{3}x^3+xy+\frac{1}{3}y^3$. The general solution is given by

$$\phi(x,y)=\frac{2}{3}x^3+xy+\frac{1}{3}y^3=C,$$

But this is difficult to solve for y explicitly, so we do not. (Note: it *is* possible using the *cubic formula*).

Integrating Factors*

3.6 Definition. An *integrating factor* $\mu(x,y)$ is a function of two variables such that

$$\mu(x,y)M(x,y)+\mu(x,y)N(x,y)=0$$

is an exact equation.

By the exactness test, it is equivalent that μ satisfies

$$\frac{\partial}{\partial y}\left(\mu(x,y)M(x,y)\right)=\frac{\partial}{\partial x}\left(\mu(x,y)N(x,y)\right);$$

or equivalently using the product rule,

$$\mu_yM+\mu M_y=\mu_xN+\mu N_x.$$

We will rewrite this for future use as

$$\mu_yM-\mu_xN=(N_x-M_y)\mu. \tag{1.1}$$

For an interesting example, let's return to the differential equation from section 1.1:

$$xy'=y^2\sqrt{x}+\frac{y}{2}.$$

It is not required that you have seen the substitution technique we used to solve it previously. We give here another way of solving it.

First, let's put it in standard form:

$$\text{original:}\quad x\frac{dy}{dx}=y^2\sqrt{x}+\frac{y}{2}$$

$$\text{rewrite:}\quad -(y^2\sqrt{x}+\frac{y}{2})+x\frac{dy}{dx}=0$$

and observe that this is not exact. Indeed, $\frac{\partial}{\partial x}N(x,y)=\frac{\partial}{\partial x}x=1$, while $\frac{\partial}{\partial y}M(x,y)=\frac{\partial}{\partial y}\left(-(y^2\sqrt{x}+\frac{y}{2})\right)=-(2y\sqrt{x}+\frac{1}{2})$; since they are not equal, the equation is not exact. Can we find an integrating factor?

3.7 Excercise. Given the differential equation

$$-(y^2\sqrt{x} + \frac{y}{2}) + x\frac{dy}{dx} = 0,$$

check that $\mu(x,y) = \frac{1}{y^2\sqrt{x}}$ is an integrating factor, and then solve the corresponding exact equation. Check your answer with the solution to the equation in section 1.1.

Finding a function $\mu(x,y)$ which satisfies the partial differential equation (1.1) is not always possible. We consider now special cases where (1.1) simplifies so that we *can* find such a function, μ.

Case 1: $\mu(x,y) = \mu(x)$ Since μ is a function only of x, $\mu_y = 0$, and $\mu_x = \mu'(x)$. Then (1.1) becomes

$$-\mu'(x)N = (N_x - M_y)\mu, \quad \text{or} \quad \frac{\mu'(x)}{\mu(x)} = \frac{N_x - M_y}{-N}.$$

Since $\mu(x)$ and $\mu'(x)$ depend only on x, we must have that $Q(x) := \frac{N_x - M_y}{-N}$ depends only on x; integrating yields

$$\ln(\mu(x)) = \int Q(x)dx \qquad \Leftrightarrow \qquad \mu(x) = e^{\int Q(x)dx}.$$

Case 2: $\mu(x,y) = \mu(y)$ Since μ is a function only of y, $\mu_x = 0$, and $\mu_y = \mu'(y)$. Then (1.1) becomes

$$\mu'(y)M = (N_x - M_y)\mu, \quad \text{or} \quad \frac{\mu'(y)}{\mu(y)} = \frac{N_x - M_y}{M}.$$

Again, since μ depends only on y, we must have that $Q(y) := \frac{N_x - M_y}{M}$ is a function of y only, and so integrating gives us that

$$\ln(\mu(y)) = \int Q(y)dy \qquad \Leftrightarrow \qquad \mu(y) = e^{\int Q(y)dy}.$$

Case 3: $\mu(x,y) = \mu(x^r y^s)$ Given real numbers r and s, we suppose $\mu(x,y)$ can be written as a function of $t := x^r \cdot y^s$. Then

$$\mu_x(t) = \mu'(t)\frac{dt}{dx} = \mu'(t)rx^{r-1}y^s$$
$$\mu_y(t) = \mu'(t)\frac{dt}{dy} = \mu'(t)sx^r y^{s-1}$$

Then (1.1) becomes

$$\mu'(t)sx^r y^{s-1}M - \mu'(t)rx^{r-1}y^s N = (N_x - M_y)\mu(t), \quad \text{or}$$
$$\frac{\mu'(t)}{\mu(t)} = \frac{N_x - M_y}{sx^r y^{s-1}M - rx^{r-1}y^s N} = \frac{1}{t}\frac{N_x - M_y}{\frac{s}{y}M - \frac{r}{x}N}$$

Again, since $\mu(t)$ depends only on t (and hence so does $\mu'(t)$), we must have that $R(t) := \frac{1}{t}\frac{N_x - M_y}{\frac{s}{y}M - \frac{r}{x}N}$ depends only on t – i.e.

$$\frac{N_x - M_y}{\frac{s}{y}M - \frac{r}{x}N} \quad \text{depends only on } t.$$

Integrating gives us that

$$\ln(\mu(t)) = \int R(t)dt \qquad \Leftrightarrow \qquad \mu(t) = e^{\int R(t)dt}.$$

The integrating factor should be expressed in terms of x and y, of course, so we need to remember to substitute back $t = x^r y^s$, i.e.

$$\mu = e^{\int R(t)dt}\Big|_{t=x^r y^s}.$$

Of special importance here is the case when $r = s = 1$, and $t = xy$. Then $R(t)$ simplifies to just

$$R(t) = \frac{N_x - M_y}{xM - yN}.$$

3.8 Note. Case 3 includes cases 1 and 2 since we can choose $(r, s) = (0, 1)$ or $(r, s) = (1, 0)$. If $s \neq 0$, then only the value $\frac{r}{s}$ should matter in case 3 – intuitively, just observe that any function of x^4y^2 is also a function of x^2y, for example.

Exercise 3.7 continued How could we have found $\mu(x, y) = \frac{1}{y^2\sqrt{x}}$ in the previous exercise? One way is to use case 3, and look for values of r and s that simplify $R(t)$. First, we calculate that

$$N_x - M_y = 1 + 2y\sqrt{x} + \frac{1}{2} = \frac{3}{2} + 2y\sqrt{x}.$$

We want to find r and s such that

$$\frac{N_x - M_y}{\frac{s}{y}M - \frac{r}{x}N} = \frac{-3/2 - 2y\sqrt{x}}{sy\sqrt{x} + \frac{s}{2} + r}$$

is a function of $x^r y^s$ only. Observe that choosing $s = -2$ and $r = -1/2$ reduces the above expression to the constant 1, which is clearly a function of $t = x^r y^s = y^{-2}x^{-1/2}$. In this case,

$$R(t) = \frac{1}{t} \qquad \text{so that} \qquad \mu(t) = e^{\int R(t)dt} = t = \frac{1}{y^2\sqrt{x}}.$$

This is precisely the integrating factor given previously.

Perhaps this seems too lucky. Maybe you would look at the expression

$$\frac{-3/2 - 2y\sqrt{x}}{sy\sqrt{x} + \frac{s}{2} + r}$$

and see terms involving only $\sqrt{x} \cdot y$; then you could pick $r = 1/2$ and $s = 1$. Indeed, we obtain (for $t = y\sqrt{x}$)

$$\begin{aligned} R(t) &= \frac{1}{t} \frac{N_x - M_y}{\frac{s}{y}M - \frac{r}{x}N} \\ &= \frac{1}{t} \frac{-3/2 - 2t}{3/2 + t} = \frac{1}{t}\left(-1 - \frac{t}{3/2 + t}\right) \\ &= -\frac{1}{t} - \frac{1}{3/2 + t}. \end{aligned}$$

Then $\int R(t)dt = -\ln(t) - \ln(3/2 + t) = -\ln(t(3/2 + t))$; we obtain then

$$\mu(t) = e^{\int R(t)dt} = \frac{1}{t(3/2 + t)}$$

so that $\mu(y\sqrt{x}) = \frac{1}{xy^2 + \frac{3}{2}y\sqrt{x}}$.

3.9 Note. Integrating factors are not unique!

Homework Problems

These problems are taken from section §2.6 of the textbook.

7. Determine whether each of the equations in Problems 1 through 12 is exact. If it is exact, find the solution.

B2) $(2x + 4y) + (2x - 2y)y' = 0$

B3) $(3x^2 - 2xy + 2)dx + (6y^2 - x^2 + 3)dy = 0$

B10) $(y/x + 6x)dx + (\ln x - 2)dy = 0, \quad x > 0$

8. In each of the following problems, find an integrating factor and solve the given equation.

B30) $[4(x^3/y^2) + (3/y)]dx + [3(x/y^2) + 4y]dy = 0$

B31) $\left(3x + \frac{6}{y}\right) + \left(\frac{x^2}{y} + 3\frac{y}{x}\right)\frac{dy}{dx} = 0.$

4 First-order Modeling

In this section, we will study some real life problems which can be modeled using first order differential equations. In all cases, we identify some quantity which we will model by a function of time. We then attempt to discover the function by finding a differential equation which it must satisfy.

Mixing Problems

4.1 Examples (A mixing problem.). Initially, a tank contains 50 gallons of a solution of water and 30 lbs of some chemical. Fresh water runs into the tank at the rate of 2 gal/min and the solution runs out at the same rate. Assume that this mixture is homogeneous at all times. How much of the chemical remains in the tank after 20 minutes?

4.2 Solution. Let $y(t)$ be the function giving the number of lbs of chemical in the tank at time t. The initial conditions given in the problem will occur when $t = 0$ (this is arbitrary, but convenient). Thus we have that $y(0) = 30$.

What can we say about $\frac{dy}{dt}$? Since only fresh water is flowing into the tank, we are never adding any additional chemical to the tank; thus, the change in $y(t)$, $\frac{dy}{dt}$, is negative (or 0). How much chemical leaves the tank in a given amount of time? Chemical can only leave via the solution flowing out of the tank. We know that 2 gal/min of solution flows out of the tank. Each gallon of solution contains precisely

$$c(t) = \frac{y(t)}{50},$$

where $c(t)$ is the *concentration* of the chemical in the tank at time t. By definition, this is the total number of lbs of the chemical at time t divided by the total number of gallons in the tank at time t.

Thus, we have

$$\frac{dy}{dt} = -2 \cdot c(t) = -2\frac{y(t)}{50}.$$

This is a separable differential equation (and is also linear), and we know how to solve it.

$$\begin{aligned} \text{separate:} \quad & \frac{dy}{y} = -\frac{2}{50}dt \\ \text{integrate:} \quad & \ln|y| = -\frac{1}{25}t + C_1 \\ \text{solve:} \quad & y = C_2 e^{-\frac{1}{25}t}. \end{aligned}$$

But how do we determine this value C_2? We know that $y(0) = 30$, so $C_2 = y(0) = 30$. Finally, we can answer the question: after 20 minutes, there will be

$$y(20) = 30e^{-20/25} = 13.480$$

4.3 Examples (Another mixing problem). Initially, a tank contains 50 gallons of a solution of water and 30 lbs of some chemical. Fresh water runs into the tank at the rate of 2 gal/min and the solution runs out at 1 gal/min. Assume that this mixture is homogeneous at all times. What is the concentration of the tank after 20 minutes?

4.4 Solution. Again, let $y(t)$ be the function giving the number of lbs of chemical in the tank at time t. Since no chemical is being added to the tank, we know that $\frac{dy}{dt}$ is again negative. At what rate is the chemical leaving the tank? We know that solution runs out at 1 gal/min, and this contains precisely $c(t)$ lbs of chemical, where $c(t)$ is the concentration at time t of the solution, which is the total amount of chemical at time t divided by the total volume at time t.

In this problem, the volume is not kept constant - indeed it increases by (2 gal/min − 1 gal/min =) 1 gal/min, and is initially 50 gallons. In summary, the volume of solution in the tank at time t is $v(t) = 50 + t$; it follows that the concentration is

$$c(t) = \frac{y(t)}{50 + t},$$

and so

$$\frac{dy}{dt} = -c(t) = -\frac{y(t)}{50 + t}.$$

This is again separable, and so we have

$$\begin{aligned} \text{separate:} \quad & \frac{dy}{y} = -\frac{dt}{50 + t} \\ \text{integrate:} \quad & \ln|y| = -ln|50 + t| + C_1 \\ \text{solve:} \quad & y = \frac{C_2}{50 + t}. \end{aligned}$$

Since $y(0) = 30$, we see that $C_2 = 150$. Rewriting, we have

$$y(t) = \frac{30}{1 + t/50}.$$

Then $y(20) = 21.429$ lbs, and the concentration after 20 minutes is

$$c(20) = \frac{y(20)}{50 + 20} = \frac{21.429}{70} = .306 \; lb/gal.$$

Cooling Problems

Newton's law of cooling states that the rate of change of temperature is directly proportional to the difference of temperatures. Let's call $u(t)$ the temperature of some object at time t, and let u_0 be the ambient temperature, then

$$\frac{du}{dt} = -k(u - u_0).$$

4.5 Examples. Suppose that an object is heated to $300°F$, and then allowed to cool in an $85°F$ warehouse. Assume that after 5 minutes, the temperature is $270°F$. What is the temperature after 20 minutes?

4.6 Solution. Let $u(t)$ be the temperature of the object. Then $u(0) = 300°F$. We know that

$$\begin{aligned} \text{original:} \quad & \frac{du}{dt} = -k(u - u_0) \\ \text{separate:} \quad & \frac{du}{u - u_0} = -kdt \\ \text{integrate:} \quad & \ln|u - u_0| = -kt + C_1 \\ \text{solve:} \quad & u(t) = C_2 e^{-kt} + u_0. \end{aligned}$$

It follows that $C_2 = u(0) - u_0 = 300°F - 85°F = 215°F$. We need to know what the constant k is, so we use our other piece of data -

$$u(5) = 270 = 215e^{-5k} + 85,$$

so that $k = \frac{-1}{5} \ln \left| \frac{185}{215} \right| = 0.030$. Finally,

$$u(20) = 215e^{-20k} + 85 = 202.86°F$$

Economics Problems

Consider some commodity, and let the price P, supply S and demand D be continuous functions of time (this is already an unrealistic assumption - but let's pretend). We will model the price P by the differential equation

$$\frac{dP}{dt} = k(D - S) \qquad 0 < k \in \mathbb{R}.$$

The heuristic here is that if the demand is greater than the supply then the price should increase; if the supply is greater than the demand then the price will decrease.

4.7 Question. Is there a simpler first-order differential equation that is suggested from supply and demand? Assuming that all functions are continuous, which other assumption is being made in the above model that is most unlikely?

4.8 Examples. Assume that $D(t) = a + b \cdot P(t)$, and $S(t) = e + f \cdot P(t) + g \cdot \cos(t)$. Find the price $P(t)$ at any time t. What is the limiting behavior?

4.9 Solution. We have that

$$\begin{aligned} \text{original:} \quad & \frac{dP}{dt} = k(D - S) \\ \text{rewrite:} \quad & \frac{dP}{dt} = k(a + b \cdot P - e - f \cdot P - g\cos(t)) \\ \text{normalize:} \quad & \frac{dP}{dt} = k(b - f)P + k(a - e - g\cos(t)) \end{aligned}$$

This is a linear equation with $p(t) = -k(b - f)$, and $g(t) = k(a - e - g\cos(t))$. The integrating factor is then

$$\mu(t) = e^{\int p(t)dt} = e^{-k(b-f)t},$$

so that the we have

$$\begin{aligned} \text{multiply:} \quad & e^{-k(b-f)t}\frac{dP}{dt} - k(b - f)e^{-k(b-f)t}P = ke^{-k(b-f)t}(a - e - g\cos(t)) \\ \text{rewrite:} \quad & \frac{d}{dt}(e^{-k(b-f)t}P) = ke^{-k(b-f)t}(a - e - g\cos(t)) \\ \text{integrate:} \quad & e^{-k(b-f)t}P = k\int e^{-k(b-f)t}(a - e - g\cos(t))dt \end{aligned}$$

Integrating by parts twice, we have

$$k \int e^{-k(b-f)t}(a - e - g\cos(t))dt = \frac{a-e}{b-f}e^{-k(b-f)t} - kg\left[\frac{1}{1+k^2(b-f)^2}e^{-k(b-f)t}\left(\sin(t) - k(b-f)\cos(t)\right)\right] + C.$$

It follows that

$$P(t) = \frac{a-e}{b-f} - \frac{kg}{1+k^2(b-f)^2}\left[\sin(t) - k(b-f)\cos(t)\right] + Ce^{k(b-f)t}.$$

Now, we can analyze the limiting behavior of $P(t)$. We basically have two cases. If $b > f$, then the exponent in the last term, $Ce^{k(b-f)t}$ is positive. Thus the price will be unbounded, and will increase exponentially (in particular, the first two terms are not significant for large values of t). But this is not very plausible; so we require $b < f$. In that case, the limiting behavior is simpler to describe. The first term is a constant, $\frac{a-e}{b-f}$ which may be the dominant term depending on the values of the constants. The second term is periodic, and the third term is negligible for large values of t. It is plausible that the constant term is dominant, while the amplitude of the periodic function

$$\frac{kg}{1+b^2(b-f)^2}[\sin(t) - k(b-f)\cos(t)]$$

is less than $2 \cdot \frac{kg}{1+b^2(b-f)^2}$, which is controlled by g. If g is much smaller than $\frac{a-e}{b-f}$, we see a stable price with small periodic fluctuation. This might describe the price for seasonal commodities, for example.

Homework Problems

These problems are taken from section §2.3 of the textbook.

9 (B2). A tank initially contains 120 L of pure water. A mixture containing a concentration of γ g/L of salt enters the tank at a rate of 2 L/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of γ for the amount of salt in the tank at any time t. Also find the limiting amount of salt in the tank as $t \longrightarrow \infty$.

10 (B3). A tank originally contains 100 gal of fresh water. Then water containing $\frac{1}{2}$ lb of salt water per gallon is poured into the tank at a rate of 2 gal/min, and the mixture is allowed to leave at the same rate. After 10 min the process is stopped, and fresh water is poured into the tank at a rate of 2 gal/min, with the mixture again leaving at the same rate. Find the amount of salt in the tank at the end of an additional 10 min.

11 (B4). A tank with a capacity of 500 gal originally contains 200 gal of water with 100 lb of salt in solution. Water containing 1 lb of salt per gallon is entering at a rate of 3 gal/min, and the mixture is allowed to flow out of the tank at a rate of 2 gal/min. Find the amount of salt in the tank at any time prior to the instant when the solution begins to overflow. Find the concentration (in pounds per gallon) of salt in the tank when it is on the point of overflowing. Compare this concentration with the theoretical limiting concentration if the tank had infinite capacity.

12 (B6). Suppose that a tank containing a certain liquid has an outlet near the bottom. Let $h(t)$ be the height of the liquid surface above the outlet at time t. Torricelli's principle states that the outflow velocity v at the outlet is equal to the velocity of a particle falling freely (with no drag) from the height h.

(a) Show that $v = \sqrt{2gh}$, where g is the acceleration due to gravity.

(b) By equating the rate of outflow to the rate of change of liquid in the tank, show that $h(t)$ satisfies the equation

$$A(h)\frac{dh}{dt} = -\alpha a\sqrt{2gh},$$

where $A(h)$ is the area of the cross section of the tank at height h and a is the area of the outlet. The constant α is a contraction coefficient that accounts for the observed fact that the cross section of the (smooth) outflow stream is smaller than a. The value of α for water is about 0.6.

(c) Consider a water tank in the form of a right circular cylinder that is 3 m high above the outlet. The radius of the tank is 1 m and the radius of the circular outlet is 0.1 m. If the tank is initially full of water, determine how long it takes to drain the tank down to the level of the outlet.

13 (B12). An important tool in archeological research is radiocarbon dating, developed by the American chemist Willard F. Libby. This is a means of determining the age of certain wood and plant remains, hence of animal or human bones or artifacts found buried at the same levels. Radiocarbon dating is based on the fact that some wood or plant remains contain residual amounts of carbon-14, a radioactive isotope of carbon. This isotope is accumulated during the lifetime of the plant and begins to decay at its death. Since the half-life of carbon-14 is long (approximately 5730 years), measurable amounts of carbon-14 remain after many thousands of years. If even a tiny fraction of the original amount of carbon-14 is still present, then by appropriate laboratory measurements the proportion of the original amount of carbon-14 that remains can be accurately determined. In other words, if $Q(t)$ is the amount of carbon-14 at time t and Q_0 is the original amount, then the ratio $Q(t)/Q_0$ can be determined, as long as this quantity is not too small. Present measurement techniques permit the use of this method for time periods of 50,000 years or more.

(a) Assuming that Q satisfies the differential equation $Q' = -rQ$, determine the decay constant r for Carbon-14.

(b) Find an expression for $Q(t)$ at any time t, if $Q(0) = Q_0$.

(c) Suppose that certain remains are discovered in which the current residual amount of carbon-14 is 20% of the original amount. Determine the age of these remains.

14 (B13). The population of mosquitoes in a certain area increases at a rate proportional to the current population, and in the absence of other factors, the population doubles each week. There are 200,000 mosquitoes in the area initially, and predators (birds, bats, and so forth) eat 20,000 mosquitoes/day. Determine the population of mosquitoes in the area at any time.

15 (B16). Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that the temperature of a cup of coffee obeys Newton's law of cooling. If the coffee has a temperature of 200°F when freshly poured, and 1 min later has cooled to 190°F in a room at 70°F determine when the coffee reaches a temperature of 150°F

5 Autonomous Equations

5.1 Definition. An *autonomous* first-order differential equation is an equation of the form

$$\frac{dy}{dt} = h(y).$$

We see in particular that autonomous equations are separable. Solving the usual way, we obtain

$$\int \frac{dy}{h(y)} = \int dt = t + C.$$

Unfortunately, it may be difficult to understand the qualitative features of an antiderivative $\int \frac{dy}{h(y)}$ – it is even more difficult to understand $y(t)$ based on this implicit equation! Our goal in this section is to understand the solutions $y(t)$ qualitatively, which requires different methods. We begin with a few examples.

5.1 Exponential Growth

The most basic family of autonomous equations are those representing *exponential growth*. These are of the form

$$\frac{dy}{dt} = ry,$$

where r is called the *rate of growth*.

(MORE TO ADD)

Homework Problems

These problems are taken from section §2.5 of the textbook.

16. Each of the following problems involve equations of the form $\frac{dy}{dt} = f(y)$. In each problem, sketch the graph of $f(y)$ vs y, determine the equilibrium points, and classify each one as asymptotically stable or unstable. Draw the phase line, and sketch a graph with several solutions in the ty-plane.

B3) $\frac{dy}{dt} = y(y-1)(y-2), \quad y_0 \geq 0$

B4) $\frac{dy}{dt} = e^y - 1, \quad -\infty < y_0 < \infty$

B6) $\frac{dy}{dt} = -2(\tan^{-1} y)/(1+y^2), \quad -\infty < y_0 < \infty$

17 (B16). Another equation that has been used to model population growth is the Gompertz equation

$$\frac{dy}{dt} = ry\ln(K/y),$$

where $r, K > 0$ are positive constants.

(a) Sketch the graph of $f(y)$ vs y, find the critical points, and determine whether each is asymptotically stable or unstable.

(b) For $0 \leq y \leq K$, determine where the graph of y vs t is concave up and where it is concave down.

(c) For each y in $0 < y \leq K$, show that $\frac{dy}{dt}$ as given by the Gompertz equation is never less than $\frac{dy}{dt}$ as given by the logistic equation.

