Your Name

Student ID #

- Do not open this exam until you are told to begin. You will have 50 minutes for the exam.
- Check that you have a complete exam. There are 4 questions for a total of 60 points.
- You are allowed to have one handwritten note sheet. Only basic non-graphing scientific calculators are allowed, though you should not need one.
- Cheating will result in a zero and be reported to the Dean's Academic Conduct Committee.
- **Show all your work**. Unless explicitly stated otherwise in a particular question, if there is no work supporting your answer, you will not receive credit for the problem. If you need more space to answer a question, continue on the back of the page, and indicate that you have done so.

Question	Points	Score
1	20	
2	16	
3	14	
4	10	
Total:	60	

1. Multiple choice and short answer. For these questions, you are **not required to show any work**.

   (a) (2 points) Two vectors are linearly dependent if and only if one is a scalar multiple of the other.

   $\surd$ **True** ◯ False

   (b) (2 points) If $V, W \subset \mathbb{R}^n$ and $\operatorname{span} V \subset \operatorname{span} W$, then $V \subset W$.

   ◯ True $\surd$ **False**

   (c) (2 points) One may choose α, β, γ so that there are exactly two quadratics $p(x) = ax^2 + bx + c$ whose graph passes through the points $(1, \alpha), (2, \beta), (3, \gamma)$.

   ◯ True $\surd$ **False**

   (d) (3 points) Check all that apply: a linearly dependent subset of $\mathbb{R}^n$...

   ◯ cannot have precisely n vectors ◯ must have precisely n vectors
   ◯ must have fewer than n vectors $\surd$ **can have more than n vectors**

   (e) (3 points) In which of the following situations *can there be* **no solutions** to a linear system? (Check all that apply.)

   $\surd$ **More variables than equations.** $\surd$ **More equations than variables.**
   ◯ Homogneous system. ◯ Triangular system. ◯ Echelon system.

   (f) (4 points) Give an example of a pair of two consistent systems each with 2 equations in 5 variables but where no solution of one system is a solution of the other system.

   Solution: There are many examples, but one is

   $$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 0 \\ x_1 &= 0 \end{aligned} \qquad \begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 0 \\ x_1 &= 1 \end{aligned}$$

   Both are clearly consistent, yet their solutions have different x_1 values.

   (g) (4 points) Give an example of a linear system whose solution set is contained in the span of a set of three vectors but where the solution set is **not** itself the span of some set of vectors.

   Solution: If we restrict to systems with three variables, all solutions are contained in $\operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \mathbb{R}^3$ automatically. A solution set is not itself a span if and only if the system is homogeneous, so any non-homogeneous system in three variables works. One example is then

   $$x_1 + x_2 + x_3 = 1.$$

2. Consider the following homogeneous linear system.

$$\begin{aligned} x_1 + x_2 + 7x_3 \quad &= 0 \\ 3x_1 + x_2 + 15x_3 + 6x_4 &= 0 \\ 2x_2 + 6x_3 + 3x_4 &= 0 \end{aligned}$$

(a) (8 points) Solve this homogeneous system and write your answer in vector form.

Solution: The corresponding augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 7 & 0 & 0 \\ 3 & 1 & 15 & 6 & 0 \\ 0 & 2 & 6 & 3 & 0 \end{array}\right]$$

reduces to

$$\left[\begin{array}{cccc|c} 1 & 0 & 4 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

The only free variable is $x_3 = s_1$. Reading off the solution in vector form yields

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s_1 \begin{bmatrix} -4 \\ -3 \\ 1 \\ 0 \end{bmatrix}.$$

(b) (2 points) Consider the non-homogeneous system

$$\begin{aligned} x_1 + x_2 + 7x_3 \quad &= 9 \\ 3x_1 + x_2 + 15x_3 + 6x_4 &= 25 \\ 2x_2 + 6x_3 + 3x_4 &= 11 \end{aligned}$$

Verify that $x_1 = x_2 = x_3 = x_4 = 1$ is a solution to this system.

