Your Name

Student ID #

- Do not open this exam until you are told to begin. You will have 1 hour and 50 minutes for the exam.
- Check that you have a complete exam. There are 7 questions for a total of 110 points.
- You are allowed to have one handwritten note sheet. Only basic non-graphing scientific calculators are allowed, though you should not need one.
- Cheating will result in a zero and be reported to the Dean's Academic Conduct Committee.
- **Show all your work**. Unless explicitly stated otherwise in a particular question, if there is no work supporting your answer, you will not receive credit for the problem. If you need more space to answer a question, continue on the back of the page, and indicate that you have done so.

Question	Points	Score
1	21	
2	21	
3	18	
4	12	
5	12	
6	11	
7	15	
Total:	110	

1. Multiple choice and short answer. For these questions, you are **not required to show any work**.

(a) (2 points) There are infinitely many one-dimensional subspaces of $\mathbb{R}^2$.

√ **True** ○ False

Solution: Infinitely many such subspaces are given as span $\left\{ \begin{bmatrix} 1 \\ t \end{bmatrix} \right\}$ for $t \in \mathbb{R}$.

(b) (2 points) If A is $n \times n$, then the reduced row echelon form of A is I_n.

○ True √ **False**

Solution: 0 is its own reduced row echelon form.

(c) (2 points) If A and B are $n \times n$ and $\det(AB) \neq 0$, then A and B are row equivalent.

√ **True** ○ False

Solution: $\det(AB) = \det(A)\det(B) \neq 0$, so A and B are invertible. Invertible square matrices row reduce to the identity, so are row equivalent.

(d) (2 points) Let A be $m \times n$. Then $A^T A$ is symmetric if and only if $m = n$. (Recall that X is symmetric if $X = X^T$.)

○ True √ **False**

Solution: $(A^T A)^T = A^T (A^T)^T = A^T A$, regardless of the size of A.

(e) (2 points) A nonsingular matrix can have 0 as an eigenvalue.

○ True √ **False**

Solution: A nonsingular matrix is a square matrix which is invertible. By the big theorem, such a matrix cannot have 0 as an eigenvalue.

(f) (2 points) If $S \subset \mathbb{R}^4$ is a subspace of dimension 2, then every $\mathbf{x} \in \mathbb{R}^4$ is in either S or $S^\perp$.

○ True √ **False**

Solution: Let $S = \text{span}\{\mathbf{e}_1\} \subset \mathbb{R}^2$, so $S^\perp = \text{span}\{\mathbf{e}_2\}$. Then $\mathbf{e}_1 + \mathbf{e}_2$ is in neither S nor $S^\perp$.

(g) (2 points) Let A be an $n \times n$ matrix with (distinct) eigenvalues $\lambda_1, \ldots, \lambda_k$ and eigenspaces $S_1, \ldots, S_k$. Then $\dim S_1 + \cdots + \dim S_k \leq n$.

√ **True** ○ False

Solution: $\dim S_i$ is less than or equal to the multiplicity of λ_i in the characteristic polynomial of A, so adding up all of these gives at most n.

(h) (2 points) A subspace $S \neq \{\mathbf{0}\}$ can have a finite number of vectors.

○ True √ **False**

Solution: Such a subspace has at least one non-zero vector, and it has all multiples of that vector, of which there are infinitely many.

(i) (3 points) Give the definition of "subspace."

Solution: A subspace S of $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ which contains $\mathbf{0}$ and is closed under addition and scalar multiplication.

(j) (2 points) Give the definition of "basis."

Solution: A basis is a linearly independent spanning set for some given subspace.

2. Provide examples meeting the given requirements. Unlike on the midterms, you **must justify your answers** on this question.

   (a) (4 points) Give a 3×3 matrix which has π as an eigenvalue where the π-eigenspace has dimension 3.

   Solution: The only example is πI_3. It clearly scales all inputs by π, so the π eigenspace is $\mathbb{R}^3$ which has dimension 3.

   (b) (4 points) Find A and B where $\det(A+B) \neq \det(A) + \det(B)$.