18 (B18). A pond forms as water collects in a conical depression of radius a and depth h. Suppose that water flows in at a constant rate k and is lost through evaporation at a rate proportional to the surface area.

(a) Show that the volume $V(t)$ of water in the pond at time t satisfies the differential equation

$$\frac{dV}{dt} = k - \alpha\pi\left(\frac{3a}{\pi h}\right)^{2/3} V^{2/3},$$

where α is the coefficient of evaporation.

(b) Find the equilibrium depth of water in the pond. Is the equilibrium asymptotically stable?

(c) Find a condition that must be satisfied if the pond is not to overflow.

6 Slope fields and Euler's Method

Example.

In the graph above, we have drawn a small line element of slope $-t_0$ at each point (t_0, y_0) in the plane. Of course, this is not really possible, so we have only drawn in the line elements along some grid which suggests how the flow lines behave.

6.1 Definition. An **integral curve** $y_1(t)$ is a curve (segment) which has the direction of the slope field at every point (t, y), where $y = y_1(t)$.

The point here is that if you start with some initial value, i.e. a point (t_0, y_0), then you can just follow the direction lines with your pencil to draw an integral curve.

Exercise. Check that $y_1(t) = -\frac{1}{2}t^2$ is an integral curve for the slope field above, by plotting it.

Exercise. Check that $y_1(t) = -\frac{1}{2}t^2$ is a **solution** to the differential equation $\frac{dy}{dt} = -t$.

Fact. A solution to a differential equation is the same as an integral curve for that differential equation's slope field.

Example.

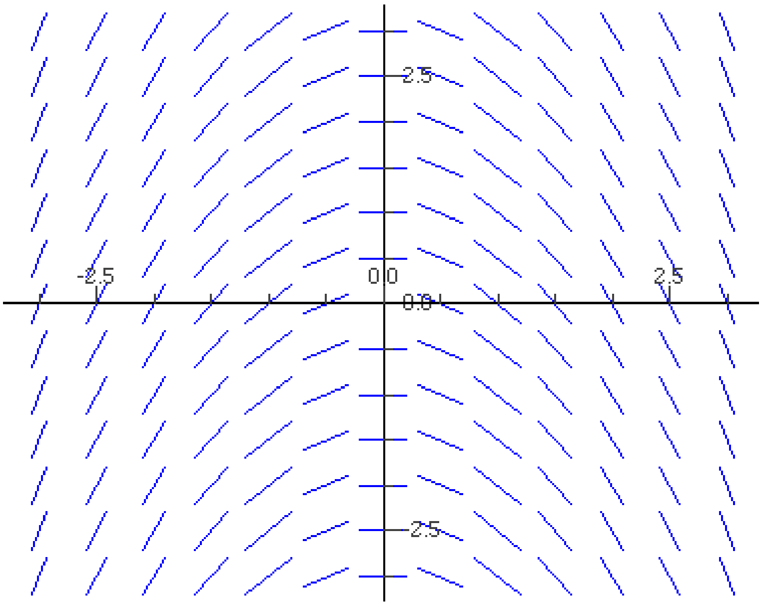

Figure 1.1: $\frac{dy}{dt} = -t$

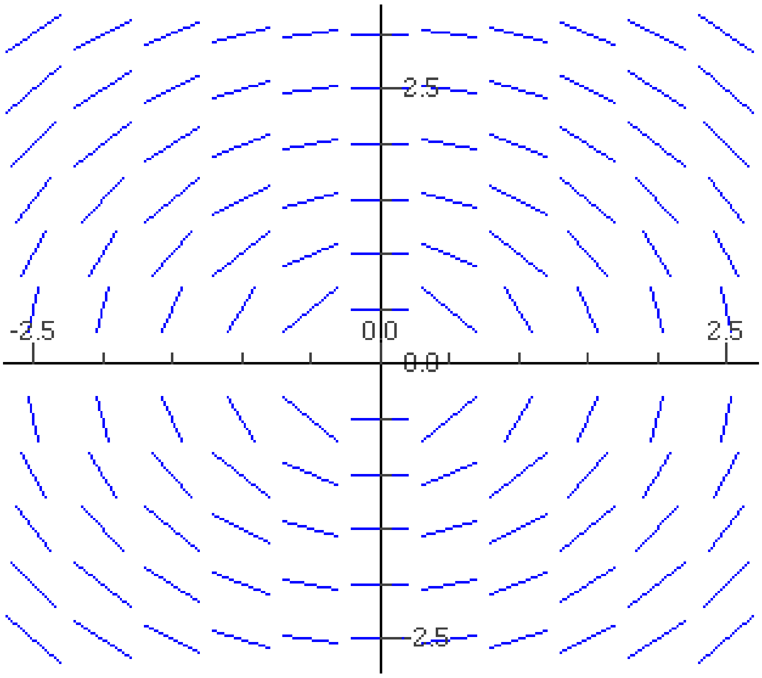

Figure 1.2: $\frac{dy}{dx} = -\frac{x}{y}$

The picture suggests that the integral curves are circles centered at zero. Since we know the formula for a general circle centered at $(0, 0)$, we can check this. So we consider the circle of radius r: it is defined by $x^2 + y^2 = r^2$. Now we want to know its derivative at any point. We can do this by differentiating implicitly:

$$\begin{aligned} 2xdx + 2ydy &= 0 \\ 2ydy &= -2xdx \\ dy &= \frac{-x}{y}dx \\ \frac{dy}{dx} &= \frac{-x}{y}. \end{aligned}$$

This shows that the curve $x^2+y^2 = r^2$ is a **solution** to the differential equation $\frac{dy}{dx} = -\frac{x}{y}$, and is therefore an integral curve for the slope field above, as we guessed.

6.2 Note. Suppose we want the solution $y_r(0) = +r$. If we solve the above equation explicitly for $y(x)$, we get

$$y_r(x) = \sqrt{r^2 - x^2} \qquad \text{for } -r < x < r.$$

We couldn't guess that the domain of the particular solution would be restricted to the finite interval $(-r, r)$ just by looking at the differential equation $\frac{dy}{dx} = -\frac{x}{y}$. However, we would have been able to guess this by looking at the slope field, since any function $y(x)$ must pass the "vertical line test" - in otherwords, it cannot be multi-valued.

Algorithm for drawing slope-fields.

Consider the differential equation $\frac{dy}{dt} = f(t, y)$. We may plot the slope field by doing the following:

Step 0 (Normalize!) Make sure that it really is in the above form. For example, if the equation were

$$x\frac{dy}{dt} = -y$$

we would first need to divide both sides by x.

Step 1 Find the *isoclines*: these are curves in the plane of constant slope m, defined implicitly by

$$f(t, y) = m.$$

Step 2 For a specific value m_0, plot the isocline and draw in the line elements of slope m along that curve.

Step 3 Repeat step 2 for various values of m.

Exercise.

i) Draw the direction field for

$$\frac{dy}{dt} = y.$$

Draw in a few **integral curves**. Write down a **solution** to the differential equation (i.e. a formula $y(x) =$?)

ii) What do you expect the integral curves for

$$\frac{dy}{dx} = -4\frac{x-1}{y-1}$$

will look like? Draw the slope field to get some ideas, and look carefully at the previous example.

Example. We look at the differential equation $\frac{dy}{dt} = 1 + t - y$. Let's draw the slope field:

Step 0 It is normalized!

Step 1 To find the isoclines, we solve $1 + t - y = m$ for y:

$$y = t + (1 - m).$$

Step 2 We can plot the isocline for $m = 0$: this is the line $y(t) = t + 1$. The line elements have slope $m = 0$ here.

Step 3 We repeat for $m = 1, -1$, etc.

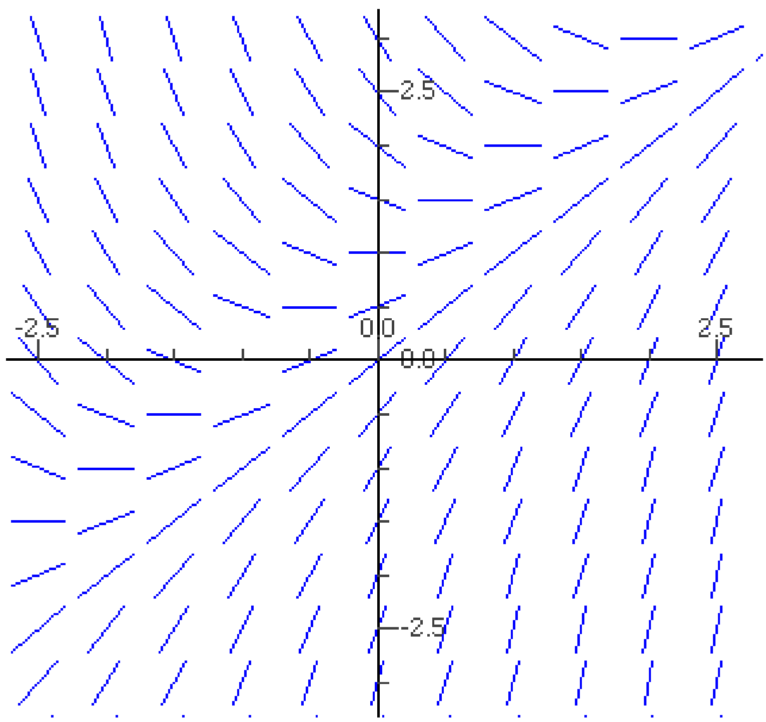

Figure 1.3: $\frac{dy}{dt} = 1 + t - y$

What can we say about the limiting behavior of any **integral curve** as $t \longrightarrow \infty$? The idea is that all integral curves fall into the diagonal stripe between the isoclines corresponding to $m = 0$ and $m = 2$. Plot a few integral curves to see why this is true.

Once an integral curve $y_1(t)$ has entered this "stripe" - we will see that it gets closer and closer to the special solution $y_0(t) = t$. There are essentially two cases: if $y_1(t_*) > t_*$, so that $y_1(t) > y_0(t)$ for some number t_*, then the slope

$$y_1'(t_*, y_1(t_*)) = 1 + t_0 - y_1(t_*) < 1.$$

Since that slope is < 1, which is the slope of $y_0(t)$, the curve $y_1(t)$ will tend closer towards the line $y_0(t) = t$.

On the otherhand, if $y_1(t_*) < t_*$, so $y_1(t) < y_0(t)$, then

$$y_1'(t_*, y_1(t_*)) = 1 + t_0 - y_1(t_*) > 1,$$

so the integral curve $y_1(t)$ will tend upwards towards $y_0(t)$. This analysis suggests that any integral curve gets arbitrarily close to $y_0(t) = t$ for large values of t. Later on in the course, we will be able to show this algebraically by finding explicit formulas for all solutions, but this is more work than is necessary! Anyway, it is good to be able to see this from the picture, since we can't always find explicit formulas for all solutions.

Facts about Integral Curves.

Consider a direction field for the differential equation

$$\frac{dy}{dt} = f(t, y).$$

6.3 Proposition.

i) Integral curves can never cross.

ii) Provided that $f(t, y)$ is "nice" in a small rectangle $t_0 < t < t_1$ and $y_0 < y < y_1$, then integral curves do not touch, i.e. they are not tangent.

Here, "nice" means that $f(t, y)$ and its partial derivative $\frac{\partial f}{\partial y}$ are both continuous functions.

We can check the first property. Consider two different integral curves $y_1(t)$ and $y_2(t)$ which cross at t_*, so that their y-values are the same: $y_1(t_*) = y_2(t_*)$; but then their slopes

$$y_1'(t_*) = f(t_*, y_1(t_*)) = f(t_*, y_2(t_*)) = y_2'(t_*)$$

are equal. This shows that they do not cross at t_* but could possibly be tangent.

The second property is more subtle; it is a statement about the uniqueness of solutions to initial value problems. We will take this for granted in this class.

To see what can go wrong, consider the following examples:

i) Consider $\frac{dy}{dt} = y^{1/3}$. Draw the slope field. If $y_1(0) = 0$, is there a unique function $y_1(x)$ starting at $x = 0$ which is an integral curve?

ii) Consider $\frac{dy}{dx} = \frac{y-1}{x-1}$, $y(1) = 1$. We can check algebraically that $y_c(x) = c(x-1) + 1$ is a **solution** for every constant c! Indeed, $y_c'(x) = c$, and

$$\frac{y_c(x) - 1}{x - 1} = c.$$

This shows that an **integral curve** starting at $y(1) = 1$ is not unique (an integral curve is a straight line starting at $(1, 1)$ in *any* direction).

Homework Problems

These problems are taken from section §1.1 of the textbook.

19. Draw the direction fields for each of the following differential equations. Determine the behavior of any solution $y(t)$ of the differential equation as $t \longrightarrow \infty$. If this behavior depends on the initial value $y(0)$, describe the dependency.

B5) $y' = 1 + 2y$

B6) $y' = y + 2$

20. Write down a differential equation of the form $\frac{dy}{dt} = ay + b$ whose solutions have the following behavior as $t \longrightarrow \infty$:

B8) All solutions approach $y = 2/3$.

B10) All other solutions diverge from $y = 1/3$.

21 (B20). Look in the book at the direction field of Figure 1.1.10, and identify which differential equation (a)-(j) it corresponds to.

6.1 Euler's Method

Every solution of a first order differential equation, which is an equation of the form

$$\frac{dy}{dt} = f(t, y),$$

can be seen by plotting the direction field for $f(t, y)$ and then tracing the integral curves. Each integral curve is the graph of a solution $y_1(t)$ to the differential equation.

If we want to specify a particular integral curve, we can prescribe the value of the function $y_1(t)$ at some specific time t_0. This is often written

$$\begin{cases} \frac{dy}{dt} = f(t, y) \\ y_1(t_0) = y_0. \end{cases}$$

The system of equations above is called an *initial value problem* or *IVP*.

Example. Consider the IVP below:

$$\begin{cases} \frac{dy}{dt} = -t \\ y_1(0) = 1. \end{cases}$$

We recall that the slope field is:

To "see" the solution $y_p(t)$ to this IVP, we place our pencil at the point $(t = 0, y = 1)$, and then trace the integral curve through that point, by following the slope field. The resulting curve should look like a parabola pointing downwards, which has been shifted up by 1 from the origin.

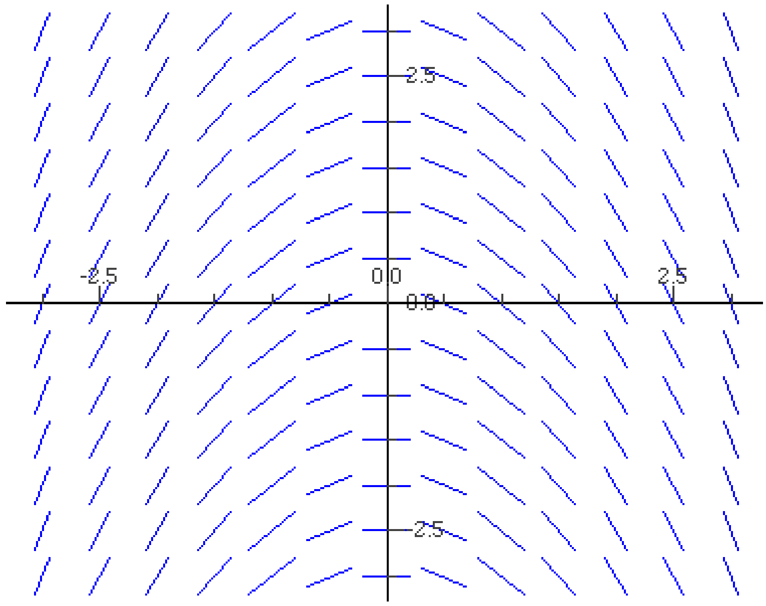

Figure 1.4: $\frac{dy}{dt} = -t$

(t, y)	$f(t, y) = -t$	$h \cdot f(t, y)$	$h \cdot f(t, y) + y$	$t + h$
$(0, 1)$	0	0	$0 + 1 = 1$	$0 + 0.1 = 0.1$
$(0.1, 1)$	-0.1	-0.01	$-0.01 + 1 = 0.99$	$0.1 + 0.1 = 0.2$
$(0.2, 0.99)$	-0.2	-0.02	$-0.02 + 0.99 = 0.97$	$0.2 + 0.1 = 0.3$
$(0.3, 0.97)$				

This graphical method for solving initial value problems is not quantitative: in the previous example, what is the value of $y_p(0.3)$? At the least, what is an approximate value?

One way to answer this question is to sketch the curve as we did above, and then use the graph to estimate the value of $y_1(0.3)$. We will now introduce an algorithm, called Euler's method, which provides an estimate using this idea.

We will illustrate the algorithm in table form, and then discuss what it is actually calculating. The step-size, denoted h, is $h = 0.1$:

6.4 Remark.

i) We initialize the first row and column with the initial value (t_0, y_0) given in the IVP.

ii) We complete the table one row at a time. For a fixed row, each column is computed by the formulas at the top. Except the second column, we only use values in the previous columns and the value of h.

iii) The first entry in each row (starting from the second row) consists of the 4th and 5th values of the previous row. These entries are approximately points on the solution curve $y_1(t)$. In otherwords, $y_1(0.1) \approx 1$, $y_1(0.2) \approx 0.99$), and $y_1(0.3) \approx 0.97$.

iv) We stop the algorithm when we have an approximation for the value at the time we care about. In this example, we wanted to know about $y_1(0.3)$, so we stopped at $(0.3, 0.97)$.

v) The algorithm is more accurate for smaller values of h. For example, if we had used $h = 0.05$, we would have computed 6 rows, and found that $y_1(0.3) \approx 0.9625$. (The exact value is $y_1(0.3) = \frac{1}{2}(0.3)^2 + 1 = 0.955$).

Let's discuss what the algorithm is doing. For each approximate point (t_*, y_*) on the curve $y_1(t)$, we use the linear approximation $L(t)$ to $y_1(t)$ to calculate the next point. In symbols:

$$\begin{aligned} t_{new} &= t_* + h \\ y_{new} &= L(t_* + h). \end{aligned}$$

Remember that the linear approximation at (t_*, y_*) of $y_1(t)$ is the line

$$L(t) = m(t - t_*) + y_*$$

whose slope m is the value of the derivative $y_1'(t_*)$. Since $y_1(t)$ is a solution to the differential equation, we know that the derivative $y_1'(t_*) = f(t_*, y_*)$. This is what we compute in the second column.

We want to evaluate this linear approximation at $t_* + h$. So we calculate

$$L(t_* + h) = m(t_* + h - t_*) + y_* = hm + y_* = h \cdot f(t_*, y_*) + y_*.$$

This is exactly what we calculate in column 4. Column 5 is the new t-coordinate, $t = t_* + h$. Thus the new y-value, y_{new}, is the value from column 4, and t_{new} is the value in column 5.

We do one more example, slightly more complicated, so you can check your understanding of the algorithm.

Example. Approximate the solution $y_1(t)$ to the IVP below at $t = 2.4$ using Euler's method with $h = 0.1$:

$$\begin{cases} \frac{dy}{dt} = sin(t) \cdot y^2 \\ y_1(2) = 3. \end{cases}$$

So we say that $y_1(2.4) \approx 10.9846$.

6.5 Theorem. *Euler's method converges to the exact value as $h \longrightarrow 0$.*

6.6 Question. Later we will see that the exact value is $y_1(2.4) = 82.7373$. Why is the numerical approximation so bad in this case? It turns out that the exact solution is

$$y_1(t) = \frac{1}{\cos(t) + \frac{1}{3} - \cos(2)},$$

(t, y)	$f(t, y)$	$h \cdot f(t, y)$	$h \cdot f(t, y) + y$	$t + h$
(t, y)	$sin(t) \cdot y^2$	$(0.1)(sin(t) \cdot y^2)$	$(0.1)(sin(t)\ y^2) + y$ $\cdot$	$t + 0.1$
$(2, 3)$	8.184	0.8184	$0.8184 + 3 = 3.8184$	$2 + 0.1 = 2.1$
$(2.1, 3.8184)$	12.586	1.2586	5.0769	2.2
$(2.2, 5.0769)$	20.840	2.0840	7.1608	2.3
$(2.3, 7.1608)$	38.2379	3.82379	10.9846	2.4
$(2.4, 10.9846)$				

as you can check. Observe that $y_1(t)$ has a singularity when $\cos(t) = \cos(2) - 1/3$, since that makes the denominator 0. This happens when $t \approx 2.418$. This means that $y_1(2.418) = +\infty$; since 2.4 is close the 2.418, the exact value is very large, and our approximation is much too small.

The moral of the story is that singularities, i.e. points where the exact solution has a vertical asymptote, make the numerical approximation very bad for nearby values of t. This is a major complication when using software to find solutions to differential equations numerically.

If you are feeling ambitious, try to find a starting value for $y_p(2)$, different from 3, so that the exact solution does not have any singularities. Can you find all such values?

Homework Problems

These problems are taken from section §2.2 of the textbook.

22 (B4). Consider the differential equation: $y' = 3\cos(t) - 2y, \quad y(0) = 0.$

a) Find approximate values of the solution of the IVP at $t = 0.1, 0.2, 0.3,$ and 0.4 using the Euler method with $h = 0.1$.

b) Repeat part (a) with $h = 0.05$. Compare with the previous approximation.

c) Repeat part (a) with $h = 0.025$. Compare with the previous approximations.

d) Find the exact solution $y(t)$ using one of our techniques for solving differential equations. Evaluate $y(t)$ at $t = 0.1, 0.2, 0.3,$ and 0.4 and compare these values with the approximations in the previous parts.

23 (B10). Consider the differential equation

$$y' = \frac{y^2 + 2ty}{3 + t^2}.$$

Draw the direction field and state whether you think the solutions are converging or diverging.

24 (B17). Consider the initial value problem

$$y' = \frac{y^2 + 2ty}{3 + t^2}, \qquad y(1) = 2.$$

Use Euler's method with $h = 0.1, 0.05,$ and 0.025 to approximate the value of y at $t = 1.3$.

Chapter 2

2^{nd}-Order Differential Equations

The general form of a second order differential equation is

$$\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt}).$$

In this chapter, we restrict our attention to *linear second order equations*, which can be written in the form

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t).$$

We will very often further restrict attention to the case where $p(t)$ and $q(t)$ are just constants.

Definition. When $g(t) = 0$, we say that the linear equation above is *homogeneous*. This just means that the right hand side (RHS) is zero.

1 Homogeneous Equations

Examples.

i) $y'' + y = 0$.

ii) $y'' - y = 0$.

iii) $y'' - 4y = 0$.

In these first examples, we can just guess the function.

i) We want a function whose second derivative is the opposite of the function. This should remind us of the trigonometric functions: $\sin(t)$, $\cos(t)$. Indeed,

$$\sin(t) \longrightarrow \cos(t) \longrightarrow -\sin(t)$$

where the arrow indicates differentiation. Likewise,

$$\cos(t) \longrightarrow -\sin(t) \longrightarrow -\cos(t).$$

This shows that both $\cos(t)$, and $\sin(t)$ are solutions.

ii) Now we want a function whose second derivative is equal to the function itself. Of course, we know that e^t satisfies this property, since

$$e^t \longrightarrow e^t \longrightarrow e^t.$$

It might take a minute to be able to guess that e^{-t} also has this property, since

$$e^{-t} \longrightarrow -e^{-t} \longrightarrow -(-e^{-t}) = e^{-t}.$$

iii) From the previous example, we easily check that e^{2t} and e^{-2t} are two solutions.

