Your Name

Student ID #

- Do not open this exam until you are told to begin. You will have 50 minutes for the exam.
- Check that you have a complete exam. There are 4 questions for a total of 60 points.
- You are allowed to have one handwritten note sheet. Only basic non-graphing scientific calculators are allowed, though you should not need one.
- Cheating will result in a zero and be reported to the Dean's Academic Conduct Committee.
- **Show all your work**. Unless explicitly stated otherwise in a particular question, if there is no work supporting your answer, you will not receive credit for the problem. If you need more space to answer a question, continue on the back of the page, and indicate that you have done so.

Question	Points	Score
1	20	
2	16	
3	14	
4	10	
Total:	60	

1. Multiple choice and short answer. For these questions, you are **not required to show any work**.

   (a) (2 points) Two vectors are linearly dependent if and only if one is a scalar multiple of the other.

   ◯ True ◯ False

   (b) (2 points) If $V, W \subset \mathbb{R}^n$ and $\operatorname{span} V \subset \operatorname{span} W$, then $V \subset W$.

   ◯ True ◯ False

   (c) (2 points) One may choose α, β, γ so that there are exactly two quadratics $p(x) = ax^2 + bx + c$ whose graph passes through the points $(1, \alpha), (2, \beta), (3, \gamma)$.

   ◯ True ◯ False

   (d) (3 points) Check all that apply: a linearly dependent subset of $\mathbb{R}^n \ldots$

   ◯ cannot have precisely n vectors ◯ must have precisely n vectors
   ◯ must have fewer than n vectors ◯ can have more than n vectors

   (e) (3 points) In which of the following situations *can there be* **no solutions** to a linear system? (Check all that apply.)

   ◯ More variables than equations. ◯ More equations than variables.
   ◯ Homogneous system. ◯ Triangular system. ◯ Echelon system.

   (f) (4 points) Give an example of a pair of two consistent systems each with 2 equations in 5 variables but where no solution of one system is a solution of the other system.

   (g) (4 points) Give an example of a linear system whose solution set is contained in the span of a set of three vectors but where the solution set is **not** itself the span of some set of vectors.

2. Consider the following homogeneous linear system.

$$\begin{aligned} x_1 + x_2 + 7x_3 \qquad &= 0 \\ 3x_1 + x_2 + 15x_3 + 6x_4 &= 0 \\ 2x_2 + 6x_3 + 3x_4 &= 0 \end{aligned}$$

(a) (8 points) Solve this homogeneous system and write your answer in vector form.

(b) (2 points) Consider the non-homogeneous system

$$\begin{aligned} x_1 + x_2 + 7x_3 \qquad &= 9 \\ 3x_1 + x_2 + 15x_3 + 6x_4 &= 25 \\ 2x_2 + 6x_3 + 3x_4 &= 11 \end{aligned}$$

Verify that $x_1 = x_2 = x_3 = x_4 = 1$ is a solution to this system.

(c) (6 points) Show that every solution $\mathbf{x}$ of the non-homogeneous system from (b) is of the form

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \mathbf{u}$$

where $\mathbf{u}$ is a solution to the homogeneous system from (a).

3. You are given the following vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \\ 6 \\ 4 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$$

   (a) (10 points) Determine which of the four standard basis vectors $\mathbf{e}_i \in \mathbb{R}^4$ are in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. (Recall that $\mathbf{e}_i \in \mathbb{R}^4$ has 0's in each coordinate except the ith, where it is 1.)

   (b) (4 points) Exhibit a vector $\mathbf{v}_4 \in \mathbb{R}^4$ such that $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \mathbb{R}^4$.

4. (10 points) Let A be an $m \times n$ matrix. For this question, you may use the following facts freely:

   (i) $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

   (ii) $A(c\mathbf{x}) = c(A\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ and scalars c.

   (iii) $A\mathbf{0} = \mathbf{0}$.

   Answer only one of the following two questions. If you answer more than one part, your *worse* answer will be ignored. Circle the question you decide to answer.

   (a) Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are non-zero. Prove that if $A\mathbf{u} = \mathbf{u}$ and $A\mathbf{v} = 2\mathbf{v}$, then $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent.

   (b) Let A be a 2×2 matrix and suppose $\{\mathbf{u}, \mathbf{v}\} \subset \mathbb{R}^2$ is linearly independent. Prove that if $A\mathbf{u} = \mathbf{u}$ and $A\mathbf{v} = \mathbf{v}$, then

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$