   Solution: Almost everything is a counterexample. For instance,

$$\det\left(\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix} + \begin{bmatrix}0 & -1\\0 & 1\end{bmatrix}\right) = \det\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix} = 2$$
$$\neq 0 = 0 + 0 = \det\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix} + \det\begin{bmatrix}0 & -1\\0 & 1\end{bmatrix}.$$

   (c) (5 points) Find a 3×3 matrix A where $A^3 = 0$ but $A^2 \neq 0$. (Hint: triangular matrices.)

   Solution: The simplest example is probably

$$A = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$$

   since

$$A^2 = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$
$$A^3 = A^2A = \begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}.$$

(d) (4 points) Find a matrix whose characteristic polynomial is $(1-\lambda)(2-\lambda)^2(3-\lambda)^3$.

Solution: Any triangular matrix with diagonal entries $1, 2, 2, 3, 3, 3$ has this property. For instance,

$$\det \begin{bmatrix} 1-\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 2-\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 2-\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 3-\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 3-\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 3-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda)^2(3-\lambda)^3.$$

(e) (4 points) Give an example of a one-to-one linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ together with another linear transformation $U\colon \mathbb{R}^m \to \mathbb{R}^n$ where $U \circ T\colon \mathbb{R}^n \to \mathbb{R}^n$ is the identity, i.e. $U(T(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

Solution: The simplest example is to let $U = T$ be the identity on $\mathbb{R}^1$. Another example is "inclusion" of $\mathbb{R}^1$ into $\mathbb{R}^2$ as the x-axis followed by "projection" of $\mathbb{R}^2$ onto the x-axis, namely

$$T\left(\begin{bmatrix} x \end{bmatrix}\right) := \begin{bmatrix} x \\ 0 \end{bmatrix}, \qquad U\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) := \begin{bmatrix} x \end{bmatrix}, \qquad \text{so} \qquad U \circ T\left(\begin{bmatrix} x \end{bmatrix}\right) = U\left(\begin{bmatrix} x \\ 0 \end{bmatrix}\right) = \begin{bmatrix} x \end{bmatrix}.$$

3. Prove each of the following statements. Hint: each part is independent of the others unless stated otherwise.

   (a) (2 points) Let A be a square matrix. Show that $A^3 - I = (A - I)(A^2 + A + I)$.

   Solution: We compute
   $$(A - I)(A^2 + A + I) = A^3 + A^2 + A - A^2 - A - I = A^3 - I.$$

   (b) (2 points) If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are (column) vectors, show that $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v}$.

   Solution: We compute
   $$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + \cdots + u_nv_n = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \mathbf{u}^T\mathbf{v}.$$

   (c) (5 points) Show that if λ is an eigenvalue of A^TA, then $\lambda \geq 0$. (Hint: if $\mathbf{v}$ is an eigenvector of A^TA with eigenvalue λ, show that $\lambda|\mathbf{v}|^2 = (A^TA\mathbf{v})\cdot\mathbf{v} = (A\mathbf{v})\cdot(A\mathbf{v}) \geq 0$ using (b) twice.)

   Solution: As the hint suggests, we compute
   $$\begin{aligned} \lambda|\mathbf{v}|^2 &= (\lambda\mathbf{v}) \cdot \mathbf{v} = (A^TA\mathbf{v}) \cdot \mathbf{v} \\ &= (A^TA\mathbf{v})^T\mathbf{v} = \mathbf{v}^TA^TA\mathbf{v} \\ &= (A\mathbf{v})^T(A\mathbf{v}) = (A\mathbf{v}) \cdot (A\mathbf{v}) \geq 0. \end{aligned}$$
   Hence $\lambda|\mathbf{v}|^2 \geq 0$, so since $\mathbf{v} \neq \mathbf{0}$, we have $\lambda \geq 0$.

(d) (2 points) Let A and P be $n \times n$ matrices with P invertible and let λ be a scalar. Show that $PAP^{-1} - \lambda I = P(A - \lambda I)P^{-1}$

Solution: We compute

$$P(A - \lambda I)P^{-1} = PAP^{-1} - \lambda PIP^{-1} = PAP^{-1} - \lambda I.$$

(e) (3 points) Let A and P be $n \times n$ matrices with P invertible. Use (d) to show that A and PAP^{-1} have the same characteristic polynomial.