In the example (1) above, $\cos(t)$ and $\sin(t)$ both satisfy the differential equation. Let's consider their sum, the function $y(t) = \cos(t) + \sin(t)$. The second derivative is $y''(t) = -\cos(t) - \sin(t)$, so $y(t)$ is also a solution. Also, if we multiply $\cos(t)$ by a constant, say π, then we would get another solution, since

$$\pi \cos(t) \longrightarrow \pi \sin(t) \longrightarrow -\pi \cos(t).$$

1.1 Definition.

- A *fundamental set of solutions* to a homogeneous differential equation is a minimal collection $\{y_1, \ldots, y_n\}$ of solutions such that **every** solution y_0 can be written as

$$y_0 = c_1y_1 + c_2y_2 + \cdots + c_ny_n, \qquad c_i \in \mathbb{R}.$$

- We say that y_0 is a *linear combination* of the functions in the fundamental set.
- The generic expression

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n$$

is called the *general solution* to the differential equation.

Solving a homogeneous linear differential equation means finding a fundamental set of solutions. But the following fact makes this much easier than you might expect:

1.2 Proposition. *The fundamental set of solutions of a second order homogeneous linear differential equation contains exactly two functions. Furthermore, any two solutions which are not constant multiples of each other form a fundamental set of solutions.*

Examples.

i) For $y'' + y = 0$, a fundamental set of solutions is $\{\cos(t), \sin(t)\}$.

ii) For $y'' - y = 0$, a fundamental set of solutions is $\{e^t, e^{-t}\}$.

iii) For $y'' - 4y = 0$, a fundamental set of solutions is $\{e^{2t}, e^{-2t}\}$.

General Solution. Now we want to find the general solution to the differential equation $ay'' + by' + cy = 0$. Based on examples (2) and (3), it's reasonable to try a guess like $y(t) = e^{rt}$, and see what happens. We calculate

$$y'(t) = re^{rt} \qquad \text{and} \qquad y''(t) = r^2 e^{rt}.$$

Then if we plug $y(t)$ into the differential equation we get

$$\begin{aligned} 0 &= a(r^2 e^{rt}) + b(re^{rt}) + c(e^{rt}) \\ &= e^{rt}\left(ar^2 + br + c\right). \end{aligned}$$

Since $e^{rt} \neq 0$, we can cancel it from the right hand side. If $ar^2 + br + c = 0$, then $y(t)$ will satisfy the differential equation.

1.3 Definition. Associated to the differential equation $ay'' + by' + cy = 0$ is its *characteristic equation* and *characteristic polynomial*:

$$ar^2 + br + c = 0 \qquad \text{and} \qquad ar^2 + br + c.$$

The two solutions r_1 and r_2 of the characteristic equation correspond exactly to the fundamental set of solutions of the differential equation: $\{e^{r_1 t}, e^{r_2 t}\}$. (These solutions r_1 and r_2 are also roots of the characteristic polynomial.)

1.1 Complex solutions to the characteristic equation

For the three simple examples we looked at:

i) $y'' + y = 0$ - fundamental set of solutions is $\{\cos(t), \sin(t)\}$

ii) $y'' - y = 0$ - fundamental set of solutions is $\{e^t, e^{-t}\}$

iii) $y'' - 4y = 0$ - fundamental set of solutions is $\{e^{2t}, e^{-2t}\}$.

In general, our strategy for solving these linear equations was to use the function $y = e^{rt}$ for values of r that satisfy the *characteristic equation*. But what about example (i)? It doesn't seem that $\cos(t)$ and $\sin(t)$ are in the form e^{rt}, and anyway the characteristic equation

$$r^2 + 1 = 0$$

has no solutions in the real numbers! In what follows, we will explain how to compute the fundamental solutions $\cos(t)$ and $\sin(t)$ from the general solution $e^{\pm it}$, where $i = \sqrt{-1}$.

Complex Numbers

A complex number is a point (or vector) $(a, b) \in \mathbb{R}^2$. We add complex numbers exactly like we do vectors in $\mathbb{R}^2$:

$$(a, b) + (a', b') = (a + a', b + b').$$

We also have a scalar multiplication by *real* numbers, $\lambda \in \mathbb{R}$:

$$\lambda \cdot (a, b) := (\lambda a, \lambda b).$$

Typically, we write $a + bi$ for the complex number (a, b). This notation suggests how we should multiply two complex numbers:

$$(a + bi) \cdot (a' + b'i) = aa' + ab'i + a'bi - bb' = (aa' - bb') + (ab' + a'b)i.$$

This defines a commutative, associative multiplication [$ab = ba$, and $a(bc) = (ab)c$.]

The *real part* of (a+bi) is denoted

$$Re(a + bi) := a,$$

while the *imaginary part* of $(a + bi)$ is

$$Im(a + bi) := b.$$

These are exactly the x- and y-components of the point (or vector) $(a, b) \in \mathbb{R}^2$. Given $a + bi$, we call $a - bi$ its *complex conjugate*. Notice that

$$(a + bi)(a - bi) = a^2 + b^2 = ||(a, b)||^2 = d(0, (a, b))^2 \in \mathbb{R}.$$

If we want to divide by $a + bi$, it is equivalent to finding $\frac{1}{a+bi}$ and then to multiply using the multiplication defined above. Indeed, we have

$$\frac{1}{a + bi} = \frac{1}{a + bi} \cdot \frac{a - bi}{a - bi} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i.$$

This is exactly "rationalizing the denominator," but it is important since it shows that we can divide by *any* non-zero complex number.

For a slightly more thorough treatment, see the section in Chapter 0.

Taylor Series

Given an infinitely differentiable function f, we can form its Taylor series based at x_0:

$$T(x) := \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

When a function is equal to its Taylor series in some interval about x_0, i.e.

$$f(x) = T(x) \qquad \text{for all } x \in (x_0 - w, x_0 + w),$$

then we say that $f(x)$ is *real analytic* at x_0.

Almost all of the elementary functions that we will work with are real analytic at every point. For example, one can calculate (just by using the above formula, and calculating the derivatives at 0):

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$
$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$
$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

We are essentially going to use Taylor series only to get the definition of e^{it}; for a slightly more thorough treatment, read the section in Chapter 0. The idea is that the Taylor series for e^x should give us a good definition, since we can just plug in "it" for "x". That's exactly what we do now:

$$\begin{aligned} e^{it} &= \sum_{n=0}^{\infty} \frac{1}{n!}(it)^n \\ &= \sum_{n=0}^{\infty} \left[\frac{t^{4n}}{(4n)!} + \frac{it^{4n+1}}{(4n+1)!} - \frac{t^{4n+2}}{(4n+2)!} - \frac{it^{4n+3}}{(4n+3)!} \right] \\ &= \sum_{n=0}^{\infty} \left[\frac{t^{4n}}{(4n)!} - \frac{t^{4n+2}}{(4n+2)!} \right] + i \sum_{n=0}^{\infty} \left[\frac{t^{4n+1}}{(4n+1)!} - \frac{t^{4n+3}}{(4n+3)!} \right] \\ &= \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p)!} t^{2p} + i \sum_{p=0}^{\infty} \frac{(-1)^p t^{2p+1}}{(2p+1)!} \\ &= \cos(t) + i \sin(t). \end{aligned}$$

To get to the 4th line, we use that $p = 2n$. Finally, in the last line, we identify the Taylor series as those of the functions $\cos(t)$ and $\sin(t)$. This gives us *Euler's formula*:

$$e^{it} = \cos(t) + i \sin(t).$$

More generally, we have that

$$e^{(a+bi)t} = e^{at+ibt} = e^{at}\left(e^{ibt}\right) = e^{at}\left(\cos(bt) + i\sin(bt)\right).$$

This is basically what we need to solve the linear differential equations in this course. But it's worth pointing out that Euler's formula carries a lot of information. For example, we can recover essentially all of the trigonometric identities that one learns before college. For example, to find the double-angle formula:

$$\begin{aligned} e^{2it} &= \cos(2t) + i \sin(2t) \\ e^{it+it} &= (\cos(t) + i\sin(t))^2 = \cos^2(t) - \sin^2(t) + i\left(2\sin(t)\cos(t)\right). \end{aligned}$$

Since $e^{2it} = e^{it+it}$, we must have that

$$(\cos(2t), \sin(2t)) = \left(\cos^2(t) - \sin^2(t), 2\sin(t)\cos(t)\right),$$

which is exactly the pair of formulas that we know:

$$\cos(2t) = \cos^2(t) - \sin^2(t)$$
$$\sin(2t) = 2\sin(t)\cos(t).$$

Using Euler's Formula

We return now to our first example to illustrate the advantage of using complex numbers and Euler's formula. Consider the differential equation $y'' + y = 0$. The characteristic equation is $r^2 + 1 = 0$; so $r = \pm i$ and the fundamental solutions are $\{e^{it}, e^{-it}\}$. Finally, we can use Euler's formula to understand this.

$$e^{it} = \cos(t) + i\sin(t),$$
$$e^{-it} = \cos(t) - i\sin(t).$$

This is perfectly correct; $\{e^{it}, e^{-it}\}$ forms a fundamental set of solutions to this differential equation. But perhaps we would be more comfortable with real-valued functions instead of these complex-valued ones. Remember the principle of superposition – since any linear combination of fundamental solutions is again a solution, we have that:

$$\frac{1}{2}\left(e^{it} + e^{-it}\right) = \cos(t)$$
$$\frac{1}{2i}\left(e^{it} - e^{-it}\right) = \sin(t).$$

are both solutions. Since $\cos(t)$ is not a multiple of $\sin(t)$, we know that $\{\cos(t), \sin(t)\}$ is also a fundamental set of solutions. Since these functions are real-valued, they are preferred for us in this class.

Notice that $\cos(t)$ and $\sin(t)$ are just the real and imaginary parts of $e^{it} = \cos(t) + i\sin(t)$. If your homogeneous linear differential equation involves only real-valued functions (as in this class), then the real and imaginary parts of any complex-valued solution will also be solutions to the differential equation.

Now consider the linear homogeneous differential equation with (real) constant coefficients:

$$y'' + by' + cy = 0;$$

now we have our general strategy for solving it. If the characteristic equation

$$r^2 + br + c = 0,$$

admits two real roots, say r_1, r_2, then the functions $e^{r_1 t}$, $e^{r_2 t}$ form a fundamental set of solutions. Otherwise, we will have roots which are complex conjugates. Then we may select either of them, say r_1, and take real and imaginary parts of the function $e^{r_1 t}$ to get a fundamental set of solutions.

1.4 Examples. Find the general solution to $y'' - 2y' + 2y = 0$.

1.5 Solution. We first find the characteristic equation: $r^2 - 2r + 2 = 0$. Then we solve for r:

$$r = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i.$$

We select either of the conjugate roots, say $r = 1+i$. Now use Euler's formula to calculate

$$e^{(1+i)t} = e^t(\cos(t) + i\sin(t)).$$

Finally, the real and imaginary parts are exactly $\{e^t\cos(t), e^t\sin(t)\}$. Thus, we have found a fundamental set of solutions, so the general solution is

$$y = c_1e^t\cos(t) + c_2e^t\sin(t).$$

1.6 Examples. Solve the initial value problem:

$$3y'' - y' + 2y = 0 \qquad \begin{cases} y(0) = 2 \\ y'(0) = 0 \end{cases}$$

1.7 Solution. We find the characteristic equation: $3r^2 - r + 2 = 0$. We solve this for r, and obtain

$$r = \frac{1 \pm \sqrt{1-24}}{6} = \frac{1}{6} \pm \frac{\sqrt{23}}{6}i.$$

Then $e^{rt} = e^{t/6}\left(\cos(\frac{\sqrt{23}}{6}t) + i\sin(\frac{\sqrt{23}}{6}t)\right)$. Thus, the fundamental set of solutions, i.e. the real and imaginary parts, are

$$\{e^{t/6}\cos\left(\frac{\sqrt{23}}{6}t\right), e^{t/6}\sin\left(\frac{\sqrt{23}}{6}t\right)\}.$$

The general solution is then

$$y(t) = c_1e^{t/6}\cos\left(\frac{\sqrt{23}}{6}t\right) + c_2e^{t/6}\sin\left(\frac{\sqrt{23}}{6}t\right).$$

We use the initial conditions to determine the constants c_1 and c_2. We have that

$$2 = y(0) = c_1.$$

Then

$$\begin{aligned} y'(t) = 1/3e^{t/6}\cos\left(\frac{\sqrt{23}}{6}t\right) - 2e^{t/6}\sin\left(\frac{\sqrt{23}}{6}t\right)\frac{\sqrt{23}}{6} \\ + c_2/6e^{t/6}\sin\left(\frac{\sqrt{23}}{6}t\right) + c_2e^{t/6}\cos\left(\frac{\sqrt{23}}{6}t\right)\frac{\sqrt{23}}{6}. \end{aligned}$$

It follows that $0 = y'(0) = 1/3 + \frac{\sqrt{23}}{6}c_2$, so that $c_2 = -\frac{2}{\sqrt{23}}$. Thus

$$y(t) = 2e^{t/6}\cos\left(\frac{\sqrt{23}}{6}t\right) - \frac{2}{\sqrt{23}}e^{t/6}\sin\left(\frac{\sqrt{23}}{6}t\right).$$

Homework Problems

These problems are taken from section §3.1 of the textbook.

25. Find the general solution to the given differential equations

B4) $2y'' - 3y' + y = 0$

B8) $y'' - 2y' - 2y = 0$

26 (B15). Solve the initial value problem

$$y'' + 8y' - 9y = 0, \qquad y(1) = 1, \quad y'(1) = 0.$$

Sketch the graph of the solution and describe its behavior as t increases.

27 (B24). Determine the value of α, if any, for which all solutions to the differential equation

$$y'' + (3 - \alpha)y' - 2(\alpha - 1)y = 0$$

tend to zero as $t \longrightarrow \infty$; also determine for which α, if any, all (non-zero) solutions become unbounded as $t \longrightarrow \infty$.

28 (B27). Consider the equation $ay'' + by' + cy = d$, where a, b, c and d are constants.

a) Find all equilibrium, or constant, solutions of this differential equation.

b) Let y_e denote an equilibrium solution, and let $Y = y - y_e$. Thus Y is the deviation of a solution y from an equilibrium solution. Find the differential equation satisfied by Y.

Homework Problems

These problems are taken from section §3.3 of the textbook.

29 (B6). Use Euler's formula to write

$$\pi^{-1+2i}$$

in the form $a + bi$.

30 (B11). Find the general solution to

$$y'' + 6y' + 13y = 0.$$

31 (B34). An equation of the form

$$t^2 \frac{d^2y}{dt^2} + \alpha t \frac{dy}{dt} + \beta y = 0, \quad t > 0,$$

where α and β are real constants, is called an Euler equation.

a) Let $x = \ln(t)$ and calculate $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ in terms of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

b) Use the results of part (a) to transform the Euler equation into

$$\frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0.$$

Observe that this equation has constant coefficients. If $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions to this equation, then $y_1(\ln t)$ and $y_2(\ln t)$ form a fundamental set of solutions of the Euler equation.

32 (B40). Using the method of problem 34, solve the equation

$$t^2y'' - ty' + 5y = 0.$$

33 (B46). Consider the differential equation

$$ty'' + (t^2 - 1)y' + t^3y = 0, \qquad 0 < t < \infty.$$

Try to transform this equation into one with constant coefficients using the method described in problem 43 of the textbook. If this is possible, find the general solution.

2 Repeated Roots

Consider the problem of solving a 2nd order linear homogeneous equation with constant coefficients, i.e. something like

$$ay'' + by' + cy = 0,$$

where a, b, and c are constants. In the previous section, we omitted a special case from the general solution. What happens when the characteristic equation has only one real root? For example, consider the differential equation

$$y'' - 2y' + y = 0,$$

which has characteristic equation $r^2 - 2r + 1 = 0$. The left-hand side factors as $(r-1)^2$, so the only solution is $r = 1$. Then we only find one solution, $y = e^{1 \cdot t}$. But a fundamental set of solutions consists of two solutions which are not constant multiples of each other! Since there is only one root of the characteristic equation, we can't immediately find another solution. What to do?

The Idea of Deformation

Imagine that the characteristic equation was slightly different, say $(r-1)(r-(1+\varepsilon))$, where $\varepsilon > 0$ is a very small positive number. Then we would have two independent solutions:

$$y_1(t) = e^t, \qquad y_2(t) = e^{(1+\varepsilon)t}.$$

Of course, these are solutions to a modified or "deformed" differential equation:

$$y'' - (2+\varepsilon)y' + (1+\varepsilon)y = 0.$$

The point here is that the smaller ε is, the closer we are to the "original" differential equation above.

Now, we will choose a particular solution, say with initial values $y(0) = 0$, and $y'(0) = 1$. Then the general solution

$$y = c_1 y_1 + c_2 y_2,$$

together with these initial conditions, yields two equations in two unknowns:

$$\begin{aligned} 0 &= y(0) = c_1 + c_2 \\ 1 &= y'(0) = c_1 + (1+\varepsilon)c_2. \end{aligned}$$

Solving this system gives us that $c_1 = -\frac{1}{\varepsilon}$, $c_2 = \frac{1}{\varepsilon}$. In otherwords,

$$y(t) = -\frac{1}{\varepsilon}e^t + \frac{1}{\varepsilon}e^{(1+\varepsilon)t} = e^t\left(\frac{-1+e^{\varepsilon t}}{\varepsilon}\right).$$

The question is, what happens as $\varepsilon \longrightarrow 0$? We take the limit

$$\lim_{\varepsilon \longrightarrow 0} \frac{-1+e^{\varepsilon t}}{\varepsilon} = \lim_{\varepsilon \longrightarrow 0} \frac{te^{\varepsilon t}}{1} = t.$$

(In evaluating the limit, we used L'Hôpital's rule, differentiating with respect to ε.) Then the solution $y(t) = e^t \cdot t$ must be a solution to the differential equation when $\varepsilon = 0$, which is exactly the original problem!

So we have done it! By deforming the original equation, we could use the usual method to find two solutions; by taking a limit we get a new solution that we didn't yet know. Indeed, $\{e^t, te^t\}$ is a fundamental set of solutions.

2.1 Proposition. *Consider the characteristic equation to a second-order homogeneous linear differential equation with constant coefficients. If r is a repeated root to the characteristic equation, then a fundamental set of solutions is*

$$\{e^{rt}, te^{rt}\}.$$

Proof. We may assume that the differential equation is of the form $y'' - 2ry' + r^2y = 0$, dividing by a constant if necessary. It is easy to check that te^{rt} is not a constant multiple of e^{rt} (do it!), so it suffices to check that $y_2(t) = te^{rt}$ is a also a solution to the differential equation. We compute that

$$y_2'(t) = (1+rt)e^{rt} \qquad \text{and} \qquad y_2''(t) = (2r + r^2t)e^{rt}.$$

If we plug this into the differential equation, we get

$$\begin{aligned} 0 &= e^{rt}\left((2r + r^2t) - 2r(1+rt) + r^2t\right) \\ &= e^{rt}\left(0\right). \end{aligned}$$

Thus te^{rt} satisfies the differential equation. □

2.2 Note. In the future, you do not need to use the idea of deformation to solve problems where the characteristic equation has only one root. After all, once we could guess the answer it was easy to verify directly (as we did above) that te^{rt} really does work. You may assume this going forward.

The previous derivation was a nice example which shows how to get information from old techniques when they do not apply to new problems. This is an elegant idea that you shouldn't forget, but it is not necessary for routine calculations now that we know the answer.

2.1 Mechanical and Electrical Vibrations

Recall from Math 125, given a spring with "spring-constant" k, the force of the spring acting on the mass is given by

$$F(u) = -ku,$$

where u is the displacement from the equilibrium position. Since force is mass times acceleration, this can be re-written as $mu'' = -ku$, or

$$mu'' + ku = 0.$$

We also allow for the possibility of a viscous resistance force. Such a force is defined as one proportional to the velocity of the mass at any time, retarding the mass. So this additional force is $-\gamma u'$, where γ is the proportionality constant. Thus $mu' + ku = -\gamma u'$, or

$$mu'' + \gamma u' + ku = 0.$$

More generally, we might want to consider applying any other arbitrary force to the system, say $g(t)$. This gives our final equation:

$$mu'' + \gamma u' + ku = g(t).$$

To summarize: the physical interpretation / definitions of the constants are:

m: Mass

γ: Damping constant

k: Spring constant

$g(t)$: Applied force.

Note that all of the constants m, γ, and k, are each non-negative: $m, \gamma, k \geq 0$.

2.2 Undamped, Free system.

That the system is "undamped" means that $\gamma = 0$. That the system is "free" means that it is free of any external or applied force, i.e. $g(t) = 0$. Thus the equation reduces to

$$mu'' + ku = 0.$$

It is easy for us to solve this equation, since it is homogeneous with constant coefficients. The characteristic equation is

$$mr^2 + k = 0 \qquad r = i\sqrt{k/m}.$$

So the complex solution $e^{rt} = \cos(w_0 t) + i\sin(w_0 t)$ for $w_0 = \sqrt{k/m}$. Then the fundamental set of real solutions is just

$$\{\cos(w_0 t), \sin(w_0 t)\},$$

and so the general solution is

$$y(t) = A\cos(w_0 t) + B\sin(w_0 t).$$

It is useful to combine this sum of trig functions into a single trig function. Recall the trig identity

$$\cos(G - H) = \cos(G)\cos(H) + \sin(G)\sin(H),$$

from which it follows that

$$R\cos(w_0 t - \delta) = R\cos(w_0 t)\cos(\delta) + R\sin(w_0 t)\sin(\delta).$$

Then $R\cos(w_0 t - \delta) = A\cos(w_0 t) + B\sin(w_0 t)$ precisely when

$$A = R\cos(\delta) \qquad \text{and} \qquad B = R\sin(\delta).$$

This means, in particular, that $R^2 = A^2 + B^2$ and $\delta = \tan^{-1}(B/A)$.

To illustrate this conversion in action, consider some initial conditions, like $y(0) = y_0$ and $y'(0) = y'_0$; we specify a particular solution by plugging in the initial conditions to get

$$\begin{aligned} y_0 &= A \\ y'_0 &= Bw_0 \end{aligned}$$

so that $y(t) = y_0\cos(w_0 t) + \frac{y'_0}{w_0}\sin(w_0 t)$. Now, we want to rewrite this in the form $y(t) = R\cos(w_0 t - \delta)$, so we calculate

$$R = \sqrt{y_0^2 + \frac{y_0'^2}{w_0^2}}, \qquad \delta = \tan^{-1}(\frac{y'_0}{y_0 w_0}).$$

2.3 Definition.

i) R is called the *amplitude* of the motion

ii) δ is called *the phase*

iii) $T = \frac{2\pi}{w_0}$ is called *the period*.

iv) w_0 is called the *natural frequency*.

The previous calculation shows precisely how the amplitude of the general solution depends on the natural frequency and the initial conditions.

2.3 Damped, Free system.

That the system is damped means that $\gamma > 0$, and it is free so that $g(t) = 0$, again. The model is then:

$$mu'' + \gamma u' + ku = 0.$$

This is a homogeneous linear equation with constant coefficients. The characteristic equation is

$$mr^2 + \gamma r + k = 0 \qquad r = \frac{\gamma}{2m}(-1 \pm \frac{1}{\gamma}\sqrt{\gamma^2 - 4km}) = \frac{\gamma}{2m}(-1 \pm \sqrt{1 - \frac{4km}{\gamma^2}}).$$

Observe that $\sqrt{1 - \frac{4km}{\gamma^2}}$ is never greater than 1 (if it is real), since $4km/\gamma^2 > 0$. This shows that the real part of r *will always be negative.*

There are essentially three cases now; let $\Delta = \gamma^2 - 4km$ be the stuff under the radical. This is often called the *discriminant*.

$\Delta > 0$: This occurs when $\gamma^2 > 4km$, so then r_1 and r_2 are distinct real numbers; thus, the exponential solutions $e^{r_1 t}$ and $e^{r_2 t}$ are real, independent solutions. As we just mentioned, these are both negative numbers, so that the homogeneous solution decays exponentially quickly to 0 as $t \longrightarrow \infty$. Observe: The solutions exhibit NO periodic behavior. This is because the damping constant, γ is so big. Physically, we refer to this situation as *overdamped*.

