Math 308 I/J **Final** **Autumn 2016**

Your Name

Student ID #

- Do not open this exam until you are told to begin. You will have 1 hour, 50 minutes for the exam.
- Check that you have a complete exam. There are 8 questions for a total of 117 points.
- You are allowed to have one handwritten note sheet. Only basic non-graphing scientific calculators are allowed, though you should not need one.
- Cheating will result in a zero and be reported to the Dean's Academic Conduct Committee.
- **Show all your work**. Unless explicitly stated otherwise in a particular question, if there is no work supporting your answer, you will not receive credit for the problem. If you need more space to answer a question, continue on the back of the page, and indicate that you have done so.

Question	Points	Score
1	21	
2	13	
3	12	
4	18	
5	12	
6	14	
7	15	
8	12	
Total:	117	

1. For true/false and multiple choice questions, you are **not required to show any work**.

   (a) (1 point) Any linearly independent set in $\mathbb{R}^n$ spans $\mathbb{R}^n$.

   ○ True √ **False**

   (b) (1 point) Is $A^T A$ symmetric?

   √ **always** ○ never ○ only if A is square

   (c) (4 points) Check all of the following properties of determinants which are *always true*. A, B are $n \times n$ matrices, B is invertible, and c is a scalar.

   √ $\det(A) = \det(A^T)$ ○ $\det(A^2 + I) = \det(A)^2 + 1$ ○ $\det(cA) = c\det(A)$
   √ $\det(B^{-1})^{-1} = \det(B)$ √ $\det(AB) = \det(A^T B^T)$ ○ $\det(A+B) = \det(A)+\det(B)$

   (d) (4 points) Define **two** of the following three terms: basis, dimension of a subspace, eigenvector. (Clearly specify which terms you are defining.)

   Solution:

   - Basis: given a subspace S, a basis for S is a linearly independent set spanning S.
   - Dimension: given a subspace S, the size of any basis for S is the dimension of S.
   - Eigenvector: given a (square) matrix A, an eigenvector $\vec{v}$ of A is any non-zero vector $\vec{v}$ where $A\vec{v} = \lambda\vec{v}$ for some scalar λ.

   (e) (4 points) Suppose a matrix A satisfies $A^3 + A = I$. Show that A is non-singular (i.e. invertible).

   Solution: A must be square for A^3 to be defined. If A were singular, by the Big Theorem, $\text{null}(A) \neq 0$, i.e. there is some $\vec{0} \neq \vec{v}$ such that $A\vec{v} = \vec{0}$. But then

   $$\vec{0} = \vec{0} + \vec{0} = A^3\vec{v} + A\vec{v} = I\vec{v} = \vec{v},$$

   a contradiction. Hence A must be non-singular.
   Alternate argument: $A(A^2 + I) = I$, so $A^{-1} = A^2 + I$ exists and A is invertible.

(f) (4 points) Suppose

$$S = \left\{ \begin{bmatrix} c_1 \\ 1 + c_2 \\ -c_1 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\} \subset \mathbb{R}^3.$$

Is S a subspace of $\mathbb{R}^3$?

Solution: Yes. Check the three conditions.

- $\vec{0} \in S$: use $c_1 = 0, c_2 = -1$.
- If $\vec{x}, \vec{y} \in S$, then $\vec{x} + \vec{y} \in S$:

$$\begin{bmatrix} c_1 \\ 1 + c_2 \\ -c_1 \end{bmatrix} + \begin{bmatrix} d_1 \\ 1 + d_2 \\ -d_1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ 1 + (c_2 + d_2 + 1) \\ -(c_1 + d_2) \end{bmatrix}.$$

- If $\vec{x} \in S$, $t \in \mathbb{R}$, then $t\vec{x} \in S$:

$$t \begin{bmatrix} c_1 \\ 1 + c_2 \\ -c_1 \end{bmatrix} = \begin{bmatrix} tc_1 \\ 1 + (t + tc_2 - 1) \\ -(tc_1) \end{bmatrix}.$$

Alternatively, S is the span of $\langle 1, 0, -1 \rangle$ and $\vec{e}_2$, and spans are subspaces.

(g) (3 points) Let $T \colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation. Suppose the range of T is a line. Describe the kernel of T geometrically.

Solution: By the rank-nullity theorem, the kernel of T has dimension $2 - 1 = 1$, so the kernel is a line.

