1. (20 points) Short answer questions. *No explanation of answers needed for this problem only. Be sure to explain your answers and show your work on all other problems!*

(a) True or False: If A, B, and X are invertible matrices such that $XA = B$, then $X = A^{-1}B$.

False. $X = BA^{-1}$

(b) True or False: If A is a square matrix whose columns add up to the zero vector, then A is invertible.

False. this means the columns are L.D. so A has no inverse.

(c) True or False: If A is an invertible matrix, then A and A^T have the same null space.

True. A invertible means $\text{null}(A) = \vec{0}$ and A^T invertible means $\text{null}(A^T) = \vec{0}$.

(d) True or False: If A is a 3×5 matrix such that $(\text{row}(A))^\perp = \mathbb{R}^5$, then A must be the zero matrix.

True. $\mathbb{R}^5 = (\text{row}(A))^\perp$ $= \text{null}(A)$ $\Rightarrow A = 0$.

(e) True or False: If an $n \times n$ matrix has n distinct eigenvalues, then it must be diagonalizable.

True. main theorem about diagonalization

(f) True or False: The transformation $T : \mathbb{R}^5 \to \mathbb{R}^5$ defined by $T(\vec{x}) = -\vec{x}$ is a linear transformation.

True. check properties.

(g) True or false: The transformation $T : \mathbb{R}^4 \to \mathbb{R}^1$ defined by $T(\vec{x}) = \|\vec{x}\|$ is a linear transformation.

False. $T(-2\bar{x}) = \|-2\bar{x}\| = 2\|\bar{x}\| \neq -2T(x)$

(h) If A is a 3×5 matrix, what are the possible values of nullity(A)?

rank + nullity = 3 so nullity = 0, 1, 2, 3

(i) Let $\vec{a_1}$, $\vec{a_2}$, $\vec{a_3}$ be linearly independent vectors in $\mathbb{R}^7$, and let $A = [\vec{a_1}\ \vec{a_2}\ \vec{a_3}]$. What are the possible values for the rank of A?

rank A = 3 ($\bar{a}_1, \bar{a}_2, \bar{a}_3$ L.I.).

(j) Let S be a subspace of $\mathbb{R}^n$, and let $\vec{v}$ and $\vec{u}$ be vectors in $\mathbb{R}^n$. If $\text{proj}_S\vec{v} = \vec{u}$, what is $\text{proj}_S\vec{u}$?

$\bar{u}$ is in S so $\text{proj}_S \bar{u} = \bar{u}$.

2. (20=6+2+8+4 points) Let $A = \begin{bmatrix} -5 & -6 & 3 \\ 3 & 4 & -3 \\ 0 & 0 & -2 \end{bmatrix}$.

(a) Compute the characteristic polynomial of A.

$$\det(A-\lambda I) = \det\begin{bmatrix} -5-\lambda & -6 & 3 \\ 3 & 4-\lambda & -3 \\ 0 & 0 & -2-\lambda \end{bmatrix}$$

$$= (-2-\lambda)((-5-\lambda)(4-\lambda)+18)$$
$$= (-2-\lambda)(\lambda+2)(\lambda-1)$$

(b) Find all of the eigenvalues of A.

eigenvalues: $\lambda = -2, 1 \leftarrow$ mult. 1

$\uparrow$ mult. 2

(c) Find a basis for each of the eigenspaces of A.

$\lambda = -2$: $A + 2I = \begin{bmatrix} -3 & -6 & 3 \\ 3 & 6 & -3 \\ 0 & 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

eigenspace: $\left\{ s_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

basis: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\lambda = 1$: $A - I = \begin{bmatrix} -6 & -6 & 3 \\ 3 & 3 & -3 \\ 0 & 0 & -3 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

eigenspace: $\left\{ s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$, basis: $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

(d) Is A diagonalizable? If yes, find P and D; if not, explain why.

Yes, $P = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

3. (9=4+5 points) Let $\vec{x} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ and $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

$\bar{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \bar{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(a) Find the coordinate vector $[\vec{x}]_{\mathcal{B}}$ of $\vec{x}$ with respect to $\mathcal{B}$.

$U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\bar{x}_{\mathcal{B}} = U^{-1}\bar{x}$

$U^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$\bar{x}_{\mathcal{B}} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \boxed{\begin{bmatrix} 7 \\ -4 \end{bmatrix}_{\mathcal{B}}}$

(b) Assume that $\mathcal{C}$ is another basis for $\mathbb{R}^2$, and that the change of basis matrix from $\mathcal{C}$ to $\mathcal{B}$ is $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$. Find the coordinate vector $[\vec{x}]_{\mathcal{C}}$ of $\vec{x}$ with respect to $\mathcal{C}$.

this says $\bar{x}_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \bar{x}_C$

so $\bar{x}_C = \left(\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\right)^{-1} \bar{x}_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ -4 \end{bmatrix} = \boxed{\begin{bmatrix} 7 \\ 3 \end{bmatrix}_C}$

4. (6 points) Find a basis for $W^{\perp}$ if

$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

$\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \bar{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}$

