1. (20 points) Indicate whether the statement is true (T) or false (F). Circle your response.

(a) A homogeneous linear system may have exactly one non-trivial solution.

If it has one non-trivial solution, it must have infinitely many.

ANSWER: (circle one) T **F**

(b) A linear system with more variables than equations must have infinitely many solutions.

~~You must have free variables in this case.~~ It may have no solutions.

ANSWER: (circle one) ~~T~~ **F**

(c) A set of two vectors is linearly dependent if and only if one is a scalar multiple of the other. $\{\bar{u}_1, \bar{u}_2\}$ is linearly dependent if and only if there are nontrivial solutions $c_1\bar{u}_1 + c_2\bar{u}_2 = \bar{0}$, if and only if c_1 or $c_2 \neq 0$, if and only if $\bar{u}_1 = -c_2/c_1\,\bar{u}_2$ or $\bar{u}_2 = -c_1/c_2\,\bar{u}_1$.

ANSWER: (circle one) **T** F

(d) The following matrix is in reduced echelon form.

see definition of REF.

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right]$$

ANSWER: (circle one) **T** F

(e) If A is a matrix with more columns than rows, then the columns of A form a linearly independent set.

example: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

ANSWER: (circle one) T **F**

(f) Every vector in $span\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is in $span\{\mathbf{u}_1, \mathbf{u}_2\}$.

If $\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ are linearly independent, $\bar{u}_3$ is not in $span\{\bar{u}_1, \bar{u}_2\}$

ANSWER: (circle one) T **F**

(g) If $\mathbf{v}$ is a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$, then $\mathbf{v}$ is also a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$. If $\bar{v} = a_1\bar{u}_1 + a_2\bar{u}_2$,
$\bar{v} = a_1\bar{u}_1 + a_2\bar{u}_2 + 0\cdot\bar{u}_3$

ANSWER: (circle one) **T** F

(h) If $\{\mathbf{u}_1, \mathbf{u}_2\}$ is linearly independent and $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly dependent, then $\mathbf{u}_3$ is in $span\{\mathbf{u}_1, \mathbf{u}_2\}$. linearly dependent means there are c_1, c_2, c_3 not all zero such that $c_1\bar{u}_1 + c_2\bar{u}_2 + c_3\bar{u}_3 = \bar{0}$. Note $c_3 \neq 0$ because $\{\bar{u}_1, \bar{u}_2\}$ is linearly independent, so $\bar{u}_3 = -c_1/c_3\,\bar{u}_1 - c_2/c_3\,\bar{u}_2$.

ANSWER: (circle one) **T** F

(i) If $m > n$, any set of m vectors in $\mathbb{R}^n$ spans $\mathbb{R}^n$.

example: $m = 3, n = 2$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

ANSWER: (circle one) T **F**

(j) Any linearly independent set of n vectors in $\mathbb{R}^n$ spans $\mathbb{R}^n$.

This is part of the big Theorem.

ANSWER: (circle one) **T** F

2. (10 points) Use any valid method to solve the system. Place your final answer in the blank below.

$$\begin{aligned} 2x_1 + x_2 + x_3 + \quad x_4 &= 2 \\ -x_1 \qquad - 2x_3 + \quad x_4 &= 5 \\ x_2 + 4x_3 + 17x_4 &= 5 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 2 & 1 & 1 & 1 & 2 \\ -1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 4 & 17 & 5 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} -1 & 0 & -2 & 1 & 5 \\ 2 & 1 & 1 & 1 & 2 \\ 0 & 1 & 4 & 17 & 5 \end{array}\right]$$

$$\xrightarrow{-R_1} \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & -5 \\ 2 & 1 & 1 & 1 & 2 \\ 0 & 1 & 4 & 17 & 5 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & -5 \\ 0 & 1 & -3 & 3 & 12 \\ 0 & 1 & 4 & 17 & 5 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & -5 \\ 0 & 1 & -3 & 3 & 12 \\ 0 & 0 & 7 & 14 & -7 \end{array}\right] \xrightarrow{\frac{1}{7}R_3} \left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & -5 \\ 0 & 1 & -3 & 3 & 12 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right]$$

$$\underset{R_2 + 3R_3}{\xrightarrow{R_1 - 2R_3}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -5 & -3 \\ 0 & 1 & 0 & 9 & 9 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right] \Bigg\} \Rightarrow x_4 \text{ is free, assign free parameter } x_4 = s$$

$$\begin{aligned} x_1 &= -3 + 5s \\ x_2 &= 9 - 9s \\ x_3 &= -1 - 2s \\ x_4 &= s, \end{aligned}$$

s any real number

ANSWER: $(x_1, x_2, x_3, x_4) =$ $(-3+5s,\ 9-9s,\ -1-2s,\ s)$

s any real number.