Solution: Using (d), we see

$$\det(PAP^{-1} - \lambda I) = \det(P(A - \lambda I)P^{-1}) = \det(P)\det(A - \lambda I)\det(P)^{-1} = \det(A - \lambda I).$$

(f) (4 points) Suppose X and Y are square matrices which commute, meaning $XY = YX$. Show that if $\mathbf{u} \in \text{null}(X)$, then $Y\mathbf{u} \in \text{null}(X)$.

Solution: We compute

$$X(Y\mathbf{u}) = (XY)\mathbf{u} = (YX)\mathbf{u} = Y(X\mathbf{u}) = Y\mathbf{0} = \mathbf{0}.$$

4. Let

$$A := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) (4 points) Compute $\det(A)$. Is A invertible?

> **Solution:** The determinant of the product is the product of the determinants. The determinant of a triangular matrix is the product of the entries on the diagonal. So, the determinant is $(1^3)(1 \cdot -3 \cdot 0) = 0$. Since the determinant is zero, A is not invertible.

(b) (4 points) Let L be the lower triangular matrix above and let U be the upper triangular matrix, so $A = LU$. Show that $\text{null}(A) = \text{null}(U)$. (Hint: L is invertible.)

> **Solution:** Note that L is invertible (for instance, it has determinant 1). Then $A\mathbf{x} = \mathbf{0}$ iff $LU\mathbf{x} = \mathbf{0}$ iff $L^{-1}LU\mathbf{x} = L^{-1}\mathbf{0}$, which says $U\mathbf{x} = \mathbf{0}$.

(c) (4 points) Show that U is an echelon form of A. Find a basis for $\text{null}(A)$.

> **Solution:** Since $\text{null}(U) = \text{null}(A)$ and U and A have the same size, they have the same reduced echelon form by the proof examples document. Alternatively, it is easy to row reduce A to U.
>
> We can find a basis for $\text{null}(A)$ by finding a basis for $\text{null}(U)$. We compute
>
> $$U \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$
>
> so reading off the general solution gives
>
> $$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ -2s \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix},$$
>
> giving basis
>
> $$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

5. (a) (5 points) Let A be a 2×2 matrix where

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \qquad A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix}.$$

What is A?

Solution: We compute

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2} A \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) = \frac{1}{2} \left(\begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 + \sqrt{2} \\ 2 \end{bmatrix}.$$
$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2} A \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) = \frac{1}{2} \left(\begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 - \sqrt{2} \\ 2 \end{bmatrix}.$$

Hence

$$A = \begin{bmatrix} 1 + \sqrt{2} & 1 - \sqrt{2} \\ 2 & 2 \end{bmatrix}.$$

(b) (3 points) Find bases $\mathcal{B}$ and $\mathcal{C}$ of $\mathbb{R}^2$ such that the matrix A from (a) is the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$. (If you did not solve (a), you may replace A with your own 2×2 matrix.)

Solution: An invertible matrix can always be viewed as a change of basis matrix from the basis consisting of the columns of that matrix to the standard basis. A is clearly invertible.

(c) (4 points) Find the change of basis matrix from the basis

$$\left\{ \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix} \right\}$$

to the basis

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.$$

Solution: From class, this matrix is

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 2\sqrt{2} \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 3 & \sqrt{2} \\ -1 & \sqrt{2} \end{bmatrix}.$$

6. Let

$$A = \begin{bmatrix} 0 & 0 & -2 & -1 \\ 1 & 1 & 6 & 5 \\ 2 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) (4 points) Compute the characteristic polynomial of A directly. (Hint: the eigenvalues of A are 1 and 2.)