$W^{\perp} = \{ \bar{u}$ such that $\bar{u} \cdot \bar{v}_1 = 0$ and $\bar{u} \cdot \bar{v}_2 = 0 \}$

$\bar{u} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$\bar{u} \cdot \bar{v}_1 = x_1 + x_2 + x_3 + x_4 = 0$

$\bar{u} \cdot \bar{v}_2 = x_1 - x_2 + x_3 + 2x_4 = 0$

$W^{\perp}$: $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 2 & 0 \end{array}\right] \longmapsto \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -2 & 0 & 1 & 0 \end{array}\right]$

$\longmapsto \left[\begin{array}{cccc|c} 1 & 0 & 1 & 3/2 & 0 \\ 0 & 1 & 0 & -1/2 & 0 \end{array}\right] \Rightarrow x_3, x_4$ free

$W^{\perp} = \left\{ s_1 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} -3/2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\boxed{\text{basis: } \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3/2 \\ 1/2 \\ 0 \\ 1 \end{bmatrix} \right\}}$

5. (5 points) Find the coordinate vector $[\vec{v}]_{\mathcal{B}}$ of $\vec{v} = \begin{bmatrix} 7 \\ -3 \\ 2 \end{bmatrix}$ with respect to *orthogonal* basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\} \text{ of } \mathbb{R}^3.$$

$\bar{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \bar{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \bar{u}_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

orthogonal basis $\Rightarrow$

$\bar{v} = c_1\bar{u}_1 + c_2\bar{u}_2 + c_3\bar{u}_3$,

$c_1 = \dfrac{\bar{v}\cdot\bar{u}_1}{\|\bar{u}_1\|^2} = \dfrac{9}{2}$ $\qquad c_3 = \dfrac{\bar{v}\cdot\bar{u}_3}{\|\bar{u}_3\|^2} = \dfrac{-11}{6}$

$c_2 = \dfrac{\bar{v}\cdot\bar{u}_2}{\|\bar{u}_2\|^2} = \dfrac{2}{3}$

$$\boxed{\bar{v}_{\mathcal{B}} = \begin{bmatrix} 9/2 \\ 2/3 \\ -11/6 \end{bmatrix}_{\mathcal{B}}}$$

6. (5 points) Compute $\det A^5$ if $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

$\cancel{de}$ $A^5 = (PDP^{-1})^5 = PD^5P^{-1}$

$\det(A^5) = \det(PD^5P^{-1}) = \det(D^5)$

$D^5 = \begin{bmatrix} 3^5 & 0 & 0 \\ 0 & 2^5 & 0 \\ 0 & 0 & (-2)^5 \end{bmatrix}$ $\qquad \boxed{\det(D^5) = (3^5 \cdot 2^5 \cdot (-2)^5)}$

7. (5 points) Show that if λ is an eigenvalue of an $n \times n$ matrix C, then λ^3 is an eigenvalue of C^3.

If λ is an eval of C,

$\det(C - \lambda I) = 0$.

Want to show $\det(C^3 - \lambda^3 I) = 0$

But $C^3 - \lambda^3 I = (C - \lambda I)(C^2 + \lambda C + \lambda^2 I)$

so $\det(C^3 - \lambda^3 I) = \underbrace{\det(C - \lambda I)}_{= 0} \det(C^2 + \lambda C + \lambda^2 I)$

so $\det(C^3 - \lambda^3 I) = 0$ so λ^3 is an eval. of C^3.

8. (5 points) Explain why a square matrix that has two equal rows, must have 0 as one of its eigenvalues.

If a square matrix A has two equal rows, $\det(A) = 0$.

But, $0 = \det(A) = \det(A - 0 \cdot I)$

so 0 is an eval. of A.

9. (5 points) Let A and B be 3×4 matrices. Show that $W = \{\vec{x} \in \mathbb{R}^4 : A\vec{x} = B\vec{x}\}$ is a subspace of $\mathbb{R}^4$.

to be in W, must satisfy this condition.

(1) $\bar{0}$ in W?

Yes: $A \cdot \bar{0} = \bar{0} = B \cdot \bar{0}$

(2) If $\bar{u}, \bar{v}$ in W, is $\bar{u} + \bar{v}$?

Yes: If $A\bar{u} = B\bar{u}$ and $A\bar{v} = B\bar{v}$,

then

$$\begin{aligned} A\bar{u} + A\bar{v} &= B\bar{u} + B\bar{v} \\ \| \quad & \quad \| \\ A(\bar{u} + \bar{v}) &= B(\bar{u} + \bar{v}) \end{aligned}$$

so $\bar{u} + \bar{v}$ is in W

(3). If $\bar{u}$ in W, is $r\bar{u}$?

Yes: If $A\bar{u} = B\bar{u}$,

then

$$\begin{aligned} rA\bar{u} &= rB\bar{u} \\ \| \quad & \quad \| \\ A(r\bar{u}) &= B(r\bar{u}) \end{aligned}$$

so $r\bar{u}$ is in W.