Your Name

Student ID #

- Do not open this exam until you are told to begin.
- You will have until 11:20 to complete this exam; there are 6 questions for a total of 48 points.
- Graphing calculators and your textbook are not allowed. You may have one page of notes, double sided and handwritten, but this page of notes may not contain any proofs. Make sure your notes have your name on them and turn them in with your test.
- **Show your work**.
- Answer in the spaces provided; if you run out of room for an answer, continue on the back of the page and indicate that you have done so.
- Ambiguous or otherwise unreadable answers will be marked incorrect. So write clearly, erase fully, and provide only one answer to each question.
- This exam has 7 pages, plus this cover sheet. Please make sure that your exam is complete.

Question	Points	Score
1	8	
2	8	
3	8	
4	8	
5	8	
6	8	
Total:	48	

1. (8 points) Select true or false for each of the following questions. There is no need to show any work.

   (a) A system of equations with more equations than variables has no solutions.
   ○ True $\surd$ **False**

   (b) If A is an $n \times n$ matrix such that A^{-1} exists, then A and A^{-1} are row equivalent.
   $\surd$ **True** ○ False

   (c) if A is an $n \times n$ matrix and $\mathcal{N}(A) = \overline{0}$, then A is nonsingular.
   $\surd$ **True** ○ False

   (d) If $S = \{\overline{v}_1, \ldots, \overline{v}_p\}$ is a set of vectors such that one vector can be written as a linear combination of the remaining vectors in S, then S is linearly dependent.
   $\surd$ **True** ○ False

   (e) If A is any matrix whose columns are linearly independent, then A has an inverse.
   ○ True $\surd$ **False**

   (f) If W is a subspace of $\mathbb{R}^n$, and $B = \{\overline{v}_1, \ldots, \overline{v}_n\}$ is a basis for $\mathbb{R}^n$, then some subset of the vectors in B form a basis for W.
   ○ True $\surd$ **False**

   (g) If $S = \{\overline{v}_1, \overline{v}_2, \overline{v}_3\}$ is a linearly independent set of vectors in $\mathbb{R}^3$, then any vector v in $\mathbb{R}^3$ can be written as a linear combination of the vectors in S.
   $\surd$ **True** ○ False

   (h) If A and B are matrices such that BA is defined, then any vector $\overline{v}$ in $\mathcal{N}(A)$ is also in $\mathcal{N}(BA)$.
   $\surd$ **True** ○ False

2. (8 points) For each set S of vectors given below, determine if S is a spanning set for $\mathbb{R}^3$. If it is **not** a spanning set, give an example of a vector $\overline{v}$ in $\mathbb{R}^3$ that **cannot** be written as a linear combination of the vectors in S.

(a)

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 7 \end{bmatrix} \right\}$$

Solution: To determine $\mathrm{Sp}(S)$, we must find all the vectors $\overline{y}$ in $\mathbb{R}^3$ such that $A\overline{x} = \overline{y}$ is consistent, where A is the matrix whose columns are the vectors in S. To determine this, we reduce the augmented matrix $[A|\overline{y}]$:

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & y_1 \\ 2 & -1 & 0 & y_2 \\ 3 & 2 & 7 & y_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & y_2 - y_1 \\ 0 & 1 & 2 & y_2 - 2y_1 \\ 0 & 0 & 0 & y_3 - 7y_1 + 5y_2 \end{array}\right]$$

This shows that

$$\mathrm{Sp}(S) = \left\{\overline{y} \in \mathbb{R}^3 : y_3 - 7y_1 + 5y_2 = 0\right\}$$

so S is **not** a spanning set for $\mathbb{R}^3$, for instance, the vector

$$\overline{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

is not in $\mathrm{Sp}(S)$ (i.e. cannot be written as a linear combination of the vectors in S) because it does not satisfy the algebraic specification of $\mathrm{Sp}(S)$.

