308 DIAGONALIZATION NOTES

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Abstract. This document summarizes diagonalization. A fuller account can be found in Holt, §6.4.

1. Diagonalization

Definition 1. An $n \times n$ matrix A is **diagonalizable** if there is a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

Example 2. A diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_n)$ has eigenvectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ with eigenvalues $d_1, \ldots, d_n$, so D is diagonalizable.

Example 3. If A has n distinct eigenvalues, their corresponding eigenvectors are linearly independent, so they form a basis, so A is diagonalizable.

Theorem 4. *An $n \times n$ matrix A is diagonalizable if and only if there is a diagonal matrix D and an invertible matrix P such that*

$$A = PDP^{-1}.$$

In this case, if the columns of P are given by $P = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_n]$ and the diagonal entries of D are given by $D = \operatorname{diag}(d_1, \ldots, d_n)$, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of $\mathbb{R}^n$ consisting of eigenvectors of A, and the eigenvalue of $\mathbf{v}_i$ is d_i.

Proof. First suppose $A = PDP^{-1}$ with D diagonal and P invertible, as above. Since P is invertible, its columns form a basis for $\mathbb{R}^n$, namely $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. We can pick off the columns of P via $P\mathbf{e}_i = \mathbf{v}_i$, so $\mathbf{e}_i = P^{-1}\mathbf{v}_i$. We verify that $\mathbf{v}_i$ indeed is an eigenvector of A with eigenvalue d_i:

$$A\mathbf{v_i} = PDP^{-1}\mathbf{v}_i = PD\mathbf{e}_i = Pd_i\mathbf{e}_i = d_iP\mathbf{e}_i = d_i\mathbf{v}_i.$$

Hence A has a basis of eigenvectors, so is diagonalizable.

Now suppose A is diagonalizable. Pick a basis of eigenvectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for $\mathbb{R}^n$ with eigenvalues $d_1, \ldots, d_n$ and set $P := [\mathbf{v}_1 \ \cdots \ \mathbf{v}_n]$ and $D := \operatorname{diag}(d_1, \ldots, d_n)$. P is evidently invertible and D is diagonal, so we must only show that $A = PDP^{-1}$. The above computation shows that PDP^{-1} has eigenvectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ with eigenvalues $d_1, \ldots, d_n$, so $PDP^{-1}\mathbf{v}_i = d_i\mathbf{v}_i$. By construction the same holds for A, namely $A\mathbf{v}_i = d_i\mathbf{v}_i$. The linear transformations of PDP^{-1} and A thus agree on a basis, so they agree everywhere, so $PDP^{-1} = A$. Alternatively, $(A - PDP^{-1})\mathbf{v}_i = 0$, so the 0-eigenspace of $A - PDP^{-1}$ has dimension at least n, so $A - PDP^{-1}$ must be the zero matrix. □

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Remark 5. Since $A = PDP^{-1}$, we have $P^{-1}AP = D$. Multiplying by P^{-1} on the left and P on the right is really computing A "in the basis of eigenvectors", and in that basis A is diagonal. More formally, P is the change of basis matrix from the basis of eigenvectors $\mathcal{B} := \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ to the standard basis $\mathcal{S}$ and P^{-1} is the change of basis matrix from $\mathcal{S}$ to $\mathcal{B}$. Hence if $\mathbf{x} \in \mathbb{R}^n$ we have

$$(P^{-1}AP)\mathbf{x}_{\mathcal{B}} = P^{-1}A\mathbf{x}_{\mathcal{S}} = P^{-1}(A\mathbf{x}) = [A\mathbf{x}]_{\mathcal{B}}.$$

Compare this to $(A)\mathbf{x}_{\mathcal{S}} = (A\mathbf{x})_{\mathcal{S}}$, which says that A takes in coordinate vectors with respect to the standard basis and returns coordinate vectors with respect to the standard basis. On the other hand, $(P^{-1}AP)x_{\mathcal{B}} = [Ax]_{\mathcal{B}}$ says that $P^{-1}AP$ takes in coordinate vectors with respect to basis $\mathcal{B}$ and returns coordinate vectors with respect to basis $\mathcal{B}$.

Remark 6. One of the main uses of diagonalization is to quickly compute powers of a matrix, which is essentially what happened in the Fibonacci number example from lecture. Indeed, if $A = PDP^{-1}$, then $A^2 = (PDP^{-1})(PDP^{-1}) = PDDP^{-1} = PD^2P^{-1}$, and more generally $A^k = PD^kP^{-1}$ for any $k \geq 0$ (and also any $k < 0$ if A or equivalently D is invertible). Computing powers of a diagonal matrix is easy since $\operatorname{diag}(d_1, \ldots, d_n)^k = \operatorname{diag}(d_1^k, \ldots, d_n^k)$.

Example 7. We can rephrase the Fibonacci computation from class in this language. With $F_0 = 0, F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$, we showed that

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}.$$

We found a basis of eigenvectors for the matrix on the left-hand side, which using the theorem shows that

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} \phi & \overline{\phi} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \phi^n & 0 \\ 0 & \overline{\phi}^n \end{bmatrix} \begin{bmatrix} \phi & \overline{\phi} \\ 1 & 1 \end{bmatrix}^{-1}$$

where $\phi := (1 + \sqrt{5})/2$ and $\overline{\phi} := (1 - \sqrt{5})/2$. Combining these two observations immediately gives Binet's formula.