$\Delta = 0$: This occurs when $\gamma^2 = 4km$. Then r is a double root, so that e^{rt} and te^{rt} are both solutions, where $r = -\gamma/(2m)$. Again, these solutions decay exponentially quickly to 0 as $t \longrightarrow \infty$. Again, the solutions exhibit NO periodic behavior, but if γ were even slightly smaller we would. We call this situation *critically damped*.

$\Delta < 0$: This occurs when $\gamma^2 < 4km$, so for γ small enough. Then $r = \frac{\gamma}{2m}(-1 + \frac{i}{\gamma}\sqrt{4km - \gamma^2})$, so that $e^{rt} = e^{-\gamma t/(2m)}(\cos(\mu t) + i\sin(\mu t))$, where $\mu = \frac{\sqrt{4km-\gamma^2}}{2m} > 0$.

We look more closely at this third case:

2.4 Definition. In the case that $\gamma^2 < 4km$ (case 3, above):

i) μ is called the *quasi-frequency* of the motion

ii) $T_d = \frac{2\pi}{\mu}$ is called the *quasi-period*.

iii) We say that the system is *underdamped*.

It is of some interest to compare the quasi-frequency with the natural frequency for a given system. Remember that the natural frequency is the frequency of the general solution for the corresponding undamped, free system (e.g. set $\gamma = 0$ and find the frequency of the solution). In particular, we can see how changing the damping constant changes the oscillatory behavior of the solution. One can calculate that

$$\frac{\mu}{w_0} = \sqrt{1 - \frac{\gamma^2}{4km}} = \frac{T}{T_d}.$$

From here we see that as γ increases towards critical damping, where $\gamma^2 = 4km$, the ratio $\frac{\mu}{w_0} \longrightarrow 0$. This means that $\mu \longrightarrow 0$, which is what we expect given the behavior of the 2nd case above. Reciprocally, the quasi-period $T_d \longrightarrow \infty$ as $\gamma^2 \longrightarrow 4km$.

2.5 Examples. A spring is stretched 10cm by a 3N force. A 2kg mass is hung and is attached to a damper which exerts 3N when the velocity of the mass is 5m/s. If the mass is pulled down 5 cm below its equilibrium position and given an initial downward velocity of 10cm/s, determine its position u at any time t. Find the quasi-frequency μ and the ratio of μ to the natural frequency of the corresponding undamped motion.

2.6 Solution. We need to figure out what the constants m, γ and k, and the function $g(t)$ are. There is no mention of any applied force, so $g(t) = 0$. Also, $m = 2$kg. To find k, we use that the spring is stretched 0.1m by a 3N force, so that $k = 3/0.1 = 30$N/m. Finally, the damping force exerts 3N when the velocity is 5m/s, so that

$$\gamma(5m/s) = 3N \qquad \gamma = 0.6.$$

So our system is modeled by

$$2u'' + 0.6u' + 30u = 0.$$

We solve the characteristic equation and get $r = -0.15 \pm 3.87i$. Thus, the general equation is given by

$$y = c_1 e^{-0.15t}\cos(3.87t) + c_2 e^{-0.15t}\sin(3.87t).$$

To find c_1 and c_2, we need to know the initial conditions. From the problem, we find that $u(0) = 5cm = 0.05m$, and $u'(0) = 10cm/s = 0.1m/s$. Solving for c_1 and c_2, we get $c_1 = 0.05$ and $c_2 = 0.0277$.

So $y = 0.05e^{-0.15t}\cos(3.87t) + 0.0277e^{-0.15t}\sin(3.87t)$. However, we need to write this as $y = R\cos(3.87t - \delta)$ for appropriate R and δ. Since $R^2 = c_1^2 + c_2^2$, we get $R = 0.0572$, and $\delta = \tan^{-1}(c_2/c_1) = 0.507$.

Finally, $\mu/w_0 = \sqrt{1 - \gamma^2/(4km)} = 0.99925$.

2.4 Electric Circuits (in Series)

We model the charge on the capacitor (and the current) for an electric circuit in series, consisting of some impressed voltage, a resistor, a capacitor and an inductor. Such a circuit is often called an *RLC Circuit*. Let Q be the total charge in Coulombs on the capacitor at time t. Then the model is

$$LQ'' + RQ' + \frac{1}{C}Q = E(t),$$

where L is the inductance of the inductor (measured in Henrys), R is the resistance of the resistor (measured in Ohms), C is the capacitance of the capacitor, and $E(t)$ is the impressed voltage.

If we let $I = \frac{dQ}{dt}$, where I is an amount of current measured in Amperes, then we get a second-order differential equation in I by differentiating the previous equation:

$$LI'' + RI' + \frac{1}{C}I = E'(t).$$

In physics, the unit of current is considered fundamental in electricity and magnetism. Conceptually, it is a rate of change of electrons flowing through a conductor, so one might think that the unit of charge is fundamental. However, it is much easier to measure current in the lab than to count electrons, and so the convention persists. Mathematically, the form of the two models is the same.

The entire discussion about the spring-mass model is applicable to this model. To be clear, L is playing the role of the mass, R is playing the role of the damping constant, and $\frac{1}{C}$ is playing the role of the spring constant k. In particular, we expect to have a critical damping level of resistance R, which differentiates the solutions into the 3 cases just as before.

2.7 Examples. If a series circuit has a capacitor $C = 0.8 \times 10^{-6}$F and an inductor of $L = 0.2$H, find the resistance R so that the circuit is critically damped.

2.8 Solution. We know the current is modeled by

$$LI'' + RI' + \frac{1}{C}I = 0;$$

the characteristic equation is then $Lr^2 + Rr + \frac{1}{C} = 0$; so $r = \frac{-R}{2L} \pm \frac{1}{2L}\sqrt{R^2 - \frac{4L}{C}}$. We recall that critical damping occurs when the discriminant $\Delta = R^2 - \frac{4L}{C}$ is 0, i.e. when

$$R = 2\sqrt{\frac{L}{C}} = 2\sqrt{\frac{0.2}{0.8 \times 10^{-6}}} = 2(500) = 1000 \text{ Ohms}.$$

Homework Problems

These problems are taken from section §3.7 of the textbook.

34 (B2). Consider the function $u(t) = -\cos t + \sqrt{3}\sin t$. Find ω_0, R, and δ to rewrite as

$$u(t) = R\cos(\omega_0 t - \delta).$$

35 (B6). A mass of 100g stretches a spring 5 cm. If the mass is set in motion from its equilibrium position with a downward velocity of 10cm/s, and if there is no damping, determine the position u of the mass at any time t. When does the mass first return to its equilibrium position?

36 (B7). A mass weighing 3 lb stretches a spring 3 in. If the mass is pushed upward, contracting the spring a distance of 1 in, and then set in motion with a downward velocity of 2 ft/s, and if there is no damping, find the position u of the mass at any time t. Determine the frequency, period, amplitude, and phase of the motion.

37 (B8). A series circuit has a capacitor of 0.25×10^{-6} F and an inductor of 1 H. If the initial charge on the capacitor is 10^{-6} C and there is no initial current, find the charge Q on the capacitor at any time t.

38 (B12). A series circuit has a capacitor of 10^{-5} F, a resistor of $3\times10^2\,\Omega$, and an inductor of $0.2\,H$. THe initial charge on the capacitor is $10^{-6}\,C$ and there is no initial current. Find the charge Q on the capacitor at any time t.

3 Reduction of Order

The general procedure for solving a characteristic equation with multiple roots can be summarized as follows: beginning with a linear homogeneous differential equation, we know one solution, $y_1(t)$. We then find another solution $y_2(t) = t * y_1(t)$ to obtain a fundamental set of solutions, $\{y_1, y_2\}$.

Now we consider a related problem. Someone tells you that $y_1(t) = t^2$ is a solution to

$$t^2 y'' - 4ty' + 6y = 0.$$

How can you find another solution to complete a fundamental set?

Of course, we should try to repeat the trick that worked before! Let $y_2(t) = t \cdot t^2 = t^3$. Does this satisfy the differential equation? We calculate:

$$y_2'(t) = 3t^2 \qquad y_2''(t) = 6t,$$

so that $t^2 y_2'' - 4ty_2' + 6y_2 = t^2(6t) - 4t(3t^2) + 6y^3 = 0$. So $y_2(t)$ is a solution! Thus,

$$\{t^2, t^3\}$$

is a fundamental set of solutions.

Unfortunately, the previous example was just luck, as the next example shows:

3.1 Examples. Someone tells you that $y_1(x) = e^x$ is a solution to

$$(x-1)y'' - xy' + y = 0.$$

Find another solution to complete a fundamental set.

Discussion. Let's see if we can just multiply by x: $y_2(x) = xe^x$. We calculate:

$$y_2'(x) = e^x(1+x) \qquad y_2''(x) = e^x(2+x).$$

Then

$$\begin{aligned}(x-1)y_2'' - xy_2' + y_2 &= (x-1)e^x(2+x) - xe^x(1+x) + xe^x \\ &= e^x(x-2) \neq 0.\end{aligned}$$

This shows that $y_2(x) = xe^x$ is NOT a solution.

Apparently, multiplication by x does not work in general. Perhaps we should multiply by some other function. Since we don't know which function to use, we just multiply by some function $w(x)$, and then see what properties $w(x)$ needs to satisfy. Observe:

3.2 Solution. We let $y_2(x) = w(x)e^x$. We calculate:

$$y_2'(x) = e^x(w + w') \qquad y_2''(x) = e^x(w + 2w' + w'').$$

Then

$$\begin{aligned}(x-1)y_2'' - xy_2' + y_2 &= (x-1)e^x(w + 2w' + w'') - xe^x(w + w') + we^x \\ &= e^x\left((x-1)w'' + (x-2)w'\right).\end{aligned}$$

The important thing to notice is that all of the w's cancelled - this happened precisely because $y_1(x) = e^x$ is a solution to the equation. The resulting equation is:

$$(x-1)w'' + (x-2)w' = 0,$$

which is really a first order differential equation for $v = w'$, i.e. $(x-1)v' + (x-2)v = 0$. This is separable (because the original differential equation was linear), thus:

$$\begin{aligned} \text{Rewrite:} \quad & \frac{v'}{v} = -\frac{x-2}{x-1} \\ \text{Integrate:} \quad & \ln(v) = -x + \ln(x-1) \\ \text{Solve:} \quad & v = (x-1)e^{-x}. \end{aligned}$$

But don't forget: $v = w'$, so we have to integrate *again*! Then

$$w(x) = \int v(x)dx = -(x-1)e^{-x} - e^{-x} = -xe^{-x}.$$

Finally, $y_2(x) = -xe^{-x}y_1(x) = -x$. (If you're not convinced, check directly that $y_2(x) = -x$ really is a solution!) Then a fundamental set of solutions is given by

$$\{e^x, -x\}.$$

Of course, we can replace either of these solutions by non-zero multiples of themselves, and we will still have a fundamental set. So $\{x, e^x\}$ is also a fundamental set, and it looks a bit more natural.

General Method

We present here a generic formula for using the reduction of order technique. The setup is as before: we have a linear 2nd order differential equation

$$y'' + p(t)y' + q(t)y = 0,$$

and we are given a solution, $y_1(t)$. We guess that $y_2(t) = w(t) \cdot y_1(t)$ is another independent solution, and we use this assumption to solve for $w(t)$.

Indeed, let's compute

$$y_2'(t) = w'y_1 + wy_1' \qquad y_2''(t) = w''y_1 + 2w'y_1' + wy_1''.$$

It follows that

$$\begin{aligned} 0 &= y_2'' + p(t)y_2' + q(t)y_2 \\ &= (w''y_1 + 2w'y_1' + wy_1'') + p(t)\,(w'y_1 + wy_1') + q(t)\,(wy_1) \\ &= w\,(y_1'' + p(t)y_1' + q(t)y_1) + w'(2y_1' + p(t)y_1) + w''y_1 \\ &= w'(2y_1' + p(t)y_1) + w''y_1. \end{aligned}$$

We re-arrange this to get the equivalent equation

$$w'' = -w' \left(2\frac{y_1'}{y_1} + p(t) \right),$$

which is a separable first-order equation for the function $w'(t)$. Indeed, we separate variables to get

$$\begin{aligned} \text{Rewrite:} \quad & d(\ln(w')) = -\left(2\frac{d(\ln(y_1))}{dt} + p(t) \right) dt \\ \text{Integrate:} \quad & \ln(w') = -2\ln(y_1) - \int p(t)dt + C_1 \\ \text{Solve:} \quad & w' = \frac{C_2}{y_1^2} \cdot e^{-\int p(t)dt} \end{aligned}$$

We can choose $C_2 = 1$, since $y_2(t)$ (and hence $w(t)$) is only desired up to a constant multiple. Thus, we find that

$$w(t) = \int \frac{e^{-\int p(t)dt}}{y_1^2} dt.$$

Again, $w(t)$ has a constant of integration, but we can choose this to be zero. Otherwise, if we add a constant C, we obtain the solution

$$y_2(t) = (w(t) + C)y_1(t) = w(t)y_1(t) + Cy_1(t).$$

But we are only interested in finding an independent solution $y_2(t)$, so we can safely ignore $Cy_1(t)$ since we already know that it is a solution to the equation.

Finally, we record that

$$y_2(t) = w(t)y_1(t) = y_1 \int \frac{e^{-\int p(t)dt}}{y_1^2} dt.$$

Homework Problems

These problems are taken from section §3.4 of the textbook.

39 (B14). Solve the initial value problem

$$y'' + 4y' + 4y = 0, \qquad y(-1) = 2, \quad y'(-1) = 1.$$

Sketch the graph of the solution and describe its behavior for increasing t.

40 (B18). Consider the initial value problem

$$9y'' + 12y' + 4y = 0, \qquad y(0) = a > 0, \quad y'(0) = -1.$$

i) Solve the initial value problem.

ii) Find the critical value of a that separates solutions that become negative from those that are always positive.

41 (B30). Use the method of reduction of order to find a second solution to

$$x^2y'' + xy' + (x^2 - 0.25)y = 0, \qquad x > 0; \quad y_1(x) = x^{-1/2}\sin(x).$$

42 (B32). The differential equation

$$y'' + \delta(xy' + y) = 0$$

arises in the study of turbulent flow of a uniform stream past a circular cylinder. Verify that $y_1(x) = e^{-\delta x^2/2}$ is one solution and then find the general solution in the form of an integral.

43 (B46). Solve the following Euler equation

$$t^2y'' + 5ty' + 13y = 0, \qquad t > 0.$$

4 Non-homogeneous Linear Equations

Consider a general second-order linear equation:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t), \qquad (NH)$$

where p, q and g are functions of the independent variable, t. Remember that the *corresponding homogeneous equation* is the same, but with $g(t)$ replaced by 0:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0 \qquad (H)$$

We now introduce a little bit of terminology to help us understand the strategy for solving non-homogeneous equations.

First, we define "L" to be the function $L = \frac{d^2}{dt^2} + p(t)\frac{d}{dt} + q(t)$. This function takes as its argument a function y, and it returns a function $Ly = \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y$. It is absolutely necessary that the domain of L contain functions which are at least twice differentiable, so that we can make sense of $\frac{d^2y}{dt^2}$. If $C^\infty(\mathbb{R})$ denotes the set of all infinitely-differentiable functions, then we can restrict L to $L : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$. Quite often, a function whose domain consists of other functions is called an *operator*; this word helps us distinguish between an operator and a function in the domain of that operator.

The operator L defined above has some very useful properties. In particular, it is *linear*:

i) $L(y_1 + y_2) = L(y_1) + L(y_2)$, any functions y_1, y_2 in the domain of L

ii) $L(c \cdot y) = c \cdot L(y)$ for all functions y in the domain and numbers $c \in \mathbb{R}$ (or even complex numbers).

Now we consider the problem of finding solutions to the non-homogeneous linear equation:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t) \qquad \text{equivalently} \qquad Ly = g.$$

Suppose first that y_1 and y_2 are two solutions of (NH). This means exactly that $Ly_1 = g = Ly_2$. Since L is a linear operator, we must have

$$L(y_1 - y_2) = L(y_1) - D(y_2) = g - g = 0.$$

This means exactly that $y_1 - y_2$ satisfies (H)!

Conversely, assume we have a solution y_g to (NH) and a solution y_0 to (H). This means that $Ly_g = g$ and $Ly_0 = 0$. Then

$$L(y_g + y_0) = Ly_g + Ly_0 = g + 0 = g.$$

So $y_g + y_0$ is another solution to (NH).

4.1 Proposition. *The general solution to a second-order linear equation can be written as*

$$y = c_1y_1 + c_2y_2 + y_g,$$

where $\{y_1, y_2\}$ *is a fundamental set of solutions to the* associated homogeneous equation*, and* y_g *is any one solution to the non-homogeneous equation.*

Proof. If y is any particular solution, then $y - y_g$ is a solution to the associated homogeneous solution. Since $\{y_1, y_2\}$ is a fundamental set of solutions to the homogeneous equation, we can find constants c_1 and c_2 such that

$$y - y_g = c_1y_1 + c_2y_2.$$

It follows immediately that $y = c_1y_1 + c_2y_2 + y_g$. □

We give now the outline for solving non-homogeneous linear equations:

Step 1 Solve the associated homogeneous equation.

Step 2 Find a single solution of the non-homogeneous equation.

Step 3 Add the particular solution from Step 2 to the general homogeneous solution to Step 1.

4.1 Undetermined Coefficients - Guess and Check

4.2 Examples. Find the general solution to $y'' - 3y' - 4y = 3e^{2t}$.

4.3 Solution.

Step 1 Solve the associated homogeneous equation. In this case, that is

$$y'' - 3y' - 4y = 0.$$

This is a linear homogeneous equation with constant coefficients, so we solve the characteristic equation:

$$r^2 - 3r - 4 = 0 \qquad r = -1, 4.$$

Then the general solution is $y = c_1 e^{-t} + c_2 e^{4t}$.

Step 2 Find a single solution to the non-homogeneous equation. In this case, we guess that $v = Ae^{2t}$. Without plugging it into the equation and solving for A, we don't know for sure that it will work. But it seems like a reasonable place to start.

Using the notation from before, $L = \frac{d^2}{dt^2} - 3\frac{d}{dt} - 4$. A calculation gives us that

$$Lv = -6Ae^{2t}.$$

Since we want v to be a solution, we want $Lv = 3e^{2t}$. Thus, we want

$$-6A = 3 \qquad \text{or} \qquad A = -1/2.$$

Summarizing, if $v = -1/2e^{2t}$ then $Dv = 3e^{2t}$, i.e. v is a solution to the non-homogeneous equation!

Step 3 The general solution to the non-homogeneous equation is then

$$y = c_1 e^{-t} + c_2 e^{4t} - \frac{1}{2}e^{2t}.$$

4.4 Examples. Find the general solution to $y'' + 3y' + 2y = \sin(t) + \cos(2t)$.

4.5 Solution. We first find the homogeneous solution. The characteristic equation is $r^2 + 3r + 2 = 0$, so $r = -1, -2$. Then

$$y_h = c_1 e^{-t} + c_2 e^{-2t}.$$

Next, we need to find a particular solution. We guess the "form" of each function and add to give

$$y_p = A\cos(t) + B\sin(t) + C\cos(2t) + D\sin(2t).$$

Differentiating, we get

$$\begin{aligned} y_p' &= -A\sin(t) + B\cos(t) - 2C\sin(2t) + 2D\cos(2t) \\ y_p'' &= -A\cos(t) - B\sin(t) - 4C\cos(2t) - 4D\sin(2t). \end{aligned}$$

Plugging into the differential equation, we get

$$\begin{aligned} Ly_p &= [-A\cos(t) - B\sin(t) - 4C\cos(2t) - 4D\sin(2t)] \\ &\quad + 3[-A\sin(t) + B\cos(t) - 2C\sin(2t) + 2D\cos(2t)] \\ &\quad + 2[A\cos(t) + B\sin(t) + C\cos(2t) + D\sin(2t)] \\ &= (-A + 3B + 2A)\cos(t) + (-B - 3A + 2B)\sin(t) \\ &\quad + (-4C + 6D + 2C)\cos(2t) + (-4D - 6C + 2D)\sin(2t). \end{aligned}$$

Remember that we want y_p to be a solution; thus we need $Ly_p = \sin(t) + \cos(2t)$. Comparing the coefficients of all the functions $\cos(t)$, $\sin(t)$, $\cos(2t)$ and $\sin(2t)$, we get the system(s) of equations:

$$\begin{aligned} A + 3B &= 0 \qquad & -2C + 6D &= 1 \\ -3A + B &= 1 \qquad & -6C - 2D &= 0. \end{aligned}$$

We solve these to find that $B = 1/10$, $A = -3/10$, $C = -1/20$, and $D = 3/20$. Thus the particular solution

$$y_p = -\frac{3}{10}\cos(t) + \frac{1}{10}\sin(t) - \frac{1}{20}\cos(2t) + \frac{3}{20}\sin(2t),$$

and the general solution is

$$y = y_h + y_p.$$

4.6 Examples.

$$y'' - 3y' - 4y = e^{4t}.$$

4.7 Solution. Since the corresponding homogeneous equation is the same as the previous one, we can re-use step 1. For step 2, let's try our naive guess that

$$v = Ae^{4t}.$$

A calculation gives us that $Lv = 0$. But we wanted $Lv = e^{4t}$! This is not true no matter which value of A we pick. This shows that our guess was bad, and we need to try something else.

If we think carefully, we notice that e^{4t} is a solution to the homogeneous equation - it is one of the elements in the fundamental set $\{e^{-t}, e^{4t}\}$. That is what went wrong.

It might be intuitive to guess the following instead: $v = Ate^{4t}$. Let's see - a calculation gives that

$$Lv = 5Ae^{4t}.$$

Since we want $Lv = e^{4t}$, we require that $5A = 1$, or $A = 1/5$. Thus, a particular solution is given by

$$v = \frac{1}{5}te^{4t}.$$

Finally, the general solution is the sum of the associated homogeneous solution and the particular solution:

$$y = c_1e{-}t + c_2e^{4t} + \frac{1}{5}te^{4t}.$$

4.8 Examples. Solve the initial value problem

$$y'' + y' - 2y = 2t \qquad \begin{cases} y(0) = 0 \\ y'(0) = 1 \end{cases}$$

4.9 Solution. We first find the general solution; only at the end can we use the initial values to find a particular solution. To find the general solution we need to find the general solution to the corresponding homogeneous equation and then find a particular solution.

The corresponding homogeneous equation is:

$$y'' + y' - 2y = 0,$$

which has characteristic equation $r^2 + r - 2 = 0$. Solving for r, we get $r = -2, 1$. Thus the general solution to the homogeneous equation is

$$y = c_1 e^{-2t} + c_2 e^t.$$

To guess a particular solution to the non-homogeneous equation, we look at $g(t) = 2t$. This is a polynomial of degree 1. It may be intuitive to guess that $v(t) = A_0 + A_1 t$, a generic first-degree polynomial, will work. Let's try it! For $L = \frac{d^2}{dt^2} + \frac{d}{dt} - 2$, we calculate

$$Lv = -2A_1 t + (A_1 - 2A_0).$$

Since we want $Lv = 2t$, we require that

$$A_1 = -1, \quad \text{and} \quad A_0 = -\frac{1}{2}.$$

Thus $v = -t - \frac{1}{2}$ works!