Solution: Plugging the variables in, we see that indeed $1+1+7 = 9$, $3+1+15+6 = 25$, $2 + 6 + 3 = 11$.

(c) (6 points) Show that every solution $\mathbf{x}$ of the non-homogeneous system from (b) is of the form

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \mathbf{u}$$

where $\mathbf{u}$ is a solution to the homogeneous system from (a).

Solution: There are several approaches. The brute force solution is to repeat the row reduction from (a) but using the rightmost column from (b) to get

$$\left[\begin{array}{cccc|c} 1 & 0 & 4 & 0 & 5 \\ 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right].$$

Using free parameter $x_3 = t_1$ yields solutions in vector form as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t_1 \begin{bmatrix} -4 \\ -3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \\ 0 \\ 1 \end{bmatrix}.$$

This is "shifted" from the suggested form. Letting $t_1 = s_1 + 1$ gives

$$\mathbf{x} = s_1 \begin{bmatrix} -4 \\ -3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -4 \\ -3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \\ 0 \\ 1 \end{bmatrix} = s_1 \begin{bmatrix} -4 \\ -3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

A shorter approach is to note that if $\mathbf{x}$ is a solution of the inhomogeneous system and $\mathbf{u}_p$ is the column vector of 1's, then subtracting $\mathbf{u}_p$ from $\mathbf{x}$ and plugging it into the left-hand side of the system yields the right-hand side of the inhomogeneous system minus itself, i.e. the zero vector. Hence the difference $\mathbf{x} - \mathbf{u}_p$ is a solution $\mathbf{u}$ of the homogeneous system. Hence $\mathbf{x} = \mathbf{u}_p + \mathbf{u}$.

Note: the question does *not* ask you to verify that vectors of the above form are solutions of the non-homogeneous system. It asks for the reverse, which is what the above arguments give.

3. You are given the following vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \\ 6 \\ 4 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 3 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$$

(a) (10 points) Determine which of the four standard basis vectors $\mathbf{e}_i \in \mathbb{R}^4$ are in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. (Recall that $\mathbf{e}_i \in \mathbb{R}^4$ has 0's in each coordinate except the ith, where it is 1.)

Solution: There are several approaches. One is to place the three vectors in the columns of a matrix together with a variable fourth column $\mathbf{b}$ and row reduce until the pivots are clear:

$$\begin{bmatrix} 1 & 2 & 0 & b_1 \\ 4 & 3 & 1 & b_2 \\ 6 & 5 & 2 & b_3 \\ 4 & 3 & 1 & b_4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & b_1 \\ 0 & -5 & 1 & b_2 - 4b_1 \\ 0 & -7 & 2 & b_3 - 6b_1 \\ 0 & 0 & 0 & b_4 - b_2 \end{bmatrix}.$$

There is already a pivot in row 1, column 1. Subtracting a multiple of the second row from the first will clearly result in pivots in row 2, column 2 and row 3, column 3. The system is consistent if and only if row reduction produces no row of the rough form $0 = 1$, which evidently occurs if and only if $b_4 - b_2 = 0$. Hence the span consists of all $\mathbf{b}$ whose second and fourth coordinates are equal. Of the four standard basis vectors in $\mathbb{R}^4$, only the first and third have this property.

A somewhat cleaner approach places the three vectors into the rows of a matrix and row reduces it to get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The row space is preserved by row reduction, so we can just consider the row span of this last matrix. Clearly $\mathbf{e}_1$ and $\mathbf{e}_3$ are in the row space (being the first and third rows). It's easy to see any linear combination of the rows of this matrix have the same second and fourth components, so neither $\mathbf{e}_2$ nor $\mathbf{e}_4$ can be in the span.

Note: placing the three vectors into a matrix as its columns and row reducing yields the matrix $[\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3]$. Since row operations preserve the row space but *not* the column space, this is irrelevant.

(b) (4 points) Exhibit a vector $\mathbf{v}_4 \in \mathbb{R}^4$ such that $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \mathbb{R}^4$.

