Math 308 M – Spring 2015
Proof Homework 3
Due Monday, June 1st, 2015

Name: ______________________________

- **Answer one** of the following two questions. Only one problem will be graded.
- Give rigorous proofs. Any skipped steps must be small enough that you could explain them to me in a few seconds. Your goal is to convince me you fully understand your argument and have not missed anything.
- You may use any theorem, proposition, etc. from lecture or the book, though when you do say at least "from the book" or "from lecture."
- For examples to model your proofs on, see the textbook, the proof examples document on the course web site, or any of the alternatives to the textbook linked from the course web site.
- You are welcome to talk to others (even outside the class) or work in groups on this assignment, though **write your final answers alone**. Keep in mind that this exercise is entirely for your benefit in becoming more comfortable with proofs.

1. Let S_1 and S_2 be subspaces of $\mathbb{R}^n$.

   Recall the *intersection* of two sets is the set of elements in both sets. For instance, in $\mathbb{R}^2$, the intersection of $\text{Span}\{\mathbf{e}_1\}$ and $\text{Span}\{\mathbf{e}_2\}$ is the intersection of the x and y-axes, namely the origin. In symbols,

$$\text{Span}\{\mathbf{e}_1\} \cap \text{Span}\{\mathbf{e}_2\} = \{\mathbf{0}\}.$$

   (a) Suppose S_1 and S_2 are subspaces of $\mathbb{R}^n$. Show that the intersection $S_1 \cap S_2$ is a subspace of $\mathbb{R}^n$.

   Solution: We verify the three properties.

   (a) (Contains $\mathbf{0}$.) $\mathbf{0} \in S_1 \cap S_2$ since $\mathbf{0}$ is in both S_1 and S_2.

   (b) (Closed under addition.) If $\mathbf{x}$ and $\mathbf{y}$ are each in $S_1 \cap S_2$, then $\mathbf{x}$ and $\mathbf{y}$ are each in S_1. Since S_1 closed under addition, $\mathbf{x} + \mathbf{y}$ is in S_1. Likewise $\mathbf{x} + \mathbf{y}$ is in S_2, so $\mathbf{x} + \mathbf{y} \in S_1 \cap S_2$.

   (c) (Closed under scaling.) If $\mathbf{x}$ is in $S_1 \cap S_2$ and c is a scalar, then $\mathbf{x}$ is in S_1, so since S_1 is closed under scaling, $c\mathbf{x}$ is in S_1. Likewise $c\mathbf{x}$ is in S_2, so $c\mathbf{x}$ is in $S_1 \cap S_2$.

   (b) Suppose S_1 and S_2 are subspaces of $\mathbb{R}^n$ and that $S_1 \cap S_2 = \{\mathbf{0}\}$. Let $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_i\}$ be a basis for S_1 and let $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_j\}$ be a basis for S_2. Show that the union $U \cup V = \{\mathbf{u}_1, \ldots, \mathbf{u}_i, \mathbf{v}_1, \ldots, \mathbf{v}_j\}$ is linearly independent.

   Solution: Suppose

$$c_1\mathbf{u}_1 + \cdots + c_i\mathbf{u}_i + d_1\mathbf{v}_1 + \cdots + d_j\mathbf{v}_j = \mathbf{0}.$$

   We must show $c_1 = \cdots = c_i = d_1 = \cdots = d_j = 0$. We can rewrite this as

$$c_1\mathbf{u}_1 + \cdots + c_i\mathbf{u}_i = -d_1\mathbf{v}_1 - \cdots - d_j\mathbf{v}_j.$$

   The left-hand side is in S_1 and the right-hand side is in S_2, so they're in the intersection $S_1 \cap S_2$. However, this intersection is just $\{\mathbf{0}\}$, so the left and right sides are each zero:

$$\begin{aligned} \mathbf{0} &= c_1\mathbf{u}_1 + \cdots + c_i\mathbf{u}_i \\ &= -d_1\mathbf{v}_1 - \cdots - d_j\mathbf{v}_j. \end{aligned}$$

   Since U and V are bases, they're linearly independent, so $c_1 = \cdots = c_i = 0$ and $-d_1 = \cdots = -d_j = 0$, i.e. $d_1 = \cdots = d_j = 0$, as required.

2. Let A and B be $n \times n$ matrices.

   (a) Suppose $A^2 = A$. Show that $\det(A)$ is either 1 or 0.

   Solution: Since the determinant of the product is the product of the determinants, we find $\det(A)^2 = \det(A^2) = \det(A)$, so $\det(A)(\det(A) - 1) = 0$, so $\det(A) = 0$ or $\det(A) = 1$.

   (b) Show $\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$ directly from the definition.

   Solution: Using sigma notation for convenience, we compute

$$\begin{aligned}\mathrm{Tr}(A+B) &= \sum_{i=1}^{n}(A+B)_{ii} = \sum_{i=1}^{n} A_{ii} + B_{ii} \\ &= \sum_{i=1}^{n} A_{ii} + \sum_{i=1}^{n} B_{ii} = \mathrm{Tr}(A) + \mathrm{Tr}(B).\end{aligned}$$

   (c) Show $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ directly from the definition. (Sigma notation might be useful.)

   Solution: We compute

$$\begin{aligned}\mathrm{Tr}(AB) &= \sum_{i=1}^{n}(AB)_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}B_{ji} \\ \mathrm{Tr}(BA) &= \sum_{j=1}^{n}(BA)_{jj} = \sum_{j=1}^{n}\sum_{i=1}^{n} B_{ji}A_{ij} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}B_{ji} = \mathrm{Tr}(AB).\end{aligned}$$

   (d) Use the previous part to show that $\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA)$. (Hint: recall that $ABC = (AB)C$. Warning: $\mathrm{Tr}(ABC) \neq \mathrm{Tr}(CBA)$ in general—only "cyclic rotations" work in general.)

   Solution: As the hint suggests, though grouping slightly differently, we have

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(A(BC)) = \mathrm{Tr}((BC)A = \mathrm{Tr}(BCA)$$

   where the middle equality uses (c) and the other two use associativity of matrix multiplication.

One can also write $\mathrm{Tr}(ABC)$ as a certain triple sum as in (c) and show it's the same as a triple sum for $\mathrm{Tr}(CBA)$.