Your Name

Student ID #

- Do not open this exam until you are told to begin. You will have 50 minutes for the exam.
- Check that you have a complete exam. There are 5 questions for a total of 54 points.
- You are allowed to have one handwritten note sheet. Only basic non-graphing scientific calculators are allowed, though you should not need one.
- Cheating will result in a zero and be reported to the Dean's Academic Conduct Committee.
- **Show all your work**. Unless explicitly stated otherwise in a particular question, if there is no work supporting your answer, you will not receive credit for the problem. If you need more space to answer a question, continue on the back of the page, and indicate that you have done so.

Question	Points	Score
1	18	
2	11	
3	8	
4	9	
5	8	
Total:	54	

1. Multiple choice and short answer. For these questions, you are **not required to show any work**.

(a) (2 points) Every subspace is the row space of some matrix.

√ **True** ◯ False

> **Solution:** Every subspace has a basis, and we can form a matrix whose rows are that basis.

(b) (2 points) If A and B are invertible $n \times n$ matrices, then $(A+B)^{-1} = A^{-1} + B^{-1}$.

◯ True √ **False**

> **Solution:** Take $A = I, B = -I$, so $A + B = 0$ is not invertible, yet $A^{-1} + B^{-1} = I + -I = 0$.

(c) (2 points) If A is $m \times n$, then $\text{nullity}(A) - \text{nullity}(A^T) = n - m$.

√ **True** ◯ False

> **Solution:** From rank-nullity applied to A, $\text{rank}(A) + \text{nullity}(A) = n$, and applied to A^T, $\text{rank}(A^T) + \text{nullity}(A^T) = m$. Note $\text{rank}(A) = \dim \text{col}(A) = \dim \text{row}(A^T) = \text{rank}(A^T)$. Subtracting these two equations now gives the suggested equation.

(d) (4 points) Let A, B be $n \times n$ matrices, let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, and let s, t be scalars. Which of the following are *always true*? (Check all that apply.)

◯ $A(s\mathbf{u} + t\mathbf{v}) = sA\mathbf{u} + tA\mathbf{v}$. ◯ $(AB)^2 = A^2B^2$.
◯ $(A+B)^2 = A^2 + 2AB + B^2$. ◯ $A\mathbf{0} = \mathbf{0}$.
◯ $A^2 = A$ implies $A(A - I) = 0$, so either $A = I$ or $A = 0$.

> **Solution:** The first is linearity; counterexamples to the second are easy to find; the left-hand side of the third is $A^2 + AB + BA + B^2$, so we need $AB = BA$, which the previous counterexample also works on; the fourth is the $s = t = 0, \mathbf{u} = \mathbf{v} = \mathbf{0}$ case of the first; counterexamples to the fifth are also easy to find.

(e) (4 points) Give an example of two subspaces S_1 and S_2 of $\mathbb{R}^4$ each of dimension 2 but where the only vector belonging to both S_1 and S_2 is $\mathbf{0}$.

> **Solution:** There are many examples, but the simplest is probably $S_1 = \text{span}\{\mathbf{e}_1, \mathbf{e}_2\}$, $S_2 = \text{span}\{\mathbf{e}_3, \mathbf{e}_4\}$.

(f) (4 points) Give an example of two linear functions $T \colon \mathbb{R}^n \to \mathbb{R}^m$ and $U \colon \mathbb{R}^m \to \mathbb{R}^\ell$ such that T is one-to-one, $\text{range}\, T = \ker U$, and U is onto. *Hint:* In your example, you'll find $m = n + \ell$.

> **Solution:** There are again many examples. A simple one with geometric motivation: let T map $\mathbb{R}^1$ to the x-axis in $\mathbb{R}^2$; let U project points in $\mathbb{R}^2$ onto the y-axis. The kernel of U is evidently the x-axis, which is the range of T, T is one-to-one, and U is onto. In coordinates, $T(x) = (x, 0), U(x, y) = y$. This is a "short exact sequence".

2. Let A be the following 3×5 matrix. Its reduced echelon form B is provided.

$$A = \begin{bmatrix} 1 & 1 & 7 & 0 & 0 \\ 3 & 1 & 15 & 6 & 0 \\ 0 & 2 & 6 & 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 & 0 & -3 \\ 0 & 1 & 3 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = B.$$

(a) (3 points) Compute $\operatorname{rank}(A)$, $\dim \operatorname{row}(A)$, $\dim \operatorname{col}(A)$, and $\operatorname{nullity}(A)$.

Solution: The first three are all the rank, which is the number of pivots in B, which is 3. The nullity is the number of columns without pivots in B, which is 2.

(b) (8 points) Find bases for $\operatorname{row}(A)$, $\operatorname{col}(A)$, and $\operatorname{null}(A)$.

Solution: A basis for $\operatorname{row}(A)$ is given by reading off the non-zero rows of B,

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 4 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

A basis for $\operatorname{col}(A)$ is given by reading off the columns of A in which a pivot appears in B,

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix} \right\}.$$

A basis for $\operatorname{null}(A)$ is given by reading off the vectors involved in the vector form of the general solution of $A\mathbf{x} = \mathbf{0}$. Here that general solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -4s_1 + 3s_2 \\ -3s_1 - 3s_2 \\ s_1 \\ -s_2 \\ s_2 \end{bmatrix} = s_1 \begin{bmatrix} -4 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 3 \\ -3 \\ 0 \\ -1 \\ 1 \end{bmatrix},$$

so a basis is given by

$$\left\{ \begin{bmatrix} -4 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

3. Let A be the following 3×3 matrix:

$$A = \begin{bmatrix} 1 & 4 & -3 \\ 1 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) (5 points) Compute A^{-1}.

