Your Name

Student ID #

- Do not open this exam until you are told to begin. You will have 50 minutes for the exam.
- Check that you have a complete exam. There are 5 questions for a total of 54 points.
- You are allowed to have one handwritten note sheet. Only basic non-graphing scientific calculators are allowed, though you should not need one.
- Cheating will result in a zero and be reported to the Dean's Academic Conduct Committee.
- **Show all your work**. Unless explicitly stated otherwise in a particular question, if there is no work supporting your answer, you will not receive credit for the problem. If you need more space to answer a question, continue on the back of the page, and indicate that you have done so.

Question	Points	Score
1	18	
2	11	
3	8	
4	9	
5	8	
Total:	54	

1. Multiple choice and short answer. For these questions, you are **not required to show any work**.

   (a) (2 points) Every subspace is the row space of some matrix.

   ◯ True ◯ False

   (b) (2 points) If A and B are invertible $n \times n$ matrices, then $(A+B)^{-1} = A^{-1} + B^{-1}$.

   ◯ True ◯ False

   (c) (2 points) If A is $m \times n$, then $\text{nullity}(A) - \text{nullity}(A^T) = n - m$.

   ◯ True ◯ False

   (d) (4 points) Let A, B be $n \times n$ matrices, let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, and let s, t be scalars. Which of the following are *always true*? (Check all that apply.)

   ◯ $A(s\mathbf{u} + t\mathbf{v}) = sA\mathbf{u} + tA\mathbf{v}$. ◯ $(AB)^2 = A^2B^2$.

   ◯ $(A+B)^2 = A^2 + 2AB + B^2$. ◯ $A\mathbf{0} = \mathbf{0}$.

   ◯ $A^2 = A$ implies $A(A-I) = 0$, so either $A = I$ or $A = 0$.

   (e) (4 points) Give an example of two subspaces S_1 and S_2 of $\mathbb{R}^4$ each of dimension 2 but where the only vector belonging to both S_1 and S_2 is $\mathbf{0}$.

   (f) (4 points) Give an example of two linear functions $T\colon \mathbb{R}^n \to \mathbb{R}^m$ and $U\colon \mathbb{R}^m \to \mathbb{R}^\ell$ such that T is one-to-one, $\operatorname{range} T = \ker U$, and U is onto. *Hint:* In your example, you'll find $m = n + \ell$.

2. Let A be the following 3×5 matrix. Its reduced echelon form B is provided.

$$A = \begin{bmatrix} 1 & 1 & 7 & 0 & 0 \\ 3 & 1 & 15 & 6 & 0 \\ 0 & 2 & 6 & 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 & 0 & -3 \\ 0 & 1 & 3 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = B.$$

(a) (3 points) Compute $\operatorname{rank}(A)$, $\dim \operatorname{row}(A)$, $\dim \operatorname{col}(A)$, and $\operatorname{nullity}(A)$.

(b) (8 points) Find bases for $\operatorname{row}(A)$, $\operatorname{col}(A)$, and $\operatorname{null}(A)$.

3. Let A be the following 3×3 matrix:

$$A = \begin{bmatrix} 1 & 4 & -3 \\ 1 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) (5 points) Compute A^{-1}.

(b) (3 points) Show that A^T is invertible, with $(A^T)^{-1} = (A^{-1})^T$.

4. Fix a 2×2 matrix A. Let $Q\colon \mathbb{R}^2 \to \mathbb{R}$ be the function given by

$$Q(\mathbf{x}) := \mathbf{x}^T A\mathbf{x}.$$

(a) (3 points) If $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, show directly that

$$Q(\mathbf{x}) = ax^2 + 2bxy + cy^2.$$

(b) (3 points) Show that $Q(s\mathbf{x}) = s^2 Q(\mathbf{x})$ for all scalars s.

(c) (3 points) Find a matrix A and vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^2$ such that $Q(\mathbf{x}_1 + \mathbf{x}_2) \neq Q(\mathbf{x}_1) + Q(\mathbf{x}_2)$.

5. In this question, you are given a proof and are asked to provide justification for individual steps. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation throughout.

Proposition. *If T is onto, then there is a linear transformation $U\colon \mathbb{R}^m \to \mathbb{R}^n$ such that*

$$T(U(\mathbf{x})) = \mathbf{x} \qquad \textit{for all } \mathbf{x} \in \mathbb{R}^m.$$

Proof. Pick $\mathbf{y}_1, \ldots, \mathbf{y}_m$ as in (a), so $T(\mathbf{y}_i) = \mathbf{e}_i$. Pick U as in (b), so $U(\mathbf{e}_i) = \mathbf{y}_i$. Then

$$T(U(\mathbf{e}_i)) = T(\mathbf{y}_i) = \mathbf{e}_i.$$

By (c), $T(U(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^m$. □

Hint: Parts (a)-(c) are independent of each other.

(a) (1 point) Show that if T is onto, then there are $\mathbf{y}_1, \ldots, \mathbf{y}_m \in \mathbb{R}^n$ such that $T(\mathbf{y}_i) = \mathbf{e}_i$.

(b) (3 points) Show that, given $\mathbf{y}_1, \ldots, \mathbf{y}_m$ in $\mathbb{R}^n$, there is some linear transformation $U\colon \mathbb{R}^m \to \mathbb{R}^n$ with

$$U(\mathbf{e}_i) = \mathbf{y}_i \qquad \text{for } i = 1, \ldots, m.$$

(c) (4 points) Show that if $U\colon \mathbb{R}^m \to \mathbb{R}^n$ is linear and $T(U(\mathbf{e}_i)) = \mathbf{e}_i$ for all $\mathbf{e}_i \in \mathbb{R}^m$, then $T(U(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^m$.