Finally, we write down the general solution:

$$y = c_1 e^{-2t} + c_2 e^t - t - \frac{1}{2}.$$

We want to solve for c_1 and c_2 using the initial conditions. We get:

$$\begin{aligned} 0 &= y(0) = c_1 + c_2 - 1/2 \\ 1 &= y'(0) = -2c_1 + c_2 - 1. \end{aligned}$$

Solving this system yields $c_1 = -1/2$, and $c_2 = 1$. Thus

$$y(t) = -\frac{1}{2}e^{-2t} + e^t - t - \frac{1}{2}.$$

4.10 Examples. Find the general solution to $y'' - 4y' + 4y = (t + 1)e^{2t}$.

4.11 Solution. The characteristic equation is $r^2 - 4r + 4 = (r - 2)^2 = 0$. So $r = 2$ with multiplicity 2. The homogeneous solution is

$$y_h = c_1 e^{2t} + c_2 t e^{2t}.$$

Since $t + 1$ is a polynomial of degree 1, we guess for the particular solution of $t + 1$ a generic polynomial $a + bt$ of degree 1. Similarly, we guess Ae^{2t} for e^{2t}. Our provisional guess is then the product

$$y_p = A(a + bt)e^{2t}.$$

Observe that the constant A is extraneous, since a and b are already arbitrary and we can just distribute. This means we can choose $A = 1$ (i.e. delete it from consideration). Finally, we note that (at least) one of the terms of our guess, e.g. ae^{2t}, is a fundamental solution to the homogeneous equation. It is necessary to multiply this by t^2 since $r = 2$ is a root of order 2. In this case, we multiply the entire guess by t^2, and so our final guess is

$$y_p = (a_0 t^2 + a_1 t^3)e^{2t}.$$

You can check that

$$\begin{aligned} y_p' &= \left[(2a)t + (2a+3b)t^2 + (2b)t^3\right] e^{2t} \\ y_p'' &= \left[2a + (8a+6b)t + (4a+12b)t^2 + (4b)t^3\right] e^{2t}. \end{aligned}$$

Plugging this into the differential equation, we get (after much simplification)

$$Ly_p = \left[2a + (6b)t\right] e^{2t}.$$

Since we want $Ly_p = (1+t)e^{2t}$, we need that $a = 1/2$, and $b = 1/6$. Thus $y_p = (t^2/2 + t^3/6)e^{2t}$, and so the general solution is

$$y = y_h + y_p = c_1 e^{2t} + c_2 t e^{2t} + (t^2/2 + t^3/6)e^{2t}.$$

4.2 Undetermined Coefficients with Complexification

4.12 Examples. Find the general solution to

$$y'' + 2y' + y = e^{-t}\cos(t).$$

4.13 Solution. Since the characteristic equation is $(r+1)^2 = 0$, we know that the homogeneous solution is $y_h = c_1 e^{-t} + c_2 t e^{-t}$. What do we guess for the form of the particular solution y_p? Here are two ways to do this problem. First, we guess for each factor $(A\cos(t) + B\sin(t))$ and Ce^{-t}. You might notice that Ce^{-t} is already a homogeneous solution; however, you would be mistaken to multiply by t^2 in this case, because neither of the products $e^{-t}\cos(t)$ nor $e^{-t}\sin(t)$ is a solution of the homogeneous equation. Thus, our final guess is

$$y_p = (A\cos(t) + B\sin(t))e^{-t}.$$

Then

$$\begin{aligned} y_p' &= (-A\sin(t) + B\cos(t))e^{-t} - (A\cos(t) + B\sin(t))e^{-t} \\ &= \left[(-A-B)\sin(t) + (B-A)\cos(t)\right] e^{-t} \\ y_p'' &= \left[(-A-B)\cos(t) + (A-B)\sin(t)\right] e^{-t} - \left[(-A-B)\sin(t) + (B-A)\cos(t)\right] e^{-t} \\ &= \left[(-2B)\cos(t) + (2A)\sin(t)\right] e^{-t}. \end{aligned}$$

Plugging this into the differential equation we get

$$\begin{aligned} Ly_p = &\left[(-2B)\cos(t) + (2A)\sin(t)\right] e^{-t} + 2\left[(-A-B)\sin(t) + (B-A)\cos(t)\right] e^{-t} \\ &+ (A\cos(t) + B\sin(t))e^{-t}. \\ = &\left[(-A)\cos(t) + (-B)\sin(t)\right] e^{-t}. \end{aligned}$$

Since we want $Ly_p = e^{-t}\cos(t)$, we need $A = -1$ and $B = 0$. Then $y_p = -\cos(t)e^{-t}$, and the general solution is

$$y = y_h + y_p = c_1 e^{-t} + c_2 t e^{-t} - \cos(t)e^{-t}.$$

Complexified Method. We will use the following proposition to find an alternative solution.

4.14 Proposition. *Suppose that L is a linear operator (*with real coefficients*) and g is a complex valued function. Then*

$$Lv = g \qquad \text{if and only if} \qquad \begin{cases} L(Re(v)) = Re(g) \\ L(Im(v)) = Im(g). \end{cases}$$

Proof. ($\Rightarrow$:) We first assume that $Lv = g$ to show that $L(Re(v)) = Re(g)$ and $L(Im(v)) = Im(g)$. Write $v = Re(v) + iIm(v)$. Then

$$Lv = LRe(v) + iLIm(v) = g = Re(g) + iIm(g).$$

By comparing real and imaginary parts, we have $LRe(v) = Re(g)$ and $LIm(v) = Im(g)$ as desired. ($\Leftarrow$:) Exercise. □

We complexify the function $e^{-t}\cos(t)$ by replacing it with $e^{-t}e^{it} = e^{(-1+i)t}$. Then the differential equation is

$$y'' + 2y' + y = e^{(-1+i)t}.$$

The homogeneous solution is the same: $y_h = c_1e^{-t} + c_2te^{-t}$, and the particular solution has the form $y_p = Ae^{(-1+i)t}$. It is now very easy to compute the derivatives:

$$\begin{aligned} y_p' &= A(-1+i)e^{(-1+i)t} \\ y_p'' &= A(-1+i)^2e^{(-1+i)t} = A(-2i)e^{(-1+i)t}. \end{aligned}$$

Plugging into the differential equation, we get

$$\begin{aligned} Ly_p &= A(-2i)e^{(-1+i)t} + 2\left[A(-1+i)e^{(-1+i)t}\right] + Ae^{(-1+i)t} \\ &= A(-2i - 2 + 2i + 1)e^{(-1+i)t} = -Ae^{(-1+i)t}. \end{aligned}$$

Since we want $Ly_p = e^{(-1+i)t}$, we must have that $A = -1$. Thus $y_p = -e^{(-1+i)t}$. Finally, since $e^{-t}\cos(t)$ is the real part of the complexified function $e^{(-1+i)t}$, we compute the real part of the complex solution y_p:

$$Re(y_p) = -e^{-t}\cos(t);$$

this is the answer to the original problem.

Discussion. What are the advantages and disadvantages of each of the previous methods? The first method has the advantage of being obvious. You guess a form for each solution and then you find the parameters. The disadvantage is that it is (slightly) more difficult to compute the derivatives y_p' and y_p'', and then you have to solve a system of equations to find the constants.

The second method has the disadvantage of being subtle. You begin by changing the problem so that the function $g(t)$ is complex valued. For this to work well one must be comfortable manipulating complex-valued expressions. The advantage is that it is easy to calculate y_p' and y_p'', and there is no system of equations to solve at the end.

4.15 Examples. Find the general solution to

$$y'' - 3y' - 4y = 2\sin(t).$$

4.16 Solution. To solve this problem, we employ the following outrageous trick: we replace $2\sin(t)$ with the complex-valued function $2e^{it}$. The complete reasoning will become clear later, but for now just observe that the imaginary part $Im(2e^{it}) = 2\sin(t)$.

So to be clear, we are now solving a *different* equation:

$$y'' - 3y' - 4y = 2e^{it}.$$

But we can at least guess what the solution is now. Let $v = Ae^{it}$. Then a calculation gives us that

$$Lv = (-5 - 3i)Ae^{it}.$$

But we want $Lv = 2e^{it}$, so we must have

$$A = \frac{2}{-5-3i} = \frac{-5}{17} + \frac{3}{17}i.$$

Using Euler's formula, we can better express $v = Ae^{it}$ as

$$\begin{aligned} v &= (-\frac{5}{17} + \frac{3}{17}i)(\cos(t) + i\sin(t)) \\ &= \left[-\frac{5}{17}\cos(t) - \frac{3}{17}\sin(t)\right] + \left[\frac{3}{17}\cos(t) - \frac{5}{17}\sin(t)\right] i. \end{aligned}$$

Since $Lv = 2e^{it}$, we know that $Im(Lv) = 2\sin(t)$, by taking imaginary parts. Now, we will show that

$$Im(Lv) = L(Im(v));$$

to do this, write $v(t) = x(t) + iy(t)$, and note that $v'(t) = x'(t) + iy'(t)$, $v''(t) = x''(t) + iy''(t)$. Then

$$\begin{aligned} Lv = v'' - 3v' - 4v &= x'' + iy'' - 3(x' + iy') - 4(x + iy) \\ &= [x'' - 3x' - 4x] + [y'' - 3y' - 4y]\, i. \end{aligned}$$

Then we see that $Im(Lv) = Ly = L(Im(v))$.

In summary, if $y_g = Im(v)$, then $Ly_g = Im(Lv) = 2\sin(t)$; thus y_g is a particular solution to the *original* differential equation. Finally, the general solution is given by

$$\begin{aligned} y &= c_1e^{-t} + c_2e^{4t} + Im(v) \\ &= c_1e^{-t} + c_2e^{4t} + \frac{3}{17}\cos(t) - \frac{5}{17}\sin(t). \end{aligned}$$

4.17 Examples. Find the general solution to

$$y'' - 3y' - 4y = -8e^t\cos(2t).$$

4.18 Solution. We want to use the same trick as before, but we need to recognize $-8e^t\cos(2t)$ as the real or imaginary part of some exponential function. The function $-8e^t$ is already of the right form. Also, $\cos(2t) = Re(e^{2it})$. Thus,

$$-8e^t\cos(2t) = Re(-8e^te^{2it}) = Re(-8e^{(1+2i)t}).$$

So now we solve the *new* differential equation

$$Ly = -8e^{(1+2i)t}.$$

The homogeneous solutions are the same as before. Then we need to guess the form of our particular solution: $v(t) = Ae^{(1+2i)t}$. Now calculate:

$$Lv = (-10-2i)Ae^{(1+2i)t}.$$

Since we want $Lv = -8e^{(1+2i)t}$, we require that

$$A = \frac{-8}{-10-2i} = \frac{8}{10+2i} = \frac{80}{104} - \frac{16}{104}i.$$

Plugging this into v, and using Euler's formula,

$$\begin{aligned} v &= \left(\frac{80}{104} - \frac{16}{104}i\right)e^t(\cos(2t) + i\sin(2t)) \\ &= e^t\left(\frac{80}{104}\cos(2t) + \frac{16}{104}\sin(2t)\right) + e^t\left(-\frac{16}{104}\cos(2t) + \frac{80}{104}\sin(2t)+\right)i. \end{aligned}$$

Since $Lv = -8e^{(1+2i)t}$, we know from the fact above that

$$L(Re(v)) = Re(-8e^{(1+2i)t}) = -8e^t\cos(2t).$$

This means that $Re(v) = e^t\left(\frac{80}{104}\cos(2t) + \frac{16}{104}\sin(2t)\right)$ is a particular solution to the *original* differential equation.

Finally, the general solution is given by

$$y = c_1e^{-t} + c_2e^{4t} + e^t\left(\frac{80}{104}\cos(2t) + \frac{16}{104}\sin(2t)\right).$$

Homework Problems

These problems are taken from section §3.5 of the textbook.

44 (B11). Find the general solution to the differential equation

$$y'' + y' + 4y = 2\sinh t.$$

Recall: $\sinh t = \frac{e^t - e^{-t}}{2}$.

45. Find the solution to the initial value problems

B15) $y'' - 2y' + y = te^t + 4, \qquad y(0) = 1, \quad y'(0) = 1$

B18) $y'' + 2y' + 5y = 4e^{-t}\cos(2t), \qquad y(0) = 1, \quad y'(0) = 0$

46 (B33). In this problem we indicate an alternative procedure for solving the differential equation

$$y'' + by' + cy = (D^2 + bD + c)y = g(t),$$

where b and c are constants, and $D = \frac{d}{dt}$ is differentiation with respect to t. Let r_1 and r_2 be the zeros of the characteristic polynomial of the corresponding homogeneous equation. These roots may be real and different, real and equal, or conjugate complex numbers.

i) Verify that the differential equation above can be written in the factored form

$$(D - r_1)(D - r_2)y = g(t),$$

where $r_1 + r_2 = -b$ and $r_1 r_2 = c$.

ii) Let $u = (D - r_2)y$. Show that the solution of the equation can be found by solving the following two first-order equations:

$$(D - r_1)u = g(t), \qquad (D - r_2)y = u(t).$$

47 (B36). Use the method problem 33 to solve the differential equation

$$y'' + 2y' + y = 2e^{-t}.$$

5 Forced Vibrations

We consider a spring-mass system like we did before; we model it with the equation:

$$mu'' + \gamma u' + ku = g(t).$$

Now, we consider what happens when the forcing function $g(t) \neq 0$. More specifically, we will see what happens when a periodic force is applied, so that $g(t) = g_0 \cos(\omega t)$.

5.1 Definition. Consider a differential equation $Ly = g(t)$, where $g(t)$ is periodic. Suppose $y(t) = y_0(t) + y_g(t)$ is a solution, where $y_0(t) \longrightarrow 0$ as $t \longrightarrow \infty$ and $y_g(t)$ is periodic with the same period as $F(t)$. Then

i) $y_0(t)$ is called the *transient solution*.

ii) $y_g(t)$ is called the *steady-state solution* or the *forced response*.

The behavior of such a physical system is eventually described by the steady-state solution, so this is what we should look to observe. An important fact is that the steady state solution does not depend on the initial conditions.

Note: This gives physical meaning to our pre-existing mathematical concepts (which we defined as an aid to solving linear equations). The homogeneous solution is the transient solution, and the periodic particular solution is the forced response. It is nice to reconcile a mathematical understanding with the physical intuition of a transient and steady-state solution.

5.1 Resonance

Consider a mass-spring system. Since the steady-state solution is what we observe, we want to know how this depends on the forcing function, which we may be able to control (or predict or whatever). So given

$$Lu := mu'' + \gamma u' + ku = g_0 \cos(wt),$$

and given a particular solution $y_g(t) = R\cos(wt - \delta)$, how does R and δ depend on the parameters of the system: $\{m, \gamma, k, g_0, \omega\}$?

We may answer this question using the method of undetermined coefficients. We guess $y_g(t) = A\cos(wt) + B\sin(wt)$, and we compute

$$\begin{aligned} Lu &:= m(-w^2 A\cos(wt) - Bw^2\sin(wt)) + \gamma(-wA\sin(wt) + Bw\cos(wt)) + k(A\cos(wt) + B\sin(wt)) \\ &= (-mw^2 A + \gamma wB + kA)\cos(wt) + (-mw^2 B - \gamma wA + kB)\sin(wt) = g_0\cos(wt). \end{aligned}$$

This gives us the system of equations:

$$\begin{aligned} (k - mw^2)A + \gamma wB &= g_0 \\ -\gamma wA + (k - mw^2)B &= 0. \end{aligned}$$

We can solve this system using Cramer's rule:

$$A = \frac{\begin{vmatrix} g_0 & \gamma w \\ 0 & (k - mw^2) \end{vmatrix}}{\begin{vmatrix} (k - mw^2) & \gamma w \\ -\gamma w & (k - mw^2) \end{vmatrix}} \qquad B = \frac{\begin{vmatrix} (k - mw^2) & g_0 \\ -\gamma w & 0 \end{vmatrix}}{\begin{vmatrix} (k - mw^2) & \gamma w \\ -\gamma w & (k - mw^2) \end{vmatrix}}.$$

It follows that

$$A = \frac{g_0(k - mw^2)}{(k - mw^2)^2 + \gamma^2 w^2} \qquad B = \frac{g_0 \gamma w}{(k - mw^2)^2 + \gamma^2 w^2}.$$

Finally, since we really want y_g in the form $y_g = R\cos(wt - \delta)$, we need to calculate $R^2 = A^2 + B^2$:

$$\begin{aligned} R &= \frac{g_0}{(k - mw^2)^2 + \gamma^2 w^2}\sqrt{(k - mw^2)^2 + \gamma^2 w^2} \\ &= \frac{g_0}{\sqrt{(k - mw^2)^2 + \gamma^2 w^2}}. \end{aligned}$$

We can also calculate δ by noting that

$$\tan(\delta) = B/A = \frac{\gamma w}{k - mw^2}.$$

Notice that as $w \longrightarrow 0$, we see that $R \longrightarrow \frac{g_0}{k}$ (and $\delta \longrightarrow 0$). This is the case when the applied force is approximately constant g_0, and in the limit (i.e. for $w = 0$) represents a *constant solution* $y(t) = g_0/k$.

It is more interesting to consider which frequency of forcing function will maximize R. Regarding all of the other parameters of the system as constant (which is a plausible assumption), we have that

$$\begin{aligned} R(w) &= \frac{g_0}{\sqrt{(k-mw^2)^2+\gamma^2 w^2}} \\ R'(w) &= \frac{-g_0((k-mw^2)(-2mw)+\gamma^2 w)}{((k-mw^2)^2+\gamma^2 w^2)^{3/2}}. \end{aligned}$$

Thus $R'(w) = 0$, implies that

$$\begin{aligned} \text{num. is 0} \qquad & 2mw(k-mw^2) = \gamma^2 w \\ \text{div by } w \qquad & 2mk - 2m^2w^2 = \gamma^2 \\ \text{solve for } w \qquad & w = \sqrt{\frac{2mk-\gamma^2}{2m^2}} \\ & \quad = \sqrt{\frac{k}{m} - \frac{\gamma^2}{2m^2}}. \end{aligned}$$

We call this $w_{max} = \sqrt{\frac{k}{m} - \frac{\gamma^2}{2m^2}}$. Recall that the natural frequency $w_0 = \sqrt{k/m}$; it is easy to check (using the formula just given) that

$$w_{max} = w_0\sqrt{1 - \frac{\gamma^2}{2mk}}$$

(this is just factoring). Furthermore, one can check that

$$R_{max} = R(w_{max}) = \frac{g_0}{\gamma w_0 \sqrt{1 - \frac{\gamma^2}{4mk}}}.$$

5.2 Definition. Consider a forced harmonic oscillator that is driven by some periodic forcing function with frequency ω. The frequency w_{max} which maximizes the amplitude of the steady-state solution is called the *resonant frequency*. The fact that the amplitude of the steady-state solution depends on ω and may achieve some maximum value R_{max} is called *resonance*.

Consider $\gamma << 1$ very small, so that $R_{max} \approx \frac{g_0}{\gamma w_0}$. Then R_{max} is very large. This shows that if we drive a very lightly damped system at a frequency $w = w_{max} \approx w_0$ then we observe a steady state solution with a very large amplitude $R_{max} \approx \frac{g_0}{\gamma w_0}$. The key point here is that this happens even if g_0 is relatively small, provided that γ is small enough.

This poses some practical design considerations. If some component of a system can only safely assume some range of values, say $[-M, M]$, then one must be sure that $R_{max} < M$. This involves knowing the range of potential forcing amplitudes g_0 and driving frequencies. It may be necessary to increase the damping constant γ in order to reduce R_{max}, even if this decreases the efficiency of the system or otherwise incurs a high cost.

Homework Problems

These problems are taken from section §3.8 of the textbook.

48 (B4). Write the expression $\sin 3t + \sin 4t$ as a product of two trigonometric functions of different frequencies.

49 (B5). A mass weighing $4\,lb$ stretches a spring $1.5\,in$. The mass is displaced $2\,in.$ in the positive direction from its equilibrium position and released with no initial velocity. Assuming that there is no damping and that the mass is acted on by an external force of $2\cos 3t\,lb$, formulate the initial value problem describing the motion of the mass.

50 (B6). A mass of $5\,kg$ stretches a spring $10\,cm$. The mass is acted on by an external force of $10\sin(t/2)\,N$ and moves in a medium that imparts a viscous force of $2\,N$ when the speed of the mass is $4\,cm/s$. If the mass is set in motion from its equilibrium position with an initial velocity of $3\,cm/s$, formulate the initial value problem describing the motion of the mass.

51 (B9). If an undamped spring-mass system with a mass that weighs $6\,lb$ and a spring constant $1\,lb/in$ is suddenly set in motion at $t = 0$ by an external force of $4\cos 7t\,lb$, determine the position of the mass at any time and draw a graph of the displacement versus t.

52 (B10). A mass that weighs $8\,lb$ stretches a spring $6\,in$. The system is acted on by an external force of $8\sin 8t\,lb$. If the mass is pulled down $3\,in$ and then released, determine the position of the mass at any time. Determine the first four times at which the velocity of the mass is zero.

6 Variation of Parameters

The general method of finding a particular solution to a non-homogeneous linear equation:

$$Ly = \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t).$$

First: Solve the corresponding homogeneous equation: $Ly = 0$. Get a fundamental set $\{y_1, y_2\}$ of solutions.

Second: Guess $y_g = u_1(t)y_1(t) + u_2(t)y_2(t)$ is the particular solution. (Instead of a linear combination of homogeneous solutions with constant coefficients, we allow the coefficients $u_1(t)$ and $u_2(t)$ to be functions of t.)

We check that

$$\begin{aligned} y_g &= u_1y_1 + u_2y_2 \\ y_g' &= u_1y_1' + u_2y_2' + u_1'y_1 + u_2'y_2 \\ y_g'' &= u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2' + \frac{d}{dt}(u_1'y_1 + u_2'y_2). \end{aligned}$$

Now, we make our lives easier by assuming that $u_1'y_1 + u_2'y_2 = 0$. This assumption imposes a relationship between u_1', u_2', but luckily we can always find functions u_1 and u_2 such that y_g *IS* a particular solution, AND $u_1'y_1 + u_2'y_2 = 0$. So this simplification does not eliminate potential solutions. (A priori this is not obvious, but good guesses usually aren't.)

Armed with our simplifying assumption, the above equations reduce to:

$$\begin{aligned} y_g &= u_1y_1 + u_2y_2 \\ y_g' &= u_1y_1' + u_2y_2' \\ y_g'' &= u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2'. \end{aligned}$$

Now, it is simple to check that

$$Ly_g = u_1Ly_1 + u_2Ly_2 + u_1'y_1' + u_2'y_2'.$$

Of course, we want this to be equal to g (because if $Ly_g = g$, then y_g is a particular solution). The point is that $Ly_1 = 0$ and $Ly_2 = 0$ precisely because y_1, y_2 are solutions to the corresponding homogeneous solution. So we want $u_1'y_1' + u_2'y_2' = g$. If we write the conditions on u_1' and u_2' in matrix form, we get:

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}.$$

We can solve this matrix equation using *Cramer's rule*, a fact from linear algebra:

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \qquad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}.$$

Here, as is the custom, the vertical bars mean that we are taking the determinant of the matrix; for example, the denominator of either fraction is

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'.$$

This denominator comes up a lot; it is non-zero (so that we can divide by it) precisely because y_1 and y_2 are not constant multiples of each other, which is true because $\{y_1, y_2\}$ is a fundamental set of solutions. So we give this a special name:

6.1 Definition. Given differentiable functions y_1 and y_2, the *Wronskian* of y_1 and y_2 is defined by

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'.$$

More generally, to check if functions are linear independent, we can compute their wronskian and check if it is non-zero.