Solution: We can do this by forming the augmented matrix $[A|I]$ and row reducing. The result (if A is invertible) will be $[I|A^{-1}]$. And indeed,

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 4 & -3 & 1 & 0 & 0 \\ 1 & 3 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 4 & -21 \\ 0 & 1 & 0 & 1 & -1 & 6 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = [I|A^{-1}].$$

(b) (3 points) Show that A^T is invertible, with $(A^T)^{-1} = (A^{-1})^T$.

Solution: We can do this by directly multiplying out $A^T(A^{-1})^T$, which will result in I. We can also repeat (a) with A^T. On the other hand, we can also show this is true abstractly in general with very little effort:

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I.$$

4. Fix a 2×2 matrix A. Let $Q\colon \mathbb{R}^2 \to \mathbb{R}$ be the function given by

$$Q(\mathbf{x}) := \mathbf{x}^T A\mathbf{x}.$$

(a) (3 points) If $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, show directly that

$$Q(\mathbf{x}) = ax^2 + 2bxy + cy^2.$$

Solution: We compute

$$\begin{aligned} Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} &= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} ax + by \\ bx + cy \end{bmatrix} \\ &= \begin{bmatrix} x(ax + by) + y(bx + cy) \end{bmatrix} \\ &= ax^2 + 2bxy + cy^2. \end{aligned}$$

(b) (3 points) Show that $Q(s\mathbf{x}) = s^2 Q(\mathbf{x})$ for all scalars s.

Solution: We compute

$$Q(s\mathbf{x}) = (s\mathbf{x})^T A(s\mathbf{x}) = s\mathbf{x}^T As\mathbf{x} = s^2\mathbf{x}^T A\mathbf{x} = s^2 Q(\mathbf{x}).$$

Note: if you deduce this from the formula in (a), you need to handle the case when A is not symmetric separately.

(c) (3 points) Find a matrix A and vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^2$ such that $Q(\mathbf{x}_1 + \mathbf{x}_2) \neq Q(\mathbf{x}_1) + Q(\mathbf{x}_2)$.

Solution: Using the formula from (a), we can choose $A = I$ to get $Q(\mathbf{x}) = x^2 + y^2$. This is not linear for many reasons, but an explicit one is that

$$Q(\mathbf{e}_1 + \mathbf{e}_1) = 2^2 = 4 \neq 2 = 1 + 1 = Q(\mathbf{e}_1) + Q(\mathbf{e}_1).$$

5. In this question, you are given a proof and are asked to provide justification for individual steps. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation throughout.

Proposition. *If T is onto, then there is a linear transformation $U\colon \mathbb{R}^m \to \mathbb{R}^n$ such that*

$$T(U(\mathbf{x})) = \mathbf{x} \qquad \text{for all } \mathbf{x} \in \mathbb{R}^m.$$

Proof. Pick $\mathbf{y}_1, \ldots, \mathbf{y}_m$ as in (a), so $T(\mathbf{y}_i) = \mathbf{e}_i$. Pick U as in (b), so $U(\mathbf{e}_i) = \mathbf{y}_i$. Then

$$T(U(\mathbf{e}_i)) = T(\mathbf{y}_i) = \mathbf{e}_i.$$

By (c), $T(U(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^m$. □

Hint: Parts (a)-(c) are independent of each other.

(a) (1 point) Show that if T is onto, then there are $\mathbf{y}_1, \ldots, \mathbf{y}_m \in \mathbb{R}^n$ such that $T(\mathbf{y}_i) = \mathbf{e}_i$.

Solution: Since T is onto, for each $\mathbf{e}_i$, there is some $\mathbf{y}_i$ such that $T(\mathbf{y}_i) = \mathbf{e}_i$ by definition.

(b) (3 points) Show that, given $\mathbf{y}_1, \ldots, \mathbf{y}_m$ in $\mathbb{R}^n$, there is some linear transformation $U\colon \mathbb{R}^m \to \mathbb{R}^n$ with

$$U(\mathbf{e}_i) = \mathbf{y}_i \qquad \text{for } i = 1, \ldots, m.$$

Solution: Let B be $n \times m$ with columns $\mathbf{y}_1, \ldots, \mathbf{y}_m$. Then $B\mathbf{e}_i = \mathbf{y}_i$, so $U(\mathbf{x}) := B\mathbf{x}$ has $U(\mathbf{e}_i) = \mathbf{y}_i$ and U is linear.

(c) (4 points) Show that if $U\colon \mathbb{R}^m \to \mathbb{R}^n$ is linear and $T(U(\mathbf{e}_i)) = \mathbf{e}_i$ for all $\mathbf{e}_i \in \mathbb{R}^m$, then $T(U(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^m$.

Solution: If A is the matrix of T and B is the matrix of U, then we have $AB\mathbf{e}_i = \mathbf{e}_i$, so the ith column of AB is $\mathbf{e}_i$, and AB is $m \times m$, so $AB = I$. Hence $T(U(\mathbf{x})) = I\mathbf{x} = \mathbf{x}$. Alternatively, for each $\mathbf{x}$ there are constants for which $\mathbf{x} = c_1\mathbf{e}_1 + \ldots + c_m\mathbf{e}_m$. Then

$$\begin{aligned} T(U(\mathbf{x})) &= T(U(c_1\mathbf{e}_1 + \cdots + c_m\mathbf{e}_m)) \\ &= T(c_1U(\mathbf{e}_1) + \cdots + c_mU(\mathbf{e}_m)) \\ &= c_1T(U(\mathbf{e}_1)) + \cdots + c_mT(U(\mathbf{e}_m)) \\ &= c_1\mathbf{e}_1 + \cdots + c_m\mathbf{e}_m \\ &= \mathbf{x}. \end{aligned}$$