2. Produce example(s) with the given properties. You are *not required* to give justification.

(a) (4 points) A square matrix A is called *orthogonal* if $A^T A = I$. It is a fact that if A is orthogonal, then $\det(A) = \pm 1$. Give examples of orthogonal matrices B and C where $\det(B) = 1$ and $\det(C) = -1$.

Solution: There are many examples. For instance, take $B = I_n$ and $C = \operatorname{diag}(-1, 1, 1, \ldots, 1)$.

(b) (4 points) Give an example of three pairwise orthogonal vectors in $\mathbb{R}^4$ with no 0 coordinates.

Solution: There are many examples. For instance, take $\vec{u} = (1, 1, 1, 1)$, $\vec{v} = (1, 1, -1, -1)$, $\vec{w} = (-1, 1, -1, 1)$.

(c) (3 points) Give a linear transformation T whose corresponding matrix is non-zero and triangular, and where T is not onto.

Solution: There are many examples. For instance, $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T(\vec{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \vec{x}.$$

(d) (2 points) Give an example of a matrix with 4 distinct eigenvalues.

Solution: There are many examples. For instance, take $\operatorname{diag}(1, 2, 3, 4)$.

3. Produce example(s) with the given properties. You are *not required* to give justification.

(a) (4 points) Suppose $V = \{\vec{v}_1, \ldots, \vec{v}_n\} \subset \mathbb{R}^n$, $A = [\vec{v}_1 \ \cdots \ \vec{v}_n]$, and $T \colon \mathbb{R}^n \to \mathbb{R}^n$ by $T(\vec{x}) = A\vec{x}$. Using this notation, give four of the equivalent conditions in the Big Theorem.

> **Solution:** Possibilities include: V spans $\mathbb{R}^n$; V is linearly independent; for all $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$; T is onto; T is one-to-one; A is invertible; $\ker(T) = \{\vec{0}\}$; V is a basis for $\mathbb{R}^n$; $\mathrm{col}(A) = \mathbb{R}^n$; $\mathrm{row}(A) = \mathbb{R}^n$; $\mathrm{rank}(A) = n$; $\det(A) \neq 0$; $\lambda = 0$ is not an eigenvalue of A.

(b) (4 points) Give examples of two-dimensional subspaces S_1 and S_2 of $\mathbb{R}^4$ where $S_1 \neq S_2$ and where S_1 and S_2 contain some common non-zero vector.

> **Solution:** One example is $S_1 = \mathrm{span}\{\vec{e}_1, \vec{e}_2\}$, $S_2 = \mathrm{span}\{\vec{e}_1, \vec{e}_3\}$.

(c) (4 points) Find some linear transformation T such that $\mathrm{range}\, T$ contains $\ker T$ and $\ker T \neq \{\vec{0}\}$. (Hint: this can be done in two dimensions.)

> **Solution:** One example is the linear transformation $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ which sends $\vec{e}_1$ to $\vec{0}$ and $\vec{e}_2$ to $\vec{e}_1$. The corresponding matrix is
>
> $$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

4. (a) (8 points) Let $S = \text{span}\{\vec{s}_1, \vec{s}_2, \vec{s}_3\} \subset \mathbb{R}^4$ and $\vec{u} \in \mathbb{R}^4$ where

$$\vec{s}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{s}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{s}_3 = \begin{bmatrix} 2 \\ 2 \\ -1 \\ -2 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}.$$

Compute $\text{proj}_S \vec{u}$.

Solution: Applying Gram-Schmidt to the basis $\{\vec{s}_1, \vec{s}_2, \vec{s}_3\}$ gives an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, namely

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ -2 \end{bmatrix}.$$

Using the projection formula gives

$$\text{proj}_S \vec{u} = \text{proj}_{\vec{v}_1} \vec{u} + \text{proj}_{\vec{v}_2} \vec{u} + \text{proj}_{\vec{v}_3} \vec{u} = \frac{3}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{-1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \frac{0}{5} \begin{bmatrix} 0 \\ 0 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}.$$

(b) (4 points) Let S be a subspace of $\mathbb{R}^n$. Suppose $\vec{u} \in S^\perp$. Show that $\text{proj}_S \vec{u} = \vec{0}$.