Summarizing, we have

$$u_1' = \frac{-gy_2}{W(y_1, y_2)} \qquad u_2' = \frac{y_1g}{W(y_1, y_2)},$$

so that

$$u_1(t) = -\int_{t_0}^{t} y_2(s)\frac{g(s)ds}{W(y_1(s), y_2(s))} \qquad u_2(t) = \int_{t_0}^{t} y_1(s)\frac{g(s)ds}{W(y_1(s), y_2(s))}.$$

For this choice of u_1 and u_2, our guess $y_g = u_1y_1 + u_2y_2$ IS a particular solution!

6.2 Examples. Find the general solution to $y'' + y = \tan(t)$, $0 < t < \pi/2$.

*6.3 Solution*Step 1 Solve the associated homogeneous equation. In this case, that is

$$y'' + y = 0.$$

We know that the a fundamental set of solutions is $\{\cos(t), \sin(t)\}$. Indeed, the wronskian is

$$W(\cos(s), \sin(s)) = \begin{vmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{vmatrix} = \cos^2(s) + \sin^2(s) = 1,$$

which is not zero.

Step 2 Find the particular solution $y_g(t) = u_1(t)\cos(t) + u_2(t)\sin(t)$. Using the formulas that we found above, we know that

$$u_1(t) = -\int_0^t \sin(s)\frac{\tan(s)ds}{1} = \tan(t)\cos(t) - \int_0^t \tan(s)\sin(s)ds = \sin(t) - \ln(\sec(t) + \tan(t)).$$

$$u_2(t) = \int_0^t \cos(s)\tan(s)ds = -\cos(t) + 1.$$

Step 3 The general solution to the non-homogeneous equation is then

$$\begin{aligned} y &= c_1\cos(t) + c_2\sin(t) + (\sin(t) - \ln(\sec(t) + \tan(t)))\cos(t) - (\cos(t) + 1)\sin(t) \\ &= c_1\cos(t) + \tilde{c}_2\sin(t) - \cos(t)\ln(\sec(t) + \tan(t)) \end{aligned}$$

6.4 Examples. Given the differential equation

$$t^2y'' - t(t+1)y' + (t+2)y = 2t^3,$$

verify that $y_1 = t$ and $y_2 = te^t$ are solutions to the corresponding homogeneous equation. Then find the particular solution.

6.5 Solution. We first verify the solutions given above:

$$\begin{aligned} Lt &= -t(t+2) + (t+2)t = 0 \qquad \text{(Success!)} \\ L(te^t) &= t^2((t+2)e^t) - t(t+2)((t+1)e^t) + (t+2)te^t \\ &= e^t[t^2(t+2) - t^2(t+2) - t(t+2) + (t+2)t] = 0 \qquad \text{(Success!)} \end{aligned}$$

Thus $\{t, te^t\}$ are a fundamental set of solutions to the homogeneous equation, with Wronskian

$$W(y_1, y_2) = \begin{vmatrix} t & te^t \\ 1 & (t+1)e^t \end{vmatrix} = t^2e^t.$$

Before using the formula that we have for u_1 and u_2, we MUST put the equation STANDARD FORM! Do not make the mistake of applying the formula to an equation that is not in the standard form

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t).$$

For example, we rewrite our equation in this example as:

$$y'' - \frac{t(t+1)}{t^2}y' + \frac{t+2}{t^2}y = 2t.$$

The homogeneous equation does not change (why not?), but the right-hand side $g(t) = 2t$ does change! Thus, a particular solution is given by

$$y_g = u_1 t + u_2 te^t$$

where

$$u_1(t) = -\int_0^t se^s \frac{2sds}{s^2e^s} = -\int_0^t 2ds = -2t$$
$$u_2(t) = \int_0^t s\frac{2sds}{s^2e^s} = \int_0^t 2e^{-s}ds = -2e^{-t} + 2.$$

Finally, the general solution is

$$\begin{aligned} y(t) &= c_1 t + c_2 te^t - 2t(t) - 2(e^{-t} - 2)te^t \\ &= \tilde{c}_1 t + \tilde{c}_2 te^t - 2t^2. \end{aligned}$$

6.6 Examples. Find the general solution to

$$y'' + 4y = g(t).$$

6.7 Solution. As usual, we solve the corresponding homogeneous equation, and get that $\{\cos(2t), \sin(2t)\}$ is a set of fundamental solutions. We calculate the wronskian

$$W(\cos(2t), \sin(2t)) = \begin{vmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{vmatrix} = 2(\cos^2(t) + \sin^2(t)) = 2.$$

Now, let $y_g = u_1\cos(2t) + u_2\sin(2t)$ be the particular solution, where

$$u_1(t) = -\int_0^t \sin(2s)\frac{g(s)ds}{2} = -\frac{1}{2}\int_0^t \sin(2s)g(s)ds$$
$$u_2(t) = \int_0^t \cos(2s)\frac{g(s)ds}{2} = \frac{1}{2}\int_0^t \cos(2s)g(s)ds.$$

We cannot simplify any further, because we don't know explicitly the function $g(s)$. In any case, we can still write down the general solution:

$$\begin{aligned} y(t) &= c_1\cos(2t) + c_2\sin(2t) - \frac{\cos(2t)}{2}\int_0^t \sin(2s)g(s)ds + \frac{\sin(2t)}{2}\int_0^t \cos(2s)g(s)ds \\ &= c_1\cos(2t) + c_2\sin(2t) + \frac{1}{2}\int_0^t g(s)\left[\cos(2s)\sin(2t) - \sin(2s)\cos(2t)\right]ds \\ &= c_1\cos(2t) + c_2\sin(2t) + \frac{1}{2}\int_0^t g(s)\sin(2(t-s))ds. \end{aligned}$$

Note: From the first line, we can multiply $\sin(2t)$ and $\cos(2t)$ into the integral because they are constant with respect to s. To get the third line from the second, just use the trig formula: $\sin(A-B) = \sin(A)\cos(B) - \sin(B)\cos(A)$.

6.8 Examples. Find the general solution to

$$Ly := t^2y'' - 2ty' + 2y = 4t^2, \qquad t > 0,$$

knowing that $Lt = 0$.

6.9 Solution. Since this is not an equation with constant coefficients, we have to try harder to find the solutions to the corresponding homogeneous equation. It is given to us that $Lt = 0$, which means that $y_1(t) = t$ is a solution to the corresponding homogeneous equation. We can then use the "reduction of order" technique to find the other homogeneous solution.

Indeed, we guess that $y_2 = w \cdot t$. Then

$$\begin{aligned} y_2(t) &= w \cdot t \\ y_2'(t) &= w't + w \\ y_2''(t) &= w''t + 2w'. \end{aligned}$$

Then we calculate that $Ly_2 = t^2(w''t + 2w') - 2t(w't + w) + 2wt = t^3w''$. Since we want $Ly_2 = 0$ (so that y_2 is a homogeneous solution), we require that $w'' = 0$. Thus $w(t) = At + B$, and $y_2(t) = (At + B)t = At^2 + Bt$. Since $y_1(t) = t$ is already a homogeneous solution, we know that $L(Bt) = 0$; it follows that $L(At^2) = AL(t^2) = 0$. Thus, $y_2(t) = t^2$ is another independent homogeneous solution. Indeed, the Wronskian is given by

$$W(y_1, y_2) = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = t^2.$$

Finally, we want to find the particular solution $y_g = u_1t + u_2t^2$; again, it is NECESSARY to put the equation in standard form in order to use the formulas for the u_i's. Our original equation becomes:

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = 4.$$

Then we use our formulas to find

$$\begin{aligned} u_1(t) &= -\int_1^t s^2\frac{4ds}{s^2} = -\int_1^t 4ds = -4(t-1) \\ u_2(t) &= \int_1^t s\frac{4ds}{s^2} = 4\int_1^t \frac{ds}{s} = 4\ln(t). \end{aligned}$$

Finally, the general solution is given by

$$\begin{aligned} y(t) &= c_1t + c_2t^2 - 4(t-1)t + 4\ln(t)t^2 \\ &= \tilde{c}_1t + \tilde{c}_2t^2 + 4t^2\ln(t). \end{aligned}$$

Homework Problems

These problems are taken from section §3.6 of the textbook.

53. Find the general solution to the given differential equation

7) $y'' + 4y' + 4y = t^{-2}e^{-2t}, \qquad t > 0$

11) $y'' - 5y' + 6y = g(t)$, where $g(t)$ is arbitrary.

54. Verify that y_1 and y_2 satisfy the corresponding homogeneous equation; find a particular solution of the non-homogeneous equation.

B15) $ty'' - (1+t)y' + y = t^2e^{2t}, \quad t > 0; \qquad y_1(t) = 1 + t, \quad y_2(t) = t$

B19) $(1-x)y'' + xy' - y = g(x), \quad 0 < x < 1; \qquad y_1(x) = e^x, \quad y_2(x) = x$, where $g(x)$ is arbitrary.

55 (B30). Use the method of reduction of order to find the general solution to

$$t^2y'' + 7ty' + 5y = t, \quad t > 0; \qquad y_1(t) = t^{-1}.$$

Chapter 3

Transform Methods

In this chapter we investigate a particular linear operator, the Laplace transform, and how it can help us solve linear differential equations. The basic idea is that a differential equation is more difficult to solve than an algebraic equation; so if it were possible to transform a problem of the first type into just an algebraic problem, then a solution would be easy to obtain. Briefly, this is what Laplace transforms allow us to do.

You should be aware that there are other "transforms" which are more common than Laplace transforms in different fields, with the most popular being the Fourier transform and its variants. If you study the techniques in this chapter, you will be well prepared to learn about other transform methods.

1 Laplace Transform

1.1 Definition. We suppose that $f = f(t)$ is

i) piece-wise continuous for $t \geq 0$, and

ii) of exponential order, i.e. $|f(t)| \leq Ke^{at}$ for $t \geq t_0$, (K and a in $\mathbb{R}$).

Then we can define the *laplace transform*

$$\mathcal{L}\{f\}(s) := \int_0^\infty e^{-st} f(t)dt.$$

Note! $\mathcal{L}\{f\}$ is a *function* of s, and is defined for all $s > a$. We emphasize this by sometimes writing $\mathcal{L}\{f\}(s)$.

We will almost always deal with functions f satisfying the criteria (i) and (ii) in the definition. These conditions are there to ensure that the improper integral $\int_0^\infty e^{-st} f(t)dt$ actually exists and converges to a finite number for each s.

The goal is first to find the transforms of our favorite elementary functions. The best way to do this is to first isolate the key properties of $\mathcal{L}$.

1.2 Theorem. *Suppose the functions f and g satisfy the conditions above. Let a and c be complex numbers, and $n \geq 0$ a natural number. Then*

(A) *Complex Linearity:*

$$\mathcal{L}\{f + c \cdot g\} = \mathcal{L}\{f\} + c \cdot \mathcal{L}\{g\};$$

(B) *Translation in s:*

$$\mathcal{L}\left\{e^{at} \cdot f\right\}(s) = \mathcal{L}\{f\}(s-a);$$

(C) *Differentiation in t, multiplication in s:*

$$\mathcal{L}\{f'\}(s) = s \cdot \mathcal{L}\{f\}(s) - f(0)$$

(D) *Differentiation in s, multiplication in t:*

$$\mathcal{L}\{t^n \cdot f\} = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f\}.$$

Proof. (A)

$$\begin{aligned}\mathcal{L}\{\cdot f + c \cdot g\} &= \int_0^\infty e^{-st}(\cdot f(t) + c \cdot g(t))dt \\ &= \int_0^\infty e^{-st} f(t)dt + c\int_0^\infty e^{-st} g(t)dt = \mathcal{L}\{f\} + c\mathcal{L}\{g\}.\end{aligned}$$

(B)

$$\begin{aligned}\mathcal{L}\left\{e^{at} \cdot f\right\}(s) &= \int_0^\infty e^{-st} \cdot e^{at} f(t)dt = \int_0^\infty e^{-(s-a)t} f(t)dt \\ &= \mathcal{L}\{f\}(s-a).\end{aligned}$$

(C)

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= \int_0^\infty e^{-st} f'(t)dt = e^{-st} f(t)|_0^\infty + s\int_0^\infty f(t)e^{-st}dt \\ &= -f(0) + s\mathcal{L}\{f(t)\}.\end{aligned}$$

(D) We begin with the right-hand side. If $F(s) = \mathcal{L}\{f\}(s)$, we compute

$$\begin{aligned}F^{(n)}(s) &:= \frac{d^n}{ds^n}\int_0^\infty e^{-st} f(t)dt = \int_0^\infty \left(\frac{d^n}{ds^n} e^{-st} f(t)\right) dt \\ &= \int_0^\infty (-t)^n e^{-st} f(t)dt = (-1)^n \int_0^\infty e^{-st} t^n f(t)dt \\ &= (-1)^n \mathcal{L}\{t^n f(t)\}.\end{aligned}$$

□

Property (C) generalizes: for example,

$$\mathcal{L}\{f''\} = \mathcal{L}\{(f')'\} = -f'(0) + s\mathcal{L}\{f'\} = -f'(0) - s \cdot f(0) + s^2\mathcal{L}\{f\}.$$

In general, repeating this process we get that

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - s \cdot f^{(n-2)}(0) - f^{(n-1)}(0).$$

With these key properties of $\mathcal{L}$, we be able to *very quickly* deduce the transform of many elementary functions. We will avoid the definition of $\mathcal{L}$ completely, because we can.

1.3 Proposition.

i) $\mathcal{L}\{0\} = 0$

ii) $\mathcal{L}\{1\} = \frac{1}{s}$

iii) $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

iv) $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

v) $\mathcal{L}\{\sin(bt)\} = \frac{b}{s^2+b^2}$

vi) $\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+b^2}$

vii) $\mathcal{L}\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}$

viii) $\mathcal{L}\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}$

ix) $\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$

Proof. (i): $\mathcal{L}\{0\} = \mathcal{L}\{0+0\} = \mathcal{L}\{0\} + \mathcal{L}\{0\}$. Subtracting $\mathcal{L}\{0\}$ from both sides gives the result. (This is the proof that any linear transformation must take send 0 to 0.)

(ii): If $f(t) = 1$, the $f'(t) = 0$. Using (i) and property (C), $0 = \mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0) = s\mathcal{L}\{1\} - 1$. Thus, $\mathcal{L}\{1\} = 1/s$.

(iii): Using property (B), $\mathcal{L}\{e^{at}\} = \mathcal{L}\{e^{at} \cdot 1\} = \mathcal{L}\{1\}(s-a) = \frac{1}{s-a}$.

(iv): Using property (D),

$$\mathcal{L}\{t^n\} = \mathcal{L}\{t^n \cdot 1\} = (-1)^n \frac{d^n}{ds^n}\frac{1}{s} = (-1)^n(-1)^n \frac{n!}{s^{n+1}} = \frac{n!}{s^{n+1}}.$$

(v) and (vi): By (iii), we know $\mathcal{L}\left\{e^{ibt}\right\} = \frac{1}{s-ib} = \frac{s+ib}{s^2+b^2}$. We also know from Euler's formula that

$$\cos(bt) + i\sin(bt) = e^{ibt}.$$

Using property (A), $\mathcal{L}\{\cos(bt)\} + i\mathcal{L}\{\sin(bt)\} = \mathcal{L}\left\{e^{ibt}\right\} = \frac{s}{s^2+b^2} + i\frac{b}{s^2+b^2}$. Comparing the real and imaginary parts proves (vi) and (v), respectively.

(vii) and $(viii)$: Using property (B) and (v), we get $\mathcal{L}\{e^{at}\sin(bt)\} = \mathcal{L}\{\sin(bt)\}(s-a) = \frac{b}{(s-a)^2+b^2}$. Similar for $(viii)$.

(ix): Now use property (B) with (iv) to get $\mathcal{L}\{e^{at}t^n\} = \mathcal{L}\{t^n\}(s-a) = \frac{n!}{(s-a)^{n+1}}$. □

Exercises. Find the following Laplace transforms:

i) $\mathcal{L}\{5t^2 - 3t + 1\}$

ii) $\mathcal{L}\{3\sin(\pi t)\}$

iii) $\mathcal{L}\{t\sin(t)\}$.

iv) $\mathcal{L}\{e^{at}\cosh(bt)\}$, where $\cosh(bt) = \frac{e^{bt}+e^{-bt}}{2}$.

1.4 Definition. The *Gamma function* $\Gamma : [0, \infty) \longrightarrow \mathbb{R}$ is defined by

$$\Gamma(p+1) := \int_0^\infty e^{-u} u^p du, \qquad \text{for } p \geq 0.$$

You will show in a homework problem that $\Gamma(n+1) = n!$ for any integer $n \geq 0$.

1.5 Proposition. *For any real number $p > -1$,*

$$\mathcal{L}\{t^p\} = \frac{\Gamma(p+1)}{s^{p+1}}.$$

Proof. We use the definition, and then try the substitution $u = st$:

$$\begin{aligned}\mathcal{L}\{t^p\}(s) &= \int_0^\infty e^{-st} t^p \, dt = \int_0^\infty e^{-u} (u/s)^p \frac{du}{s} \\ &= \frac{1}{s^{p+1}} \int_0^\infty e^{-u} u^p du = \frac{\Gamma(p+1)}{s^{p+1}}.\end{aligned}$$

□

1.6 Examples. This is an example of what can go wrong.

$$\mathcal{L}\left\{\frac{1}{t}\right\} = \int_0^\infty e^{-st} \frac{1}{t} dt.$$

1.7 Solution. For each $\delta > 0$, we can break the integral up into two pieces, say

$$\int_0^\infty e^{-st} \frac{1}{t} dt = \int_0^\delta e^{-st} \frac{1}{t} dt + \int_\delta^\infty e^{-st} \frac{1}{t} dt.$$

The third integral definitely converges: the function $\frac{1}{t}$ is no bigger than $\frac{1}{\delta}$ on the interval (δ, ∞), so

$$\int_\delta^\infty e^{-st} \frac{1}{t} dt < \frac{1}{\delta} \int_\delta^\infty e^{-st} dt < \frac{1}{\delta s},$$

since $\int_\delta^\infty e^{-st} dt < \int_0^\infty e^{-st} dt = \mathcal{L}\{1\} = \frac{1}{s}$.

Unfortunately, for each choice of $\delta > 0$ the integral $\int_0^\delta e^{-st} \frac{1}{t} dt$ does not converge to a finite number for any value of s. Let's check this. Since the integrand is a positive function, it suffices to show this for a particular choice of δ. But notice that when δ is very small, e^{-st} is very close to 1 when $0 < t < \delta$. This is because e^{-st} is a continuous function, and $e^0 = 1$. So we will choose a particular δ such that $e^{-st} > \frac{1}{2}$ when $0 < t < \delta$. (The number $\frac{1}{2}$ is arbitrary: we could have chosen any number in $(0, 1)$; also, δ depends on s). Then we have:

$$\int_0^\delta e^{-st} \frac{1}{t} dt > \frac{1}{2} \int_0^\delta \frac{1}{t} dt = \frac{1}{2} (\ln(\delta) - \ln(0)) = +\infty.$$

This proves that $\mathcal{L}\left\{\frac{1}{t}\right\} = +\infty$ for every value of s. This is a useless function for our purposes, and so we say that $\frac{1}{t}$ has no Laplace transform.

Homework Problems

These problems are taken from section §6.1 of the textbook.

56 (B6). Find the Laplace transform of $f(t) = \cos at$, where a is a real constant.

57. Find the Laplace transform of the following functions where a and b are real constants, and n is a positive integer

B10) $f(t) = e^{at} \sinh bt$

B14) $f(t) = e^{at} \cos bt$

B18) $f(t) = t^n e^{at}$

58 (B26). The gamma function is denoted $\Gamma(p)$ and is defined by the integral

$$\Gamma(p+1) := \int_0^\infty e^{-x} x^p dx.$$

This is an improper integral for two reasons; first, the upper limit of integration is ∞. This does not effect convergence of the integral since $e^{-x}x^p \longrightarrow 0$ rapidly as $x \longrightarrow \infty$ for all values of p.

However, we also see that for $p < 0$ the integrand is unbounded as $x \longrightarrow 0$; to get convergence of the integral we must require that $p > -1$.

a) Show that, for $p > 0$,
$$\Gamma(p+1) = p\Gamma(p).$$

b) Show that $\Gamma(1) = 1$.

c) If p is a positive integer n, show that
$$\Gamma(n+1) = n!$$

SInce $\Gamma(p)$ is also defined when p is not an integer, this function provides an extension of the factorial function to non-integral values of the independent variable. Note that it is also consistent with the convention that $0! = 1$.

d) Show that, for $p > 0$,
$$p(p+1)(p+2)\cdots(p_n - 1) = \Gamma(p+n)/\Gamma(p).$$

Thus $\Gamma(p)$ can be determined for all positive values of p if $\Gamma(p)$ is known in a single interval of unit length, say $0 < p \leq 1$. It is possible to show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. FInd $\Gamma\left(\frac{3}{2}\right)$ and $\Gamma\left(\frac{11}{2}\right)$.

2 Solving IVP's with $\mathcal{L}$

We know that $\mathcal{L}$ is linear, and that we can express $\mathcal{L}\{y''\}$ and $\mathcal{L}\{y'\}$ in terms of $\mathcal{L}\{y\}$. So if we apply $\mathcal{L}$ to both sides of a differential equation, we will get an algebraic equation for the function $\mathcal{L}\{y\}$. Solving this equation for $\mathcal{L}\{y\}$ is easy as it uses only basic algebra.

2.1 Examples. Solve for $\mathcal{L}\{y\}$, assuming that y satisfies the IVP

$$y'' - 3y' + 2y = 0 \qquad \begin{cases} y(0) = 2 \\ y'(0) = 3. \end{cases}$$

2.2 Solution. We apply $\mathcal{L}$ to both sides and get:

$$\begin{aligned} \mathcal{L}\{0\} = 0 &= \mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} \\ &= (s^2\mathcal{L}\{y\} - 2s - 3) - 3(s\mathcal{L}\{y\} - 2) + 2\mathcal{L}\{y\} \\ &= \mathcal{L}\{y\}(s^2 - 3s + 2) - 2s + 3. \end{aligned}$$

Then we solve this equation for $\mathcal{L}\{y\}$:

$$\mathcal{L}\{y\} = \frac{2s-3}{s^2-3s+2}.$$

We're actually interested in y; so we want to identify the function whose Laplace transform is $\frac{2s-3}{s^2-3s+2}$. It helps to rewrite this slightly:

$$\frac{2s-3}{s^2-3s+2} = \frac{1}{s-1} + \frac{1}{s-2}.$$

using the method of partial fractions. Using the table we started developing last lecture, it is clear that

$$\mathcal{L}\left\{e^t + e^{2t}\right\} = \frac{1}{s-1} + \frac{1}{s-2} = \frac{2s-3}{s^2-3s+2},$$

so that $y = e^t + e^{2t}$.

There are essentially three steps in this process:

i) Calculate the Laplace transform of both sides of the differential equation

ii) Solve this equation for $\mathcal{L}\{y\}$

iii) Compute the inverse Laplace transform of $\mathcal{L}\{y\}$.

2.1 Computing the inverse transform

Refer to the last page in the notes (page 317 in your textbook) for a table of Laplace transforms. This is essential for computing inverse transforms. Print it out and keep a copy with you. This will be provided for you during the final exam (it will be the last page).

2.3 Examples. Compute the inverse transform of $\frac{4}{(s-1)^3}$.

2.4 Solution. From the table we recognize the most relevant formula:

$$\mathcal{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}}.$$

Identifying the given function with the right hand side of the formula, we see that $a = 1$ and $n = 2$. So we have

$$\mathcal{L}\left\{t^2 e^t\right\} = \frac{2}{(s-1)^3}.$$

This is still not exactly what was given. The given function is twice this. So we multiply both sides of the equation by 2:

$$2\mathcal{L}\left\{t^2 e^t\right\} = \mathcal{L}\left\{2t^2 e^t\right\} = \frac{4}{(s-1)^3},$$

so that $2t^2 e^t = \mathcal{L}^{-1}\left\{\frac{4}{(s-1)^3}\right\}$. (Notice how we use linearity of the Laplace transform.)

