Your Name

Student ID #

- Do not open this exam until you are told to begin. You will have 50 minutes for the exam.
- Check that you have a complete exam. There are 4 questions for a total of 50 points.
- You are allowed to have one single sided, handwritten note sheet. Calculators are not allowed.
- Cheating will result in a zero and be reported to the Dean's Academic Conduct Committee.
- **Show all your work**. With the exception of True/False questions, if there is no work supporting your answer, you will not receive credit for the problem. If you need more space to answer a question, continue on the back of the page, and indicate that you have done so.

| Question | Points | Score |
|---|---|---|
| 1 | 18 | |
| 2 | 8 | |
| 3 | 12 | |
| 4 | 12 | |
| Total: | 50 | |

1. (18 points) True/False and short answer. For these questions, you are not required to show any work.

   (a) A system of equations with more variables than equations **cannot** have a unique solution.

   ◯ True ◯ False

   (b) If $m < n$, a set of $m$ vectors in $\mathbb{R}^n$ **cannot** span $\mathbb{R}^n$.

   ◯ True ◯ False

   (c) If $m < n$, any set of $m$ vectors in $\mathbb{R}^n$ is linearly independent.

   ◯ True ◯ False

   (d) If the equation $A\mathbf{x} = \mathbf{0}$ has a unique solution, then the columns of $A$ are linearly independent.

   ◯ True ◯ False

   (e) If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly dependent, then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly dependent (where all vectors have the same dimension).

   ◯ True ◯ False

   (f) If $\mathbf{u}_4$ is **not** a linear combination of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly independent (where all vectors have the same dimension).

   ◯ True ◯ False

   (g) Give an example of a linear system with more variables than equations that has no solution.

   (h) Give an example of three distinct nonzero vectors in $\mathbb{R}^2$ that don't span $\mathbb{R}^2$.

   (i) Give an example of two vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ in $\mathbb{R}^3$ such that $\text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$ is the set of all vectors $\mathbf{v}$ of the form $\mathbf{v} = \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix}$.

2. Consider the linear system

$$\begin{aligned} x_1 \qquad - 2x_3 + x_4 &= 0 \\ x_1 + x_2 - 2x_3 \qquad &= 0 \\ x_1 - 2x_2 - 2x_3 + 3x_4 &= 0 \end{aligned}$$

   (a) (5 points) Solve the system and write your answer in vector form.

   (b) (3 points) Find vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ such that the solution set is given by $\text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$. (Hint: your answer to (a) may be helpful).

3. Consider the vector equation

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -4 \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ b \end{bmatrix}$$

(a) (4 points) For what values of $a$ and $b$ does the equation have no solution?

(b) (4 points) For what values of $a$ and $b$ does the equation have infinitely many solutions?

(c) (4 points) Give an example of $a$ and $b$ where the equation has exactly one solution, and solve for $(x_1, x_2, x_3)$ in that case.

4. Consider the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}$$

   (a) (4 points) Show that the vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ do NOT span $\mathbb{R}^3$ and give an example of a vector $\mathbf{v}$ that is not in their span.

   (b) (4 points) Are the vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}\}$ linearly independent? (Note $\mathbf{v}$ should be the vector you found in part (a).) Justify your answer.

   (c) (4 points) Do the vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}\}$ span $\mathbb{R}^3$? Justify your answer.