(b)

$$S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\}$$

Solution: As above, we begin reducing $[A|\overline{y}]$ where A is the matrix whose columns are the given vectors:

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & y_1 \\ -1 & 1 & 0 & 3 & y_2 \\ 1 & 2 & 2 & 1 & y_3 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & y_1 \\ 0 & 1 & -2 & 4 & y_2 + y_1 \\ 0 & 0 & 8 & -8 & y_3 - 2y_2 - 3y_1 \end{array}\right]$$

Even though this is not in reduced echelon form, we can see that there will be no row of zeros so this matrix is always consistent, meaning $\mathrm{Sp}(S) = \mathbb{R}^3$.

3. (8 points) Determine if the following matrices are nonsingular and find their inverse, if it exists. If it does not exist, express one column of the matrix as a linear combination of the remaining columns.

(a)

$$A = \begin{bmatrix} 2 & 5 \\ 4 & 11 \end{bmatrix}$$

Solution: For 2×2 matrices, we have a formula to find inverses, and can use it to compute

$$A^{-1} = \begin{bmatrix} 11/2 & -5/2 \\ -2 & 1 \end{bmatrix}.$$

Because A^{-1} exists, that implies A is nonsingular.

(b)

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & -4 & -10 \\ -1 & 3 & 9 \end{bmatrix}$$

Solution: Because A reduces to

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

it is singular hence does not have an inverse. To write one column as a combination of the other two, we want to solve

$$x_1\overline{A}_1 + x_2\overline{A}_2 + x_3\overline{A}_3 = \overline{0}$$

where the $\overline{A}_i$ are the columns of A. To do so, we reduce $[A|0]$ to get

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which says

$$\begin{aligned} x_1 &= -3x_3 \\ x_2 &= -4x_3 \end{aligned}$$

where x_3 is arbitrary. Choosing $x_3 = 1$ implies

$$\overline{A}_3 = 3\overline{A}_1 + 4\overline{A}_2$$

or

$$\begin{bmatrix} -1 \\ -10 \\ 9 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 4\begin{bmatrix} -1 \\ -4 \\ 3 \end{bmatrix}.$$

4. (8 points) For each of the following subsets W, either prove that W is a subspace of $\mathbb{R}^3$ by checking the required conditions or prove that it is not a subspace by providing a counterexample to one of those conditions.

(a)

$$W = \left\{ \overline{x} : \overline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\ x_1 \leq x_2 \right\}$$

Solution: This is not a subspace because it is not closed under scalar multiplication:

$$(-1)\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \notin W.$$

(b)

$$W = \left\{ \overline{x} : \overline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\ 3x_1 - 2x_2 + 2x_3 = 0 \right\}$$

Solution: This is a subspace. First we note that $3(0) - 2(0) + 2(0) = 0$ so the zero vector is contained in W. Next let

$$\overline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \text{and} \qquad \overline{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

and assume $\overline{x}, \overline{y} \in W$, which means $3x_1 - 2x_2 + 2x_3 = 0$ and $3y_1 - 2y_2 + 2y_3 = 0$. Then

$$\overline{x} + \overline{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$$

and

$$3(x_1 + y_1) - 2(x_2 + y_2) + 2(x_3 - y_3) = (3x_1 - 2x_2 + 2x_3) + (3y_1 - 2y_2 + 2y_3) = 0 + 0 = 0$$

therefore $\overline{x} + \overline{y} \in W$; so W is closed under addition. Finally let $c \in \mathbb{R}$ be a scalar. Then

$$c\overline{x} = \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix}$$

and

$$3(cx_1) - 2(cx_2) + 2(cx_3) = c(3x_1 - 2x_2 + 2x_3) = c(0) = 0$$

therefore $c\overline{x} \in W$. Thus W is closed under scalar multiplication. This proves W is a subspace.