2.5 Examples. Compute $\mathcal{L}^{-1}\left\{\frac{2}{s^2+3s-4}\right\}$.

2.6 Solution. If we look at the entries in our table of Laplace transforms, we allow for the denominator to be either a linear function OR an irreducible quadratic (i.e. a quadratic polynomial with imaginary roots.) In this example, the denominator is not irreducible – it factors as $s^2 + 3s - 4 = (s+4)(s-1)$. So we need to rewrite the function as a sum of fractions where the denominator is linear (this is just partial fractions).

$$\frac{2}{s^2+3s-4} = \frac{2/5}{s-1} - \frac{2/5}{s+4} = \frac{2}{5}\frac{1}{s-1} - \frac{2}{5}\frac{1}{s+4}.$$

From our table we identify the relevant entry:

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}.$$

Then we see that

$$\frac{2}{s^2+3s-4} = \frac{2}{5}\mathcal{L}\left\{e^t\right\} - \frac{2}{5}\mathcal{L}\left\{e^{-4t}\right\},$$

so that

$$\mathcal{L}^{-1}\left\{\frac{2}{s^2+3s-4}\right\} = \frac{2}{5}e^t - \frac{2}{5}e^{-4t}.$$

2.7 Examples. Compute $\mathcal{L}^{-1}\left\{\frac{8s^2-4s+12}{s(s^2+2s+5)}\right\}$.

2.8 Solution. The strategy is to use partial fractions. The subtlety here is the form of the partial fractions expansion. First, the quadratic term $s^2 + 2s + 5$ in the denominator does not factor since the discriminant $\Delta = 2^2 - 4(1)(5)$ is negative. Consider all of the entries in the table of Laplace transforms which contain irreducible quadratic terms; they all have the form of a sum of squares. So to recognize our given function in the table, we must complete the square: we get

$$s^2 + 2s + 5 = (s+1)^2 + 4.$$

Then the form for the partial fraction expansion is generically

$$\frac{8s^2 - 4s + 12}{s(s^2 + 2s + 5)} = \frac{A}{s} + \frac{Bs + C}{(s+1)^2 + 4}.$$

However!! Here is a *useful trick*: rewrite the numerator $Bs + C$ as $B(s+1) + C$. This rearrangement is useful because ultimately we want to identify the function in our table, and we know that $\mathcal{L}\left\{e^{-t}\cos(2t)\right\} = \frac{s+1}{(s+1)^2+4}$, for example. Think about this for a minute.

Using the better partial fraction expansion:

$$\frac{8s^2 - 4s + 12}{s(s^2 + 2s + 5)} = \frac{A}{s} + \frac{B(s+1) + C}{(s+1)^2 + 4},$$

we need some way to determine the constants. Certainly, we can multiply both sides by s and evaluate at 0. We get

$$\frac{12}{5} = A + 0,$$

so that $A = \frac{12}{5}$. (This is sometimes called the cover-up method, since you can do the same thing by "covering-up" the term "s" in the denominator and then evaluating the fraction on the left hand side at $s = 0$.)

Copying the previous idea, we multiply both sides by the other term, $(s+1)^2 + 4$, and then let this term go to 0. Equivalently, we let s go to $-1 + 2i$. Then the equation that we have is:

$$\left.\frac{8s^2 - 4s + 12}{s}\right|_{s=-1+2i} = 0 + B(s+1) + C|_{s=-1+2i}\,.$$

Notice then that the right hand side is just $B(2i) + C$. Thus C will be the real part of the left hand side, and B with be one-half of the imaginary part of the left hand side. To simplify the left hand side we need to compute:

$$\begin{aligned}\frac{8(-1+2i)^2 - 4(-1+2i) + 12}{-1+2i} &= \frac{8(1-4-4i) + 4 - 8i + 12}{-1+2i} = \frac{-8-40i}{-1+2i}\\ &= \frac{-8-40i}{-1+2i}\frac{-1-2i}{-1-2i} = \frac{-72+56i}{5}.\end{aligned}$$

It follows that $C = -72/5$ and $B = \frac{1}{2}\frac{56}{5} = \frac{28}{5}$. Finally:

$$\mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B(s+1) + C}{(s+1)^2 + 4}\right\} = A + Be^{-t}\cos(2t) + \frac{C}{2}e^{-t}\sin(2t)$$

so that

$$\mathcal{L}^{-1}\left\{\frac{8s^2 - 4s + 12}{s(s^2 + 2s + 5)}\right\} = \frac{12}{5} + \frac{28}{5}e^{-t}\cos(2t) - \frac{36}{5}e^{-t}\sin(2t).$$

Exercises.

i)

$$\mathcal{L}^{-1}\left\{\frac{3s - 2}{s^5}\right\}$$

ii)

$$\mathcal{L}^{-1}\left\{\frac{2s+2}{s^2+2s+5}\right\}$$

iii)

$$\mathcal{L}^{-1}\left\{\frac{8s^2-4s+12}{s(s^2+4)}\right\}.$$

Challenge Problem

You should not get the idea that computing Laplace transforms (or their inverses) is easy. For many elementary functions, we cannot find explicit formulas for the Laplace transform (in terms of elementary functions). Even when we can, it can be challenging to do so. Here is an example of the latter case:

59. Compute $\mathcal{L}\{\ln(t)\}$

We will show two methods for computing this, to help build your intuition about which tricks will work. First, recall how we computed $\mathcal{L}\{t^p\}$: using a u-substitution we were able to relate it to the Gamma function, via

$$\mathcal{L}\{t^p\} = \frac{\Gamma(p+1)}{s^{p+1}}.$$

We try the same approach - again $u = st$:

$$\begin{aligned}\mathcal{L}\{\ln(t)\} &= \int_0^\infty e^{-ts}\ln(t)dt = \int_0^\infty e^{-u}\ln(u/s)\frac{du}{s}\\ &= \frac{1}{s}\int_0^\infty e^{-u}(\ln(u)-\ln(s))du = \frac{1}{s}\int_0^\infty e^{-u}\ln(u)du - \frac{\ln(s)}{s}\int_0^\infty e^{-u}du.\end{aligned}$$

It is possible to show that $\int_0^\infty e^{-u}\ln(u)du$ is a finite number - the only thing to worry about is what happens near $u=0$. Intuitively, when u is close to zero, e^{-u} is very close to 1, and we know that $\int_0^\delta \ln(u)du = \delta(\ln(\delta)-1)$ exists and is finite, so it should not be surprising that the same is true for $\int_0^\delta e^{-u}\ln(u)du$.

Actually, $\int_0^\infty e^{-u}\ln(u)du$ is the negative of a special constant γ called the *Euler-Mascheroni* constant. We don't worry too much about what that means, but we might as well abbreviate $\int_0^\infty e^{-u}\ln(u)du = -\gamma$, since to us it is just some real number. It is also clear that $\int_0^\infty e^{-u}du = 1$; thus,

$$\mathcal{L}\{\ln(t)\} = -\frac{\gamma+\ln(s)}{s}.$$

Now, we try a somewhat different method. We begin with the formula we derived last lecture:

$$\mathcal{L}\{t^p\} = \int_0^\infty e^{-st}t^p dt = \frac{\Gamma(p+1)}{s^{p+1}}.$$

Now, we differentiate this formula *with respect to p*. This should seem strange! It's a clever trick:

$$\frac{d}{dp}\int_0^\infty e^{-st}t^p dt = \frac{d}{dp}\frac{\Gamma(p+1)}{s^{p+1}}$$

$$\int_0^\infty e^{-st}t^p \ln(t)dt = \frac{\Gamma'(p+1)s^{p+1} - \Gamma(p+1)s^{p+1}\ln(s)}{(s^{p+1})^2}.$$

Finally, what we are really interested in is

$$\mathcal{L}\{\ln(t)\} = \int_0^\infty e^{-st}\ln(t)dt.$$

This is just the previous line evaluated at $p = 0$:

$$\begin{aligned}\mathcal{L}\{\ln(t)\} = \int_0^\infty e^{-st}\ln(t)dt &= \frac{\Gamma'(1)s^1 - \Gamma(1)s^1\ln(s)}{s^2} \\ &= \frac{\Gamma'(1) - \ln(s)}{s}.\end{aligned}$$

Comparing the two formulas, we see that $\Gamma'(1) = -\gamma$, which is something that can be checked directly as well.

2.9 Remark. You can also see from this formula that $\mathcal{L}\left\{\frac{1}{t}\right\}$ does not exist. We know that $\frac{1}{t}$ is the derivative of $\ln(t)$ for $t > 0$. Then using the formula at the beginning of this lecture, we get

$$\mathcal{L}\left\{\frac{1}{t}\right\} = s\mathcal{L}\{\ln(t)\} - \mathcal{L}\{\ln(t)\}(0) = (-\gamma - \ln(s)) - \left.\frac{-\gamma - \ln(s)}{s}\right|_{s=0},$$

which is not defined.

Homework Problems

These problems are taken from section §6.2 of the textbook.

60 (B). Find the inverse Laplace transform of the given function.

B1) $F(s) = \frac{3}{s^2+4}$.

B6) $F(s) = \frac{2s-3}{s^2-4}$.

61. Use Laplace transforms to solve the given initial value problems:

B22) $y'' - 2y' + 2y = e^{-t}$; $\quad y(0) = 0, \quad y'(0) = 1$

B23) $y'' + 2y' + y = 4e^{-t}$; $\quad y(0) = 2, \quad y'(0) = -1$

62 (B27). Laplace transforms of some functions can be found easily from their Taylor series expansions.

a) Using the Taylor series

$$\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}$$

and assuming that the Laplace transform of this series can be computed term by term, verify that

$$\mathcal{L}\{\sin t\} = \frac{1}{1+s^2}, \qquad s > 1.$$

b) Define the function $\operatorname{sinc} t$, the "sine cardinal" (pronounced "sink") of t as

$$\operatorname{sinc} t = \begin{cases} \frac{\sin(t)}{t} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$$

Compute the Taylor series for $\operatorname{sinc}(t)$ at 0, and use it to verify that

$$\mathcal{L}\{\operatorname{sinc} t\} = \tan^{-1}(1/s), \qquad s > 1.$$

c) The Bessel function of the first kind of order 0, J_0, has the Taylor series

$$J_0(t) := \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}.$$

Verify that

$$\mathcal{L}\{J_0(t)\} = \sqrt{1+s^2}, \qquad s > 1,$$

and

$$\mathcal{L}\{J_0(\sqrt{t})\} = s^{-1}e^{-1/(4s)}, \quad s > 0.$$

3 Step Functions

3.1 Definition. The *unit step function* at $c \geq 0$ is denoted $u_c(t)$ and is defined by

$$u_c(t) = \begin{cases} 0 & 0 \leq t < c \\ 1 & c \leq t. \end{cases}$$

If we draw the graph of $u_c(t)$, it looks like one step "up" at $t = c$ of unit height. We are mostly interested in the following property of $u_c(t)$: if we multiply a function $f(t)$ by $u_c(t)$, then the resulting function $u_c(t)f(t)$ is identically 0 until $t = c$ and then it is identically $f(t)$.

In fact, consider the following statement:

$$f(t) + u_c(t) \cdot (-f(t) + g(t)) = \begin{cases} f(t) & t < c \\ g(t) & c \leq t. \end{cases}$$

The left hand side is identically $f(t)$ for $t < c$, since $u_c(t) = 0$ there. If $t \geq c$, then $u_c(t) = 1$ so the left hand side is exactly $f(t) - f(t) + g(t) = g(t)$. This shows that the two sides of the equation are equal.

Example. What is the Laplace transform of $u_c(t)$?

We just use the definition of $\mathcal{L}\{u_c(t)\}$:

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t) dt.$$

To compute this, we need to use the definition of $u_c(t)$, a piecewise-defined function. Whenever we integrate a piecewise-defined function, we break up the domain of integration into the pieces where the function is defined. In this case, that is $(0, c)$ and (c, ∞). Then we have

$$\begin{aligned} \mathcal{L}\{u_c(t)\} &= \int_0^c e^{-st} u_c(t) dt + \int_c^\infty e^{-st} u_c(t) dt \\ &= \int_0^c e^{-st} \cdot 0 \, dt + \int_c^\infty e^{-st} \cdot 1 \, dt \\ &= 0 + \frac{-1}{s} e^{-st}|_c^\infty = \frac{e^{-cs}}{s}, \quad s > 0. \end{aligned}$$

Example. What is the Laplace transform of $f(t) = \begin{cases} 0 & t < 0 \\ 2 & 0 \le t < 4 \\ 5 & 4 \le t < 7 \\ -1 & 7 \le t < 9 \\ 1 & 9 \le t. \end{cases}$?

Our goal is to rewrite $f(t)$ so that it's not piecewise-defined, but rather is defined using products and sums of unit-step functions, $u_c(t)$.

We work on intervals of t from left to right. Notice that for all values of $t < 0$, the function $f(t)$ is 0. So we start off with $f(t) = 0$.

The next interval is $0 \le t < 4$. We want to "modify" our first guess ($f(t) = 0$) so that it agrees with $f(t)$ on the first interval $0 \le t < 4$. Intuitively, we can just "add 2" - but we must NOT change any of the values of $f(t)$ for $t < 0$. To accomplish this, we multiply our change by $u_0(t)$. This has the effect of adding 2 for $t \ge 0$, but doing nothing when $t < 0$. So our new guess is $f(t) = 0 + u_0(t) \cdot 2$.

The next interval is $4 \le t < 7$. Here the function value is 5, which is 3 more than the previous value. So we want to add 3, but only for $t \ge 4$. Thus we add $u_4(t) \cdot 3$, so our new guess is $f(t) = 0 + u_0(t) \cdot 2 + u_4(t) \cdot 3$.

The next interval is $7 \le t < 9$. Similarly, our new guess is $f(t) = 0 + u_0(t) \cdot 2 + u_4(t) \cdot 3 + u_7(t) \cdot (-6)$.

You should verify that with the final interval, we have that

$$f(t) = 0 + u_0(t) \cdot 2 + u_4(t) \cdot 3 + u_7(t) \cdot (-6) + u_9(t) \cdot 2.$$

Finally, we can compute the Laplace transform of $f(t)$ using the formula from the previous example:

$$\begin{aligned}\mathcal{L}\{f(t)\} &= 2\mathcal{L}\{u_0\} + 3\mathcal{L}\{u_4\} - 6\mathcal{L}\{u_7\} + 2\mathcal{L}\{u_9\} \\ &= \frac{1}{s}\left(2e^0 + 3e^{-4s} - 6e^{-7s} + 2e^{-9s}\right) \\ &= \frac{2 + e^{-4s} - 6e^{-7s} + 2e^{-9s}}{s}.\end{aligned}$$

In practice, rewriting the piecewise functions goes very quickly. You should practice going back and forth between piecewise-defined functions and functions defined with unit-step functions.

3.1 Translation in t

We want to have a formula for $\mathcal{L}\{f(t-c)\}$ in terms of $\mathcal{L}\{f(t)\}$; recall that $f(t-c)$ is the *translation* of $f(t)$ to the right by c. However, the Laplace transform $\mathcal{L}\{f(t)\}$ only "sees" the values of the function $f(t)$ for $t > 0$, since the domain of integration is $(0, \infty)$. Since $f(t-c)$ contains the data of $f(t)$ for $-c < t < 0$, we cannot hope for a formula for $\mathcal{L}\{f(t-c)\}$ in terms of $\mathcal{L}\{f(t)\}$. Intuitively, $\mathcal{L}\{f(t)\}$ is missing information.

However, if we "forget" the values of $f(t-c)$ for $t < c$, then there might be some hope. So we instead look at $\mathcal{L}\{u_c(t)f(t-c)\}$. This does NOT contain any information about $f(t)$ for $t \leq 0$. Now we check our guess:

$$\mathcal{L}\{u_c(t)f(t-c)\} = \int_0^\infty e^{-st}u_c(t)f(t-c)dt.$$

Remember that $u_c(t)f(t-c)$ is piecewise-defined, so we break up the integral accordingly.

$$\begin{aligned}\mathcal{L}\{u_c(t)f(t-c)\} &= \int_0^c e^{-st}\cdot 0\cdot f(t-c)dt + \int_c^\infty e^{-st}\cdot 1\cdot f(t-c)dt \\ &= \int_0^\infty e^{-s(u+c)}f(u)du = e^{-cs}\int_0^\infty e^{-su}f(u)du \\ &= e^{-cs}\int_0^\infty e^{-st}f(t)dt = e^{-cs}\mathcal{L}\{f(t)\}.\end{aligned}$$

3.2 Note. To get from the first line to the second, we use the u-substitution $u = t - c$. Finally, in the second line the "dummy-variable" u can be replaced by t.

We can also apply $\mathcal{L}^{-1}$ to this formula:

$$u_c(t)f(t-c) = \mathcal{L}^{-1}\left\{e^{-sc}F(s)\right\} \qquad \text{where } F(s) = \mathcal{L}\{f(t)\}.$$

3.3 Examples. Find $\mathcal{L}^{-1}\left\{\frac{1-e^{-2s}}{s^2}\right\}$

3.4 Solution. Let $F(s) = \frac{1}{s^2}$. Then we remember that $\mathcal{L}^{-1}\{F(s)\} = t = f(t)$, from the table on page 5.1 (we have calculated this before). By linearity of the inverse transform:

$$\mathcal{L}^{-1}\left\{\frac{1-e^{-2s}}{s^2}\right\} = \mathcal{L}^{-1}\{F(s)\} - \mathcal{L}^{-1}\left\{e^{-2s}F(s)\right\} = t - u_2(t)f(t-2)$$

where the last equality uses the formula from above. Finally,

$$\mathcal{L}^{-1}\left\{\frac{1-e^{-2s}}{s^2}\right\} = t - u_2 f(t-2) = t - u_2(t-2) = \begin{cases} t & t < 2 \\ 2 & 2 \le t. \end{cases}$$

3.5 Examples. Compute $\mathcal{L}\left\{u_3(t)e^{4t}\right\}$.

3.6 Solution. We would like to use the formula, $\mathcal{L}\{u_3(t)f(t-3)\} = e^{-3s}F(s)$, where $F(s) = \mathcal{L}\{f(t)\}$. Let $g(t) = e^{4t}$. Then we want to find a function $f(t)$ such that $f(t-3) = g(t)$. Observe that if we had such a function $f(t)$, then

$$f(t) = f((t+3)-3) = g(t+3).$$

So in fact,

$$f(t) = g(t+3) = e^{4(t+3)} = e^{12}e^{4t}.$$

Then $F(s) = \mathcal{L}\{f(t)\} = e^{12}\mathcal{L}\left\{e^{4t}\right\} = e^{12}\frac{1}{s-4}$. Finally, using the formula,

$$\mathcal{L}\{u_3(t)g(t)\} = \mathcal{L}\{u_3(t)f(t-3)\} = e^{-3s}e^{12}\frac{1}{s-4} = \frac{e^{-3(s-4)}}{s-4}.$$

3.7 Remark. We knew that this was the answer, because $\mathcal{L}\left\{e^{4t}u_3(t)\right\}(s) = \mathcal{L}\{u_3(t)\}(s-4)$. The point of the example is to see how to find $f(t)$ when given $g(t)$ such that $f(t-c) = g(t)$.

3.8 Examples. Find $\mathcal{L}^{-1}\left\{\frac{2e^{-2s}}{s^2-4}\right\}$.

3.9 Solution. Let $F(s) = \frac{2}{s^2-4}$. We will compute $f(t) = \mathcal{L}^{-1}\{F(s)\}$, in order to use the translation formula to find $\mathcal{L}^{-1}\left\{e^{-2s}F(s)\right\}$.

To compute $\mathcal{L}^{-1}\left\{\frac{2}{s^2-4}\right\}$, we first use partial fractions to express

$$\frac{2}{s^2-4} = \frac{1/2}{s-2} - \frac{1/2}{s+2}.$$

Then by linearity of $\mathcal{L}^{-1}$, we have

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = 1/2\left(e^{2t} - e^{-2t}\right) = \sinh(2t).$$

What we really want is $\mathcal{L}^{-1}\left\{e^{-2s}F(s)\right\}$, so we use the translation formula

$$\mathcal{L}^{-1}\left\{e^{-2s}F(s)\right\} = u_2(t)f(t-2) = u_2(t)\sinh(2(t-2)).$$

Exercises.

i) (a) Sketch the graph of the given function
   (b) Express the function as a combination of unit step functions $u_c(t)$
   (c) Compute the Laplace transform of the function

$$f(t) = \begin{cases} t & 0 \le t < 2, \\ 2, & 2 \le t < 5, \\ 7-t, & 5 \le t < 7, \\ 0, & t \ge 7. \end{cases}$$

ii) Find the inverse Laplace transform of

(a) $F(s) = \frac{e^{-2s}}{s^2+s-2}$

(b) $F(s) = \frac{e^{-s}+e^{-2s}+e^{-3s}-3^{-4s}}{s}$.

3.2 Periodic Functions

We all know what a periodic function is. But here is a formal definition:

3.10 Definition. Suppose there exists some positive number P such that $f(t) = f(t+P)$ for all values of t. If P is the *smallest* such number for which this is true, then we say that $f(t)$ is *periodic* with period P.

The basic examples to remember are $\sin(t), \cos(t), \tan(t)$, with periods $2\pi, 2\pi, \pi$, respectively. Notice that constant functions are NOT considered periodic under this definition.

3.11 Question. Assuming that $f(t)$ is periodic with period P, what can we say about $\mathcal{L}\{f(t)\}$?

3.12 Solution.

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t)dt = \int_0^P e^{-st} f(t)dt + \int_P^\infty e^{-st} f(t)dt \\ &= \int_0^P e^{-st} f(t)dt + \int_0^\infty e^{-s(u+P)} f(u+P)du \\ &= \int_0^P e^{-st} f(t)dt + e^{-Ps} \int_0^\infty e^{-su} f(u)du \\ &= \int_0^P e^{-st} f(t)dt + e^{-Ps} \mathcal{L}\{f(t)\}.\end{aligned}$$

At this point, we have $\mathcal{L}\{f(t)\}$ on both sides of the equation, so we can move them to one side and solve:

$$(1-e^{-sP})\mathcal{L}\{f(t)\} = \int_0^P e^{-st} f(t)dt \quad \Rightarrow \quad \mathcal{L}\{f(t)\} = \frac{\int_0^P e^{-st} f(t)dt}{(1-e^{-Ps})}.$$

3.13 Examples. Let

$$f(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & 1 \le t < 2. \end{cases},$$

and let $f(t+2) = f(t)$. Find $\mathcal{L}\{f(t)\}$.

3.14 Solution. Observe that the function is periodic with period 2, so we may use the formula just derived. Then

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \frac{\int_0^2 e^{-st} f(t)dt}{1-e^{-2s}} = \frac{\int_0^1 e^{-st}\cdot 1\,dt + \int_1^2 e^{-st}\cdot 0\,dt}{1-e^{-2s}} \\ &= \frac{(1-e^{-s})/s + 0}{1-e^{-2s}} = \frac{1}{s}\frac{1-e^{-s}}{(1-e^{-s})(1+e^{-s})} = \frac{1}{s(1+e^{-s})}.\end{aligned}$$

Discussion. Without knowing about periodic functions, and the simplifying formula we derived above... what would we have done? I will demonstrate what is possible - not so that you try this in the future, but rather so you notice the time saved by the preceding formula.