Solution: Recall that $\text{proj}_S \vec{u}$ is characterized by the property that

$$\text{proj}_S \vec{u} \in S, \quad \vec{u} - \text{proj}_S \vec{u} \in S^\perp.$$

Here we have $\vec{0} \in S$ and $\vec{u} - \vec{0} = \vec{u} \in S^\perp$, so that $\vec{0} = \text{proj}_S \vec{u}$.
Alternatively, suppose $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a basis for S. By Gram-Schmidt, we can assume our basis is orthogonal. Since $\vec{u} \in S^\perp$, we have $\vec{u} \cdot \vec{v}_i = 0$ for each i, so that $\text{proj}_{\vec{v}_i} \vec{u} = \vec{0}$. From the projection formula, we then have

$$\text{proj}_S \vec{u} = \text{proj}_{\vec{v}_1} \vec{u} + \cdots + \text{proj}_{\vec{v}_k} \vec{u} = \vec{0}.$$

(c) (6 points) Suppose $S = \text{span}\{\vec{s}_1, \vec{s}_2\}$ where

$$\vec{s}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \quad \vec{s}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}.$$

Compute a basis for $S^\perp$.

Solution: Recall that $\text{null}(A) = \text{row}(A)^\perp$. If A's rows are $\vec{s}_1^T$ and $\vec{s}_2^T$, then $\text{null}(A) = \text{row}(A)^\perp = S^\perp$, so compute a basis for $\text{null}(A)$. Row reducing gives

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -4 & -5 \\ 0 & 1 & 2 & 3 \end{pmatrix}.$$

so that the general solution of the corresponding homogeneous linear system is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4s_1 + 5s_2 \\ -2s_1 - 3s_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 1 \\ 0 \end{bmatrix} s_1 + \begin{bmatrix} 5 \\ -3 \\ 0 \\ 1 \end{bmatrix} s_2.$$

Thus a basis is

$$\left\{ \begin{bmatrix} 5 \\ -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 1 \\ 0 \end{bmatrix}. \right\}$$

5. (a) (2 points) What is the (smaller) angle between the vectors

$$\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad \text{and} \qquad \vec{v} = \begin{bmatrix} -3 \\ 2 \\ -1 \\ 0 \end{bmatrix}?$$

Solution: Their dot product is 0, so they form a 90° angle.

(b) (6 points) Show that $\text{null}(A) \subset \text{null}(A^T A)$. Conclude that $\text{rank}(A) \geq \text{rank}(A^T A)$.

Solution: If $\vec{x} \in \text{null}(A)$, then $A\vec{x} = \vec{0}$, so $A^T A\vec{x} = A^T\vec{0} = \vec{0}$, so $\vec{x} \in \text{null}(A^T A)$, giving the first part. Hence

$$\text{nullity}(A) = \dim(\text{null}(A)) \leq \dim(\text{null}(A^T A)) = \text{nullity}(A^T A).$$

Note that A and $A^T A$ have the same number of columns, say m. By the rank-nullity theorem,

$$m - \text{rank}(A) \leq m - \text{rank}(A^T A),$$

so that $\text{rank}(A^T A) \leq \text{rank}(A)$. (In fact, $\text{null}(A^T A) = \text{null}(A)$, as follows. If $A^T A\vec{x} = \vec{0}$, then

$$|A\vec{x}|^2 = (A\vec{x}).(A\vec{x}) = (A\vec{x})^T(A\vec{x}) = \vec{x}^T A^T A\vec{x} = \vec{x}^T\vec{0} = 0$$

so that $A\vec{x} = \vec{0}$. Using the same argument, it follows that $\text{rank}(A^T A) = \text{rank}(A)$.)

(c) (4 points) Find all least squares solutions to the system $A\vec{x} = \vec{y}$ where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Solution: We solve the normal equations $A^T A\vec{x} = A^T\vec{y}$. Since the columns of A are linearly independent, there is a unique solution, namely

$$\vec{x} = (A^T A)^{-1} A^T\vec{y} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Alternatively, the original system is consistent, the columns of A are linearly independent, and we can notice that the sum of the columns of A is $\vec{y}$. That is, $\widehat{y} = \vec{y}$ and $A\vec{x} = \widehat{y}$ has the suggested unique solution.

6. (a) (4 points) Consider the two linear systems $A\vec{x} = \vec{y}$ and $A\vec{u} = \vec{0}$. Suppose that $\vec{x}_p$ is a solution to the first system. For any solution $\vec{u}$ of the second system, show that $\vec{x} = \vec{x}_p + \vec{u}$ is a solution of the first system.