5. (8 points) Let A be the matrix below whose reduced echelon form, ref(A), is also given:

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & -2 & 5 & 4 \\ 1 & -1 & 0 & 7 \end{bmatrix} \qquad \text{ref}(A) = \begin{bmatrix} 1 & -1 & 0 & 7 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for $\mathcal{N}(A)$.

Solution: To find $\mathcal{N}(A) = \{\overline{x} \in \mathbb{R}^4 : A\overline{x} = \overline{0}\}$, we reduce the augmented matrix $[A|\overline{0}]$. But,

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 0 \\ 2 & -2 & 5 & 4 & 0 \\ 1 & -1 & 0 & 7 & 0 \end{array}\right]$$

reduces to

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 7 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

hence

$$\begin{aligned} x_1 &= x_2 - 7x_4 \\ x_3 &= 2x_4 \end{aligned}$$

where x_2 and x_4 are arbitrary. Writing the solution in vector form,

$$\mathcal{N}(A) = \left\{ \overline{x} : \overline{x} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \ x_2, x_4 \text{ any real numbers} \right\}$$

so a basis for the nullspace is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

(b) Find a basis for $\mathcal{R}(A)$.

Solution: You can do this in two ways. One way is to recall that the columns of A corresponding to the leading ones in the reduced echelon form of A form a basis for the range, so because A reduces to

$$\begin{bmatrix} 1 & -1 & 0 & 7 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and there are leading ones in the first and third columns, the first and third columns of A form a basis for $\mathcal{R}(A)$, so

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} \right\}$$

is a basis for $\mathcal{R}(A)$. Alternatively, you can find a basis for the range by first determining an algebraic specification for $\mathcal{R}(A)$ and writing it in vector form. To find this, we reduce $[A|\overline{y}]$ and see that

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & y_1 \\ 2 & -2 & 5 & 4 & y_2 \\ 1 & -1 & 0 & 7 & y_3 \end{array}\right]$$

reduces to

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 7 & 5y_1 - 2y_2 \\ 0 & 0 & 1 & -2 & y_2 - 2y_1 \\ 0 & 0 & 0 & 0 & y_3 + 2y_2 - 5y_1 \end{array}\right].$$

This means that a vector $\overline{y}$ is in the range if and only if $y_3 + 2y_2 - 5y_1 = 0$, or $y_3 = 5y_1 - 2y_2$. Writing this in vector form,

$$\mathcal{R}(A) = \left\{ \overline{y} : \overline{y} = y_1 \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} + y_2 \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix},\ y_1, y_2 \text{ any real numbers} \right\}$$

so a basis for $\mathcal{R}(A)$ is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

6. (8 points) Prove the following theorem:

Theorem. Let W be a subspace of $\mathbb{R}^n$. If $B = \{\overline{v}_1, \dots, \overline{v}_p\}$ is a basis for W, then any vector $\overline{x}$ in W can be expressed uniquely in terms of the basis B. That is, there are unique scalars $a_1, \dots, a_p$ such that

$$\overline{x} = a_1\overline{v}_1 + \cdots + a_p\overline{v}_p.$$

Solution: Because B is a basis for W (and hence a spanning set), we can write

$$\overline{x} = a_1\overline{v}_1 + \cdots + a_p\overline{v}_p$$

for some scalars $a_1, \dots, a_p$. If $\overline{x}$ can also be expressed as

$$\overline{x} = b_1\overline{v}_1 + \cdots + b_p\overline{v}_p$$

for scalars $b_1, \dots, b_p$, then

$$\overline{0} = \overline{x} - \overline{x} = (a_1 - b_1)\overline{v}_1 + \cdots + (a_p - b_p)\overline{v}_p.$$

Because $\{\overline{v}_1, \dots, \overline{v}_p\}$ is a basis, hence a linearly independent set, we must have

$$a_1 - b_1 = 0$$

$$\dots$$

$$a_p - b_p = 0$$

or

$$a_1 = b_1$$

$$\dots$$

$$a_p = b_p.$$

Thus, the expression is unique.