Solution: We expand $A - \lambda I$ along the second column and expand along the third row afterwards:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} -\lambda & 0 & -2 & -1 \\ 1 & 1-\lambda & 6 & 5 \\ 2 & 0 & 4-\lambda & 1 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix} \\ &= (1-\lambda) \det \begin{bmatrix} -\lambda & -2 & -1 \\ 2 & 4-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} \\ &= (1-\lambda)(1-\lambda)(-\lambda(4-\lambda)+4) \\ &= (1-\lambda)^2(\lambda^2 - 4\lambda + 4) = (1-\lambda)^2(2-\lambda)^2. \end{aligned}$$

(b) (5 points) Compute a basis for the eigenspace of 1.

Solution: We must compute $(A - I)$. We row reduce as usual:

$$A - I = \begin{bmatrix} -1 & 0 & -2 & -1 \\ 1 & 0 & 6 & 5 \\ 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which gives solutions

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s_2 \\ s_1 \\ -s_2 \\ s_2 \end{bmatrix} = s_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

so we have a basis

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

(c) (2 points) Let $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$. Compute $A^{100}\mathbf{x}$.

Solution: Clearly $\mathbf{x}$ is in the 1-eigenspace. Hence $A\mathbf{x} = \mathbf{x}$, so $A^{100}\mathbf{x} = \mathbf{x}$.

7. The following is a basis for $\mathbb{R}^3$:

$$\left\{ \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 5 \end{bmatrix} \right\}.$$

(a) (5 points) Find an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for $\mathbb{R}^3$ where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Solution: We apply Gram-Schmidt after reordering the above vectors; say they are $\mathbf{s}_2, \mathbf{s}_1, \mathbf{s}_3$, respectively. Now

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{s}_1 \\ \mathbf{v}_2 &= \mathbf{s}_2 - \operatorname{proj}_{\mathbf{v}_1} \mathbf{s}_2 \\ \mathbf{v}_3 &= \mathbf{s}_3 - \operatorname{proj}_{\mathbf{v}_1} \mathbf{s}_3 - \operatorname{proj}_{\mathbf{v}_2} \mathbf{s}_3 \end{aligned}$$

which works out to be

$$\begin{aligned} \mathbf{v}_1 &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ \mathbf{v}_2 &= \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} - \frac{6}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \\ \mathbf{v}_3 &= \begin{bmatrix} 3 \\ 5 \\ 5 \end{bmatrix} - \frac{8}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix}. \end{aligned}$$

(b) (3 points) Find an ortho**normal** basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for $\mathbb{R}^3$ where $\mathbf{u}_1 \cdot \mathbf{v}_1 = |\mathbf{u}_1||\mathbf{v}_1|$ where $\mathbf{v}_1$ is as in (a).

Solution: Simply divide each vector from (a) by its magnitude:

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{|\mathbf{v}_1|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{u}_2 = \frac{\mathbf{v}_2}{|\mathbf{v}_2|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad \mathbf{u}_3 = \frac{\mathbf{v}_3}{|\mathbf{v}_3|} = \frac{1}{\sqrt{24}} \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix}.$$

(c) (3 points) Let S be $\text{span}\{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_5\} \subset \mathbb{R}^5$ and let $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \\ 5 \end{bmatrix}$. Compute $\text{proj}_S \mathbf{y}$.

Solution: Using the formula from class (or geometric intuition) immediately gives

$$\text{proj}_S \mathbf{y} = \begin{bmatrix} 3 \\ 0 \\ 4 \\ 0 \\ 5 \end{bmatrix}.$$

(d) (4 points) Find a least squares solution $\widehat{\mathbf{x}}$ to the system $A\mathbf{x} = \mathbf{y}$ given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \\ 5 \end{bmatrix}.$$

Are there any other least squares solutions?

Solution: We can use either the original definition of least squares solutions or we can solve the normal equations in this case since we've computed $\widehat{\mathbf{y}}$.

The original definition: solve $A\mathbf{x} = \widehat{\mathbf{y}}$. The columns of A are linearly independent, so there is at most one solution, and $\widehat{\mathbf{y}} = 3\mathbf{e}_1 + 4\mathbf{e}_3 + 5\mathbf{e}_5$, giving a unique solution of

$$\widehat{\mathbf{x}} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}.$$

Normal equations: solve $A^T A\mathbf{x} = A^T\mathbf{y}$, so solve

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix},$$

which gives the same answer as before as well as uniqueness.