Math 308 H **Midterm 1** **Winter 2015**

Your Name

Student ID #

- Do not open this exam until you are told to begin. You will have 50 minutes for the exam.
- Check that you have a complete exam. There are 4 questions for a total of 50 points.
- You are allowed to have one single sided, handwritten note sheet. Calculators are not allowed.
- Cheating will result in a zero and be reported to the Dean's Academic Conduct Committee.
- **Show all your work**. With the exception of True/False questions, if there is no work supporting your answer, you will not receive credit for the problem. If you need more space to answer a question, continue on the back of the page, and indicate that you have done so.

Question	Points	Score
1	18	
2	8	
3	12	
4	12	
Total:	50	

1. (18 points) True/False and short answer. For these questions, you are not required to show any work.

   (a) A system of equations with more variables than equations **cannot** have a unique solution.

   √ **True** ○ False

   (b) If $m < n$, a set of m vectors in $\mathbb{R}^n$ **cannot** span $\mathbb{R}^n$.

   √ **True** ○ False

   (c) If $m < n$, any set of m vectors in $\mathbb{R}^n$ is linearly independent.

   ○ True √ **False**

   (d) If the equation $A\mathbf{x} = \mathbf{0}$ has a unique solution, then the columns of A are linearly independent.

   √ **True** ○ False

   (e) If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly dependent, then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly dependent (where all vectors have the same dimension).

   √ **True** ○ False

   (f) If $\mathbf{u}_4$ is **not** a linear combination of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly independent (where all vectors have the same dimension).

   ○ True √ **False**

   (g) Give an example of a linear system with more variables than equations that has no solution.

   Solution: One such example is the system

$$x_1 + x_2 + x_3 = 0$$

$$x_1 + x_2 + x_3 = 1$$

   (h) Give an example of three distinct nonzero vectors in $\mathbb{R}^2$ that don't span $\mathbb{R}^2$.

   Solution: One such example is

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right\}$$

   (i) Give an example of two vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ in $\mathbb{R}^3$ such that span$\{\mathbf{u}_1, \mathbf{u}_2\}$ is the set of all vectors $\mathbf{v}$ of the form $\mathbf{v} = \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix}$.

   Solution: One example is

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

2. Consider the linear system

$$\begin{aligned} x_1 \qquad - 2x_3 + x_4 &= 0 \\ x_1 + x_2 - 2x_3 \qquad &= 0 \\ x_1 - 2x_2 - 2x_3 + 3x_4 &= 0 \end{aligned}$$

(a) (5 points) Solve the system and write your answer in vector form.

Solution: The matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 \\ 1 & -2 & -2 & 3 & 0 \end{array}\right]$$

reduces to

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The variables x_3 and x_4 are free, so assigning the free parameters $x_3 = s_1$ and $x_4 = s_2$, the solution is

$$\begin{aligned} x_1 &= 2s_1 - s_2 \\ x_2 &= s_2 \\ x_3 &= s_1 \\ x_4 &= s_2 \end{aligned}$$

where s_1 and s_2 are any real numbers. In vector form, the solutions are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s_1 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

where s_1 and s_2 are any real numbers.

(b) (3 points) Find vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ such that the solution set is given by $\text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$. (Hint: your answer to (a) may be helpful).

Solution: The span of a set of vectors is all linear combinations of those vectors, and our answer to (a) tells us that every solution can be written as a sum of

$$\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

so we can set

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

3. Consider the vector equation

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ -4 \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ b \end{bmatrix}$$

(a) (4 points) For what values of a and b does the equation have no solution?

Solution: To solve the equation, we reduce

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 1 & 1 & -4 & 2 \\ 2 & -1 & a & b \end{array}\right]$$

to get

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & a+5 & b-1 \end{array}\right].$$

The only time this has no solution is when $a + 5 = 0$ and $b - 1 \neq 0$, or $a = -5$ and $b \neq 1$ (so we get a row of zeros with a nonzero entry in the last column).

(b) (4 points) For what values of a and b does the equation have infinitely many solutions?

Solution: Using the reduced matrix from part (a), this has infinitely many solutions whenever the bottom row is entirely zero, so $a = -5$ and $b = 1$.

(c) (4 points) Give an example of a and b where the equation has exactly one solution, and solve for (x_1, x_2, x_3) in that case.

Solution: One example: $a = -4$ and $b = 1$. Then, the matrix becomes

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

which reduces to

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

so $(x_1, x_2, x_3) = (1, 1, 0)$.

4. Consider the vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}$$

(a) (4 points) Show that the vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ do NOT span $\mathbb{R}^3$ and give an example of a vector $\mathbf{v}$ that is not in their span.

Solution: Begin by reducing

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & a \\ -1 & 2 & 3 & b \\ -2 & 2 & 4 & c \end{array}\right]$$

to

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 2a+b \\ 0 & 1 & 1 & a+b \\ 0 & 0 & 0 & 2a+c \end{array}\right].$$

This tells us the only vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ that are in $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ are ones such that $2a + c = 0$. To find a vector not in the span, we can choose any $\mathbf{v}$ that does not satisfy this, such as

$$\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

(b) (4 points) Are the vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}\}$ linearly independent? (Note $\mathbf{v}$ should be the vector you found in part (a).) Justify your answer.

Solution: These are not linearly independent because $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is not a linearly independent set, which we know because the Big Theorem tells us $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ spans $\mathbb{R}^3$ if and only if $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly independent. (Note: we could also show this by reducing a matrix.)

(c) (4 points) Do the vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}\}$ span $\mathbb{R}^3$? Justify your answer.

Solution: These vectors do span $\mathbb{R}^3$. We could show this by reducing a matrix or noting that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is linearly independent because $\mathbf{u}_1$ and $\mathbf{u}_2$ are not multiples of each other, and $\mathbf{v}$ is not a linear combination of $\{\mathbf{u}_1, \mathbf{u}_2\}$ (which we know from part (a)). Therefore, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}\}$ is linearly independent, so by the Big Theorem, spans $\mathbb{R}^3$. Therefore, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}\}$ spans $\mathbb{R}^3$.