The function defined above is a step-function, and can be written as a linear combination of $u_c(t)'s$ for various values of c. Indeed,

$$f(t) = [u_0(t) - u_1(t)] + [u_2(t) - u_3(t)] + [u_4(t) - u_5(t)] + \cdots = \sum_{c=0}^{\infty} u_{2c}(t) - u_{2c+1}(t).$$

Then by linearity of $\mathcal{L}$, we have

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \sum_{c=0}^{\infty} \mathcal{L}\{u_{2c}\} - \mathcal{L}\{u_{2c+1}\} = \sum_{c=0}^{\infty} \frac{e^{-2cs} - e^{-(2c+1)s}}{s} = \sum_{c=0}^{\infty} \frac{(1-e^{-s})}{s} e^{-2cs} \\
&= \frac{(1-e^{-s})}{s} \sum_{c=0}^{\infty} \left(e^{-2s}\right)^c = \frac{1-e^{-s}}{s} \frac{1}{1-e^{-2s}} = \frac{1-e^{-s}}{s(1-e^{-s})(1+e^{-s})} \\
&= \frac{1}{s(1+e^{-s})}.
\end{aligned}$$

To get to the second line, we use that $\frac{(1-e^{-s})}{s}$ is constant with respect to c, so we can factor it out of the sum. Notice that in the second line we use the geometric series

$$\frac{1}{1-r} = \sum_{c=0}^{\infty} r^c \qquad |r| < 1.$$

Obviously, this is a lot more work than using the previous formula.

Exercise. Find the Laplace transform of the function $f(t) = \sin(t)$, for $0 \le t < \pi$, and $f(t+\pi) = f(t)$.

Homework Problems

These problems are taken from section §6.3 of the textbook.

63 (B7). Consider the function

$$f(t) = \begin{cases} 0, & 0 \le t < 3, \\ -2, & 3 \le t < 5, \\ 2, & 5 \le t < 7, \\ 1, & 7 \le t. \end{cases}$$

a) Sketch the graph of the given function.

b) Express $f(t)$ in terms of the unit step function $u_c(t)$.

64 (B17). Find the Laplace transform of the function

$$f(t) = (t-3)u_2(t) - (t-2)u_3(t)$$

65 (B23). Find the inverse Laplace transform of the function

$$F(s) = \frac{(s-2)e^{-t}}{s^2 - 4s + 3}.$$

66 (B33). Find the Laplace transform of

$$f(t) = 1 + \sum_{k=1}^{\infty} (-1)^k u_k(t)$$

by integrating the series term-by-term.

67 (B37). Use the facts about periodic functions to evaluate the Laplace transform of

$$f(t) = \begin{cases} t, & 0 \le t < 1; \\ f(t-1) & t \ge 1. \end{cases}$$

4 Discontinuous Forcing Functions

In this section, we aim to solve initial value problems (IVP's) where the forcing function is piecewise-defined. So while the forcing function is piecewise-continuous, it need not be continuous. We already have all of the tools needed, so we begin with a simple example.

4.1 Examples. Solve the IVP

$$y'' + 4y = \sin(t) + u_\pi(t)\sin(t-\pi) \qquad \begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$$

using Laplace transforms.

4.2 Solution. First, observe that the RHS, $g(t) := \sin(t) + u_\pi(t)\sin(t-\pi)$ is continuous. Indeed,

$$g(t) = \begin{cases} \sin(t) & t < \pi \\ 0 & t \ge \pi \end{cases}$$

since $\sin(t-\pi) = -\sin(t)$. However, the continuity plays no role in our method of solution.

We apply $\mathcal{L}$ to both sides of the differential equation:

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\sin(t)\} + \mathcal{L}\{u_\pi(t)\sin(t-\pi)\}.$$

We use the various formulas to simplify this equation. For example, we know that

$$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0) \qquad \mathcal{L}\{u_\pi(t)\sin(t-\pi)\} = e^{-\pi s}\mathcal{L}\{\sin(t)\} = e^{-\pi s}\frac{1}{s^2+1}.$$

The resulting equation is:

$$\mathcal{L}\{y\}(s^2+4) = \frac{1}{s^2+1} + e^{-\pi s}\frac{1}{s^2+1}.$$

We solve this for $\mathcal{L}\{y\}$, and get:

$$\mathcal{L}\{y\} = \frac{1+e^{-\pi s}}{(s^2+1)(s^2+4)}.$$

We already know the Laplace transform of the solution! So now we just need to calculate the inverse transform. Looking at the RHS in the last equation, we can isolate $F(s) = \frac{1}{(s^2+1)(s^2+4)}$ as the function whose inverse transform we need to know. Indeed, if we knew $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then we could use the translation formula to find

$$y = \mathcal{L}^{-1}\{F(t)\} + \mathcal{L}^{-1}\left\{e^{-\pi s}F(s)\right\} = f(t) + u_\pi(t)f(t-\pi).$$

To find $\mathcal{L}^{-1}\{F(s)\}$, we use partial fractions to re-write

$$F(s) = \frac{1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}.$$

This is the general formula when we have irreducible quadratic factors. However, in the special case that both irreducible factors can be written as $(s-a)^2 + b_i^2$, we can just use the cover-up method. In this example, $a = 0$ for both quadratic factors, so we can use the cover-up method (plugging in i and $2i$ for s) to find

$$F(s) = \frac{1}{(s^2+1)(s^2+4)} = \frac{1/3}{s^2+1} + \frac{-1/3}{s^2+4} = \frac{1}{3}\cdot\frac{1}{s^2+1} - \frac{1}{6}\cdot\frac{2}{s^2+2^2}.$$

Notice that we had to re-write the terms to put them in the form we could identify from our table. It follows from the table that

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{3}\sin(t) - \frac{1}{6}\sin(2t),$$

so that

$$\begin{aligned} y(t) &= \frac{1}{3}\sin(t) - \frac{1}{6}\sin(2t) + u_\pi(t)\left(\frac{1}{3}\sin(t-\pi) - \frac{1}{6}\sin(2(t-\pi)\right) \\ &= \frac{1}{6}(2\sin(t) - \sin(2t)) - \frac{u_\pi(t)}{6}(2\sin(t) + \sin(2t)) \\ &= \begin{cases} \frac{1}{6}(2\sin(t) - \sin(2t)) & t < \pi \\ -\frac{1}{3}\sin(2t) & \pi \le t. \end{cases} \end{aligned}$$

4.3 Note. We have used that $\sin(t-\pi) = -\sin(t)$ and $\sin(2t - 2\pi) = \sin(2t)$.

4.4 Examples. Solve the following IVP using Laplace transforms:

$$y'' + 2y' + 2y = \begin{cases} 1 & \pi \le t < 2\pi \\ 0 & 0 \le t < \pi, \quad 2\pi \le t. \end{cases} \qquad \begin{cases} y(0) = 0 \\ y'(0) = 1. \end{cases}$$

4.5 Solution. Let $g(t) = \begin{cases} 1 & \pi \le t < 2\pi \\ 0 & 0 \le t < \pi, \quad 2\pi \le t. \end{cases}$. We first apply $\mathcal{L}$ to both sides:

$$\mathcal{L}\{y\}(s^2 + 2s + 2) - 1 = \mathcal{L}\{g(t)\}.$$

To solve for $\mathcal{L}\{y\}$, we need to know $\mathcal{L}\{g(t)\}$. You can check that $g(t) = u_\pi(t) - u_{2\pi}(t)$. Then using the linearity of $\mathcal{L}$ and the formula for $\mathcal{L}\{u_c(t)\}$ we get

$$\mathcal{L}\{g(t)\} = e^{-\pi s}s - e^{-2\pi s}s = \frac{e^{-\pi s} - e^{-2\pi s}}{s}.$$

Solving for $\mathcal{L}\{y\}$:

$$\mathcal{L}\{y\} = \frac{1 + \frac{e^{-\pi s} - e^{-2\pi s}}{s}}{s^2 + 2s + 2} = \frac{1}{(s+1)^2 + 1} + \frac{e^{-\pi s} - e^{-2\pi s}}{s((s+1)^2 + 1)}.$$

We will solve the problem as soon as we know the inverse Laplace transform of the right hand side above. Since $\mathcal{L}^{-1}$ is linear, we may do this term-by-term. Using our table, we know that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2 + 1}\right\} = e^{-t}\sin(t).$$

To evaluate the other terms, we really need to know the inverse transform of $F(s) = \frac{1}{s((s+1)^2+1)}$, and then we can use the transform formula. Indeed,

$$y(t) = e^{-t}\sin(t) + u_\pi(t)f(t - \pi) - u_{2\pi}(t)f(t - 2\pi) \qquad \text{when } f(t) = \mathcal{L}^{-1}\{F(s)\}.$$

To find $f(t)$, we must use partial fractions to re-write $F(s)$:

$$F(s) = \frac{1}{s((s+1)^2 + 1)} = \frac{A}{s} + \frac{B(s+1) + C}{(s+1)^2 + 1}.$$

Pay special attention to how we wrote the linear term above the irreducible quadratic. To solve for A, we multiply both sides by s and evaluate at $s = 0$: $A = \frac{1}{2}$. To get B and C, we multiply both sides by $(s+1)^2 + 1$ and evaluate at $s = -1 + i$, since that is what makes $(s+1)^2 + 1 = 0$. We get the equation

$$\frac{1}{-1 + i} = B(i) + C.$$

Since $\frac{1}{-1+i} = -\frac{1}{2}i - \frac{1}{2}$, we see that $B = -\frac{1}{2}$, and $C = -1/2$. Then

$$F(s) = \frac{1}{2}\left(\frac{1}{s} - \frac{s+1}{(s+1)^2 + 1} - \frac{1}{(s+1)^2 + 1}\right).$$

Using the table, we get

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}\left(1 - e^{-t}\cos(t) - e^{-t}\sin(t)\right).$$

Finally,

$$\begin{aligned} y(t) &= e^{-t}\sin(t) + \frac{u_\pi(t)}{2}\left(1 - e^{-(t-\pi)}\cos(t-\pi) - e^{-(t-\pi)}\sin(t-\pi)\right) \\ &\quad - \frac{u_{2\pi}(t)}{2}\left(1 - e^{-(t-2\pi)}\cos(t-2\pi) - e^{-(t-2\pi)}\sin(t-2\pi)\right) \\ &= e^{-t}\sin(t) + \frac{1}{2}\left(u_\pi(t) - u_{2\pi}(t)\right) + \frac{1}{2}e^{-t}\cos(t)\left(u_\pi(t)e^{\pi} - u_{2\pi}(t)e^{2\pi}\right) \\ &\quad + \frac{1}{2}e^{-t}\sin(t)\left(u_\pi(t)e^{\pi} - u_{2\pi}(t)e^{2\pi}\right) \\ &= e^{-t}\sin(t) + \frac{1}{2}\left(u_\pi(t) - u_{2\pi}(t)\right) + \frac{e^{-t}}{2}\left(u_\pi(t)e^{\pi} - u_{2\pi}(t)e^{2\pi}\right)\left(\cos(t) + \sin(t)\right). \end{aligned}$$

Homework Problems

These problems are taken from section §6.4 of the textbook.

68. Use Laplace transforms to find the solutions of the given initial value problems:

B1) $y'' + y = f(t); \qquad y(0) = 0, \quad y'(0) = 1;$

$$f(t) = \begin{cases} 1, & 0 \le t < 3\pi \\ 0, & 3\pi \le t < \infty \end{cases}$$

B3) $y'' + 4y = \sin t - u_{2\pi}(t)\sin(t - 2\pi); \qquad y(0) = 0, \quad y'(0) = 0$

B5) $y'' + 3y' + 2y = f(t); \qquad y(0) = 0, \quad y'(0) = 0;$

$$f(t) = \begin{cases} 1, & 0 \le t < 10 \\ 0, & 10 \le t < \infty \end{cases}$$

B7) $y'' + y = u_{3\pi}(t); \qquad y(0) = 1, \quad y'(0) = 0$

B10) $y'' + y' + \frac{5}{4}y = g(t); \qquad y(0) = 0, \quad y'(0) = 0;$

$$g(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & \pi \le t < \infty \end{cases}$$

5 Convolution Product

When we use the Laplace transform to solve initial value problems, the last step is evaluating the inverse Laplace transform of some function of s. For example, suppose we find that

$$F(s) = \frac{1}{s(s^2+1)}$$

and we needed to find the inverse transform, $\mathcal{L}^{-1}\{F\}$. The usual method is to do partial fractions, and then compute the inverse transform of each term. It is tempting to think of how $\mathcal{L}^{-1}\{F\}$ relates to $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$ and $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$, because we can readily identify these functions as 1 and $\sin(t)$. But it is obvious that

$$\mathcal{L}\{1 \cdot \sin(t)\}(s) \neq F(s).$$

This demonstrates the fact that for functions f and g,

$$\mathcal{L}\{f \cdot g\} \neq \mathcal{L}\{f\} \cdot \mathcal{L}\{g\},$$

generally. However, there is another product called the *convolution product* which makes this work.

5.1 Definition. The *convolution product* of f and g is denoted $f * g$ and is defined by

$$f * g(t) = \int_{x=0}^{t} f(x)g(t-x)dx.$$

5.2 Theorem. *Let f and g be nice functions. Then*

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\}.$$

5.3 Remark. For different transforms, i.e. the Fourier transform, you will want to consider a different convolution product. The point is to get a convolution which produces a theorem like this one.

Proof. We write $F(s) \cdot G(s) = \int_0^\infty e^{-sx} f(x)dx \int_0^\infty e^{-sy} g(y)dy$. Re-write as an iterated integral, and substitute $t = x + y$:

$$\begin{aligned} F(s) \cdot G(s) &= \int_{x=0}^{\infty}\int_{y=0}^{\infty} e^{-s(x+y)} f(x)g(y)dydx = \int_{x=0}^{\infty}\int_{t=x}^{\infty} e^{-st} f(x)g(t-x)dtdx \\ &= \int_{t=0}^{\infty} e^{-st}\int_{x=0}^{x=t} f(x)g(t-x)\,dxdt = \mathcal{L}\{f * g\}. \end{aligned}$$

In the second line, we switched the order of integration, which changes the bounds as indicated. □

Returning to our example, we may compute

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 * \sin(t) = \int_0^t \sin(t-x)dx = \cos(t-x)|_{x=0}^{x=t} = 1 - \cos(t).$$

Did it work? We can check:

$$\mathcal{L}\{1 - \cos(t)\} = \frac{1}{s} - \frac{s}{s^2+1} = \frac{(s^2+1) - s^2}{s(s^2+1)} = \frac{1}{s(s^2+1)}.$$

This example may not be impressive, since we had the technique of partial fractions already available. The following example might shed some light on the utility of the convolution product.

5.4 Examples. Solve the IVP

$$y'' + by' + cy = g(t), \qquad b, c \in \mathbb{R}, \qquad \begin{cases} y(0) = 0 \\ y'(0) = 1, \end{cases}$$

where $g(t)$ is any nice function.

5.5 Solution. If we take the Laplace transform, we get

$$(s^2 + bs + c)\mathcal{L}\{y\} = \mathcal{L}\{g\} + 1.$$

In particular, if we take the Laplace transform of the associated homogeneous equation we obtain

$$(s^2 + bs + c)\mathcal{L}\{y_h\} = 1.$$

This means that $\mathcal{L}\{y_h\} = \frac{1}{(s^2+bs+c)}$, so that it suffices to find a function y_p such that

$$(s^2 + bs + c)\mathcal{L}\{y_p\} = \mathcal{L}\{g\},$$

since then $(s^2 + bs + c)\mathcal{L}\{y_p + y_h\} = \mathcal{L}\{g\} + 1$. Notice that

$$\mathcal{L}\{y_p\} = \mathcal{L}\{g\} \cdot \frac{1}{s^2 + bs + c} = \mathcal{L}\{g\} \cdot \mathcal{L}\{y_h\};$$

it follows that $y_p = g * y_h$.

We compare this formula, $y_p = g * y_h$ with the formula given to us by variation of parameters. Assume for concreteness that $b = 3$ and $c = 2$, so that $y_h = e^{3t} - e^{2t}$. Then,

$$\begin{aligned} y_p(t) = g * y_h(t) &= \int_0^t g(s) y_h(t-s) ds = \int_0^t g(s)\left(e^{3(t-s)} - e^{2(t-s)}\right) ds \\ &= e^{3t} \int_0^t \frac{g(s)}{e^{3s}} ds - e^{2t} \int_0^t \frac{g(s)}{e^{2s}} ds. \end{aligned}$$

If we instead use the variation of parameters formula, with $y_1 = e^{2t}$, $y_2 = e^{3t}$, $W(y_1, y_2) = e^{5t}$, we get

$$\begin{aligned} y_p &= y_2 \int_0^t y_1 \frac{g(s)}{W(y_1, y_2)(s)} ds - y_1 \int_0^t y_2 \frac{g(s)}{W(y_1, y_2)(s)} ds \\ &= e^{3t} \int_0^t e^{2s} \frac{g(s)}{e^{5s}} ds - e^{3t} \int_0^t e^{3s} \frac{g(s)}{e^{5s}} ds \\ &= e^{3t} \int_0^t \frac{g(s)}{e^{3s}} ds - e^{2t} \int_0^t \frac{g(s)}{e^{2s}} ds. \end{aligned}$$

Of course, we get the same thing. But it is much easier to think of the particular solution as just the *convolution product* of the driving function $g(t)$ with the homogeneous solution y_h rather than remembering the clunky variation of parameters technique.

5.6 Theorem. *Let $f(t)$, $g(t)$, and $h(t)$ be nice functions, and $c \in \mathbb{C}$ a complex number. Then* *i) (Commutativity)* $f * g = g * f$;

ii) (Associativity) $f * (g * h) = (f * g) * h$;

iii) (Distributivity) $f * (g + h) = f * g + f * h$;

iv) ($\mathbb{C}$-Linearity) $(cf) * g = c(f * g) = f * (cg)$;

v) (Differentiation) $[f * g(t)]' = (f' * g)(t) + f(0) \cdot g(t) = (f * g')(t) + f(t) \cdot g(0)$.

Proof. (i) This is proved using the substitution $u = t - s$:

$$f * g(t) = \int_{s=0}^{t} f(s)g(t-s)ds = \int_{u=t}^{0} f(t-u)g(u)(-du) = \int_{0}^{t} g(u)f(t-u)du = g * f(t).$$

(ii) Exercise. (iii) and (iv) follow from the linearity of the integral and property (i). To prove (v), we use $\mathcal{L}$:

$$\mathcal{L}\{(f * g)'\} = s\mathcal{L}\{f * g\} - (f * g)(0) = s\mathcal{L}\{f\}\mathcal{L}\{g\}.$$

We compare this to

$$\mathcal{L}\{f' * g\} = \mathcal{L}\{f'\}\mathcal{L}\{g\} = (s\mathcal{L}\{f\} - f(0))\mathcal{L}\{g\} = s\mathcal{L}\{f\}\mathcal{L}\{g\} - f(0) \cdot \mathcal{L}\{g\}.$$

It follows that $(f * g)' = \mathcal{L}^{-1}\{\mathcal{L}\{f' * g\} + f(0)\mathcal{L}\{g\}\} = f' * g + f(0) \cdot g$. The second equality follows analogously. □

Observe that $1 * f \neq f$, in general. We will find the "multiplicative identity" of $*$ next.

5.1 The Dirac Delta

Consider a nice function $g(t)$. We first compute the convolution product $g * u_c(t)$ for future reference for $c > 0$. By definition, we get

$$g * u_c(t) := \int_0^t g(s)u_c(t-s)ds = \begin{cases} \int_0^{t-c} g(s)ds & \text{if } c < t \\ 0 & \text{if } t \leq c. \end{cases} \tag{3.1}$$

5.7 Definition. The *impulse* of a system is defined to be the integral of a force with respect to time.

Impulse is an idea from classical mechanics (physics). Mathematically, if a function $g(t)$ represents an applied force of a system at some time t, then the impulse produced from time t_0 to t_1 is $I = \int_{s=t_0}^{t_1} g(s)ds$. Recall that if $p(t)$ denotes the momentum of a particle, then $p'(t) = g(t)$ is the applied force. By the fundamental theorem of calculus, the impulse

$$I = \int_{t_0}^{t_1} g(s)ds = \int_{t_0}^{t_1} p'(s)ds = p(t_1) - p(t_0) = \Delta p$$

is the change in momentum during that interval of time. In summary, force is the *instantaneous* rate of change in momentum, while impulse is the net difference in momentum during some interval.

Now, consider adding 1 unit of momentum (say a $N \cdot s$) to some particle (by applying some force). This could be accomplished by applying a force of 1 Newton for 1 second, or 2 Newtons for 0.5 seconds, or 4 Newtons for 0.25 seconds, etc. We can model these situations using the following force functions:

$$u_0(t) - u_1(t); \qquad 2u_0(t) - 2u_{0.5}(t); \qquad 4u_0(t) - 4u_{0.25}(t); \qquad \text{etc.}$$

Generally, adding 1 unit of momentum using a constant force F is give by $F(u_0(t) - u_{1/F}(t))$. If $c = 1/F$, then we define

$$\delta_c(t) = \frac{u_0(t) - u_c(t)}{c},$$

so that the previous functions are exactly δ_1, $\delta_{1/2}$, $\delta_{1/4}$, etc.

Now we would like to compute the convolution of $\delta_c(t)$ with some applied force $g(t)$. Using (3.1), we obtain

$$\begin{aligned} g * \delta_c(t) &= \frac{1}{c}\left(\int_0^t g(s)ds - \int_0^{t-c} g(s)ds\right) \\ &= \frac{1}{c}\left(\int_{t-c}^t g(s)ds\right). \end{aligned}$$

In words, the value at time t of $g * \delta_c$ is the average applied force during the previous c seconds. Consider the idealized situation of instantaneously changing the momentum of our system by 1 unit. This is the limiting case as c tends to 0. We obtain

$$\lim_{c \longrightarrow 0} g * \delta_c(t) = \lim_{c \longrightarrow 0} \frac{G(t) - G(t-c)}{c} = G'(t) = g(t),$$

where $G(t)$ is an antiderivative of $g(t)$.

It is important to notice that $\lim_{c \longrightarrow 0} \delta_c(t)$ does not exist. The value of this function at 0 becomes unbounded as $c \longrightarrow 0$. However, the limit of the convolution product $g * \delta_c(t)$ does exist and is expressed very simply. We define an "idealized function" δ, called the *Dirac delta* function by defining

$$\delta \quad \text{“} := \text{”} \quad \lim_{c \longrightarrow 0} \delta_c;$$

this actually means to replace δ with δ_c in any convolution involving δ, and then take the limit as c tends to 0. This idealized function δ represents an idealized instantaneous unit change in momentum, and so is often called the *unit impulse function*. We have already proved the most important property:

5.8 Theorem. *For any nice function $g(t)$,*

$$\delta * g(t) = g(t) = g * \delta(t).$$

Thus, $\mathcal{L}\{\delta\} = 1$ and δ is the "multiplicative identity" for the convolution product.

5.9 Theorem. *The following initial value problems are equivalent* *i)* $y'' + p(t)y' + q(t)y = g(t)$ $\begin{cases} y(0) = y_0 \\ y'(0) = y'_0. \end{cases}$

ii) $y'' + p(t)y' + q(t)y = g(t) + y'_0 \cdot \delta(t)$ $\begin{cases} y(0) = y_0 \\ y'(0) = 0. \end{cases}$

Table of Laplace transforms:

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}, \quad s > 0$
2. e^{at}	$\frac{1}{s-a}, \quad s > a$
3. $t^n, \quad n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
4. $t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5. $\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
6. $\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
7. $\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \|a\|$
8. $\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \|a\|$
9. $e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
10. $e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
11. $t^n e^{at}, \quad n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}$
12. $u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$
14. $e^{ct}f(t)$	$F(s-c)$
15. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), c > 0$
16. $\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
17. $\delta(t-c)$	e^{-cs}
18. $f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19. $(-t)^n f(t)$	$F^{(n)}(s)$