3. (10 points)

(a) Find all values of h so that the columns of the following matrix span $\mathbb{R}^2$.

$$\begin{bmatrix} 4 & 10 \\ 3 & h \end{bmatrix}$$

want: $\begin{bmatrix} 4 & 10 \\ 3 & h \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ is consistent for all $\begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbb{R}^2$

$$\begin{bmatrix} 4 & 10 \\ 3 & h \end{bmatrix} \xrightarrow{\frac{1}{4}R_1} \begin{bmatrix} 1 & 5/2 \\ 3 & h \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & 5/2 \\ 0 & h - 15/2 \end{bmatrix}$$

This is consistent whenever $h - 15/2 \neq 0$, or $h \neq \frac{15}{2}$

Ans: whenever $h \neq \frac{15}{2}$

(b) Find all values of k so that the following linear system has infinitely many solutions.

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_1 + 4x_2 - x_3 &= k \\ 2x_1 - x_2 + 4x_3 &= k^2 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 4 & -1 & k \\ 2 & -1 & 4 & k^2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 3 & -2 & k-2 \\ 0 & -3 & 2 & k^2-4 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 3 & -2 & k-2 \\ 0 & 0 & 0 & k^2+k-6 \end{array}\right]$$

This has <u>no</u> <u>solutions</u> whenever $k^2+k-6 \neq 0$

and has <u>infinitely</u> many whenever $k^2+k-6=0$,

or $k^2+k-6 = (k+3)(k-2) = 0$

$k = -3$, $k = 2$.

Ans: $k = -3, 2$.

4. (10 points) Let $\mathbf{u}_1 = \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 8 \\ 5 \\ -3 \end{bmatrix}$.

(a) Let $\mathbf{v} = \begin{bmatrix} 26 \\ 17 \\ -8 \end{bmatrix}$. Write $\mathbf{v}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

Goal: solve

$$x_1 \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 8 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 26 \\ 17 \\ -8 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 4 & 8 & 26 \\ 4 & 5 & 17 \\ 2 & -3 & -8 \end{array}\right] \xrightarrow{1/2 R_1} \left[\begin{array}{cc|c} 2 & 4 & 13 \\ 4 & 5 & 17 \\ 2 & -3 & -8 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left[\begin{array}{cc|c} 2 & 4 & 13 \\ 0 & -3 & -9 \\ 0 & -7 & -21 \end{array}\right]$$

$$\xrightarrow[-1/7 R_3]{-1/3 R_2} \left[\begin{array}{cc|c} 2 & 4 & 13 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{array}\right] \xrightarrow[R_1 - 4R_2]{R_3 - R_2} \left[\begin{array}{cc|c} 2 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} x_1 = \frac{1}{2} \\ x_2 = 3 \end{array}$$

check: $\frac{1}{2} \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 8 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 26 \\ 17 \\ -8 \end{bmatrix}$

(b) Give an example of a vector $\mathbf{w}$ that is not in $span\{\mathbf{u}_1, \mathbf{u}_2\}$. You must show some work to justify your answer.

$span\{u_1, u_2\}$ = all vecs $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ where $x_1 \bar{u}_1 + x_2 \bar{u}_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is consistent.

$$\left[\begin{array}{cc|c} 4 & 8 & \cancel{26}\ a \\ 4 & 5 & \cancel{17}\ b \\ 2 & -3 & \cancel{-8}\ c \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 2 & 4 & a/2 \\ 4 & 5 & b \\ 2 & -3 & c \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 2 & 4 & a/2 \\ 0 & -3 & b-a \\ 0 & -7 & c - a/2 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|c} 2 & 4 & a/2 \\ 0 & 1 & -\frac{b}{3} + \frac{a}{3} \\ 0 & 1 & -\frac{c}{7} + \frac{a}{14} \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 2 & 4 & a/2 \\ 0 & 1 & -b/3 + a/3 \\ 0 & 0 & \underbrace{-\frac{c}{7} + \frac{a}{14} + \frac{b}{3} - \frac{a}{3}}_{\text{only consistent when this is } 0} \end{array}\right]$$

choose a vector such that this is nonzero, i.e. $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not in $span\{\bar{u}_1, \bar{u}_2\}$