Solution: Since $\mathbf{e}_1$ and $\mathbf{e}_3$ are in the span, using either of the remaining standard basis vectors, say $\mathbf{e}_2$, as $\mathbf{v}_4$ allows us to get vectors whose first three components are arbitrary and whose fourth component is zero. Since, say, $\mathbf{v}_1$ has a non-zero entry in its fourth component, we may zero out the other coordinates by subtracting off multiples of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, which leaves a multiple of $\mathbf{e}_4$ within the span. Since the span hence contains all four standard basis vectors, it's clearly all of $\mathbb{R}^4$.

More generally, a vector in $\mathbb{R}^4$ is a correct answer if and only if its second and fourth components are unequal.

4. (10 points) Let A be an $m \times n$ matrix. For this question, you may use the following facts freely:

   (i) $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

   (ii) $A(c\mathbf{x}) = c(A\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ and scalars c.

   (iii) $A\mathbf{0} = \mathbf{0}$.

   Answer only one of the following two questions. If you answer more than one part, your *worse* answer will be ignored. Circle the question you decide to answer.

   (a) Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are non-zero. Prove that if $A\mathbf{u} = \mathbf{u}$ and $A\mathbf{v} = 2\mathbf{v}$, then $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent.

   Solution: There are several solutions. One is to use the observation that two vectors are linearly dependent if and only if one is a scalar multiple of the other. If $\mathbf{u} = c\mathbf{v}$, then from (ii) we have

$$A\mathbf{u} = A(c\mathbf{v}) = c(A\mathbf{v}) = c(2\mathbf{v}) = 2\mathbf{u},$$

   But then $\mathbf{u} = A\mathbf{u} = 2\mathbf{u}$, so $\mathbf{u} = \mathbf{0}$, a contradiction. Essentially the same reasoning works if $\mathbf{v} = c\mathbf{u}$.

   Another solution is to suppose $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{0}$ and show that $c_1 = c_2 = 0$. For that, apply A to both sides of this equation and use the three listed properties to get

$$c_1 A\mathbf{u} + c_2 A\mathbf{v} = \mathbf{0},$$

   which says that $c_1\mathbf{u} + 2c_2\mathbf{v} = \mathbf{0}$. This gives a system of two (vector) equations in two unknowns:

$$\begin{aligned} c_1\mathbf{u} + c_2\mathbf{v} &= \mathbf{0} \\ c_1\mathbf{u} + 2c_2\mathbf{v} &= \mathbf{0} \end{aligned}$$

   Subtracting the first from the second gives $c_2\mathbf{v} = \mathbf{0}$, so since $\mathbf{v} \neq \mathbf{0}$, we have $c_2 = 0$. The first equation then forces $c_1 = 0$.

   (b) Let A be a 2×2 matrix and suppose $\{\mathbf{u}, \mathbf{v}\} \subset \mathbb{R}^2$ is linearly independent. Prove that if $A\mathbf{u} = \mathbf{u}$ and $A\mathbf{v} = \mathbf{v}$, then

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

   Solution: Using (i) and (ii) above gives

$$A(c_1\mathbf{u} + c_2\mathbf{v}) = c_1(A\mathbf{u}) + c_2(A\mathbf{v}) = c_1\mathbf{u} + c_2\mathbf{v}.$$

   Since $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent, by the big theorem it spans $\mathbb{R}^2$. In particular, every vector in $\mathbb{R}^2$ is of the form $c_1\mathbf{u} + c_2\mathbf{v}$, so by the above computation $A\mathbf{e}_i = \mathbf{e}_i$. Writing the columns of A as $A = [\mathbf{a}_1\ \mathbf{a}_2]$ and using the definition of matrix multiplication, $A\mathbf{e}_i = \mathbf{e}_i$ says precisely that $\mathbf{a}_1 = \mathbf{e}_1$ and $\mathbf{a}_2 = \mathbf{e}_2$, which is the desired conclusion.