Solution: We have $A\vec{x}_p = \vec{y}$ and $A\vec{u} = \vec{0}$. Then

$$A\vec{x} = A(\vec{x}_p + \vec{u}) = A\vec{x}_p + A\vec{u} = \vec{y} + \vec{0} = \vec{y}.$$

(b) (5 points) Solve the two linear systems $A\vec{x} = \vec{u}$ and $A\vec{x} = \vec{v}$ where

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}, \quad \vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

(Hint: inverse.)

Solution: Row reduction gives

$$\begin{pmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 2 & 3 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -2 & 1 \end{pmatrix}$$

so the matrix is invertible, giving solutions

$$\vec{x} = A^{-1}\vec{u} = \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix}$$

and

$$\vec{x} = A^{-1}\vec{v} = \begin{bmatrix} 11 \\ 2 \\ -6 \end{bmatrix}$$

(c) (5 points) Let

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\}, \quad \mathcal{B}_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Given that $\mathcal{B}_1, \mathcal{B}_2$ are bases for the same subspace of $\mathbb{R}^3$, compute the change of basis matrix C from $\mathcal{B}_1$ to $\mathcal{B}_2$.

Solution: We can row reduce the matrix whose columns are the vectors from $\mathcal{B}_2$ and $\mathcal{B}_1$, in that order, to get

$$\begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 \\ 4 & 0 & 0 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The change of basis matrix is the upper right corner,

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}.$$

Alternatively, it's easy to compute the coordinate vectors of the first basis in the second basis directly in this case, e.g. the first vector in $\mathcal{B}_1$ is the second vector in $\mathcal{B}_2$, and the second vector in $\mathcal{B}_1$ is the first vector in $\mathcal{B}_2$ plus twice the second vector in $\mathcal{B}_2$.

7. The matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

has characteristic polynomial

$$\det(A - \lambda I) = (1-\lambda)^3(1+\lambda)^2.$$

(a) (8 points) Find bases for the eigenspaces of A.

Solution: From the given characteristic polynomial, the eigenvalues are ± 1.

- For $\lambda = 1$, we find

$$A - I = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

  The general solution of the corresponding homogeneous linear system gives a basis for this null space as

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

- For $\lambda = -1$, we find

$$A + I = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

  The general solution of the corresponding homogeneous linear system gives a basis for this null space as

$$\left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(b) (2 points) What are the geometric multiplicities of the eigenvalues of A?

Solution: The geometric multiplicity of 1 is 3; the geometric multiplicity of -1 is 2.

(c) (2 points) Is A diagonalizable?

Solution: The algebraic multiplicity of 1 is 3; the algebraic multiplicity of -1 is 2. These agree with the geometric multiplicity, so from class A is indeed diagonalizable.

(d) (3 points) If some matrix B is diagonalizable, show that B^2 is diagonalizable.

Solution: We have $B = PDP^{-1}$, so $B^2 = PD^2P^{-1}$.

8. (a) (4 points) Determine the *eigenvalues* and *algebraic multiplicities* of the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Solution: Expanding $A - \lambda I$ along the third column gives

$$\det(A - \lambda I) = (1-\lambda)((1-\lambda)(2-\lambda) - 6) = (1-\lambda)(4-\lambda)(-1-\lambda),$$

so the eigenvalues are $1, 4, -1$, and each has algebraic multiplicity 1.

(b) (3 points) Suppose some matrix B has eigenvalue λ. Show that B^2 has eigenvalue λ^2.

Solution: Let $B\vec{v} = \lambda\vec{v}$ for $\vec{v} \neq \vec{0}$. Then

$$B^2\vec{v} = B(B\vec{v}) = B(\lambda\vec{v}) = \lambda(B\vec{v}) = \lambda^2\vec{v}.$$

(c) (5 points) Let

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

For which value(s) of $0 \leq \theta < 2\pi$ does R_θ have (real) eigenvalues? What are those eigenvalues?

Solution: R_θ is a 2×2 rotation matrix. The corresponding linear transformation almost never just scales some non-zero vector. It only does so when rotating by 0 or π, in which case it has eigenvalue 1 or -1, respectively.

More algebraically, the characteristic equation is

$$\lambda^2 - (2\cos\theta)\lambda + 1 = 0$$

which has discriminant $4\cos^2\theta - 4 = -4\sin^2\theta$. This is almost always negative, in which case the eigenvalues are not real; they're real when it's zero, namely when $\theta = 0$ or π, which gives roots 1 and -1, respectively.