Math 308 M – Spring 2015
Proof Homework 1
Due Wednesday, April 15th, 2015

Name: ______________________________

- **Answer two** of the following three questions. If you answer them all, your *best* answer will be ignored.
- Give rigorous proofs. Any skipped steps must be small enough that you could explain them to me in a few seconds. Your goal is to convince me you fully understand your argument and have not missed anything.
- You may use any theorem, proposition, etc. from lecture or the book, though when you do say at least "from the book" or "from lecture."
- For examples to model your proofs on, see the textbook, the proof examples document on the course web site, or any of the alternatives to the textbook linked from the course web site.
- You are welcome to talk to others (even outside the class) or work in groups on this assignment, though write your final answers alone. Keep in mind that this exercise is entirely for your benefit in becoming more comfortable with proofs.

1. Write down all the reduced row echelon forms for 2×3 matrices. How many distinct sets of possible pivot positions are there? (See Corollary 2 of the proof examples document for an example proof by cases.)

Solution: There are three columns and two rows. In reduced row echelon form, all zero rows are at the bottom. If there are two zero rows, the matrix is just

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

If there is a single zero row, it must be the second row, and the first row must be non-zero. The first row can have its leading term in any position, and the value of the leading terms is 1, giving three possible forms:

$$\begin{pmatrix} 1 & * & * \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(Here $*$ indicates any value is allowed in that position.) If there are no zero rows, then both the first and second rows have leading terms. The second row's leading term is either in the first, second, or third column. We consider these cases in turn.

- Suppose the second row's leading term is in the first column. The first entry in the first row is above this leading term, so is zero. The first row's leading term is then in the second or third column, but matrices in reduced row echelon form have leading terms which go right rather than left as you read down rows, so there are no solutions of this form.
- Suppose the second row's leading term is in the second column. The first row's leading term must be left of the second column, so it must be in the first column, giving the form
$$\begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \end{pmatrix}$$
- Suppose the second row's leading term is in the third column. In essentially the same manner as before, we find two forms,
$$\begin{pmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In all, there are 7 possible pivot sets.

2. Show that the system of equations

$$a_{11}x_1 + a_{12}x_2 = 0$$
$$a_{21}x_1 + a_{22}x_2 = 0$$

has more than one solution if and only if $a_{11}a_{22} = a_{12}a_{21}$. (Hint: row reduction. You may also solve the easier version where each a_{ij} is non-zero.)

Solution: First consider the easier version. We "abstractly row reduce" the corresponding augmented matrix

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \end{pmatrix}$$

by applying $R_2 - (a_{21}/a_{11})R_1 \Rightarrow R_2$ to get

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} - a_{21}a_{12}/a_{11} & 0 \end{pmatrix}.$$

Since $a_{11} \neq 0$, this matrix is in echelon form. If the second row's second entry is non-zero, the corresponding linear system is triangular, so it has a unique solution from class. However, it is also homogeneous, so that unique solution is just $x_1 = x_2 = 0$. On the other hand, if the second row's second entry is zero, the corresponding linear system has a free variable, so from class there are solutions with x_2 arbitrary, say $x_2 = 1$. Hence we've shown that there is more than one solution if and only if $a_{22} - a_{21}a_{12}/a_{11} = 0$, which is equivalent to saying $a_{11}a_{22} - a_{12}a_{21} = 0$.

Now for the harder version. We only ended up dividing by a_{11}, and the reasoning above still applies whenever $a_{11} \neq 0$. So, we need only consider the case when $a_{11} = 0$. There are several ways to proceed, but a quick one is to observe that we may interchange the two rows in the system. If $a_{21} \neq 0$, the preceding reasoning again applies, so we may suppose $a_{21} = 0$ on top of $a_{11} = 0$. But then $a_{11}a_{22} = 0 = a_{12}a_{21}$ and there are infinitely many solutions (namely $(x_1, x_2) = (s, 0)$ for arbitrary s), which completes the proof.

3. Suppose the solutions of a homogeneous linear system of two equations in three variables are all the multiples of $(-3, 1, 0)$, i.e. are precisely of the form $x_1 = -3s, x_2 = s, x_3 = 0$ for arbitrary s. Show that the corresponding augmented matrix's reduced row echelon form is

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(You may either follow the challenge problem outline from class, follow the proof of Theorem 6 from the proof examples document, or create your own proof. If you create your own proof, you may assume there are two pivots, one in the first column and the other in the third column. Extra credit will be given to anyone who produces an original argument which includes the computation of pivot positions and number.)

Solution: Let's follow Theorem 6. It starts by considering the reduced row echelon form of the augmented matrix of the linear system. We delete the rightmost column since it's essentially irrelevant for homogeneous systems. The result is of the form

$$\begin{pmatrix} * & * & * \\ * & * & * \end{pmatrix}.$$

The general solution of a (consistent) echelon form system groups variables into either leading variables (which come from pivots) or "free parameters," where there is always a solution with the free parameters set arbitrarily.

The proof of Theorem 6 next describes how to compute pivots from right to left. In the rightmost column, it says we can detect a pivot by looking for solutions with $x_3 = 1$—why? If x_3 is a free variable, there is a solution with $x_3 = 1$. However, we're told every solution has $x_3 = 0$, so x_3 is not a free variable, so the third column has a pivot.

The proof then describes determining whether the next column has a pivot. If the second column has a pivot, then that pivot's row corresponds to an equation of the form $x_2 + a_{i2}x_3 = 0$. All of our solutions have $x_3 = 0$, so to satisfy this equation, they must also have $x_2 = 0$, which is false (ex. $x_1 = -3, x_2 = 1, x_3 = 0$ works). Hence the second column does not have a pivot, so x_2 is a free variable.

For the first column, if it did not have a pivot, x_1 would be a free variable. In that case we may pick arbitrary values for the free variables x_1 and x_2 to get a solution, say $x_1 = 1, x_2 = 0$, but none exist. Hence the first column also has a pivot.

Since leading terms go right as we read down rows, the matrix is of the form

$$\begin{pmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

All that remains is to determine the second entry in the first row. The proof describes how to do this too. Since x_2 is a free variable, there is a solution with $x_2 = 1$—in fact, this must be $x_1 = -3, x_2 = 1, x_3 = 0$. The first row's underlying equation is of the form $x_1 + a_{12}x_2 = 0$, so $-3 + a_{12} = 0$, so $a_{12} = 3$. This completes the proof.