Your Name

Student ID #

- Do not open this exam until you are told to begin. You will have 1 hour, 50 minutes for the exam.
- Check that you have a complete exam. There are 8 questions for a total of 117 points.
- You are allowed to have one handwritten note sheet. Only basic non-graphing scientific calculators are allowed, though you should not need one.
- Cheating will result in a zero and be reported to the Dean's Academic Conduct Committee.
- **Show all your work**. Unless explicitly stated otherwise in a particular question, if there is no work supporting your answer, you will not receive credit for the problem. If you need more space to answer a question, continue on the back of the page, and indicate that you have done so.

Question	Points	Score
1	21	
2	13	
3	12	
4	18	
5	12	
6	14	
7	15	
8	12	
Total:	117	

1. For true/false and multiple choice questions, you are **not required to show any work**.

   (a) (1 point) Any linearly independent set in $\mathbb{R}^n$ spans $\mathbb{R}^n$.

   ◯ True ◯ False

   (b) (1 point) Is $A^T A$ symmetric?

   ◯ always ◯ never ◯ only if A is square

   (c) (4 points) Check all of the following properties of determinants which are *always true*. A, B are $n \times n$ matrices, B is invertible, and c is a scalar.

   ◯ $\det(A) = \det(A^T)$ ◯ $\det(A^2 + I) = \det(A)^2 + 1$ ◯ $\det(cA) = c\det(A)$
   ◯ $\det(B^{-1})^{-1} = \det(B)$ ◯ $\det(AB) = \det(A^T B^T)$ ◯ $\det(A+B) = \det(A)+\det(B)$

   (d) (4 points) Define **two** of the following three terms: basis, dimension of a subspace, eigenvector. (Clearly specify which terms you are defining.)

   (e) (4 points) Suppose a matrix A satisfies $A^3 + A = I$. Show that A is non-singular (i.e. invertible).

(f) (4 points) Suppose

$$S = \left\{ \begin{bmatrix} c_1 \\ 1 + c_2 \\ -c_1 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\} \subset \mathbb{R}^3.$$

Is S a subspace of $\mathbb{R}^3$?

(g) (3 points) Let $T \colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation. Suppose the range of T is a line. Describe the kernel of T geometrically.

2. Produce example(s) with the given properties. You are *not required* to give justification.

   (a) (4 points) A square matrix A is called *orthogonal* if $A^T A = I$. It is a fact that if A is orthogonal, then $\det(A) = \pm 1$. Give examples of orthogonal matrices B and C where $\det(B) = 1$ and $\det(C) = -1$.

   (b) (4 points) Give an example of three pairwise orthogonal vectors in $\mathbb{R}^4$ with no 0 coordinates.

   (c) (3 points) Give a linear transformation T whose corresponding matrix is non-zero and triangular, and where T is not onto.

   (d) (2 points) Give an example of a matrix with 4 distinct eigenvalues.

3. Produce example(s) with the given properties. You are *not required* to give justification.

   (a) (4 points) Suppose $V = \{\vec{v}_1, \ldots, \vec{v}_n\} \subset \mathbb{R}^n$, $A = [\vec{v}_1 \ \cdots \ \vec{v}_n]$, and $T\colon \mathbb{R}^n \to \mathbb{R}^n$ by $T(\vec{x}) = A\vec{x}$. Using this notation, give four of the equivalent conditions in the Big Theorem.

   (b) (4 points) Give examples of two-dimensional subspaces S_1 and S_2 of $\mathbb{R}^4$ where $S_1 \neq S_2$ and where S_1 and S_2 contain some common non-zero vector.

   (c) (4 points) Find some linear transformation T such that $\operatorname{range} T$ contains $\ker T$ and $\ker T \neq \{\vec{0}\}$. (Hint: this can be done in two dimensions.)

4. (a) (8 points) Let $S = \text{span}\{\vec{s}_1, \vec{s}_2, \vec{s}_3\} \subset \mathbb{R}^4$ and $\vec{u} \in \mathbb{R}^4$ where

$$\vec{s}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{s}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{s}_3 = \begin{bmatrix} 2 \\ 2 \\ -1 \\ -2 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}.$$

Compute $\text{proj}_S \vec{u}$.

(b) (4 points) Let S be a subspace of $\mathbb{R}^n$. Suppose $\vec{u} \in S^\perp$. Show that $\text{proj}_S \vec{u} = \vec{0}$.

(c) (6 points) Suppose $S = \text{span}\{\vec{s}_1, \vec{s}_2\}$ where

$$\vec{s}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \quad \vec{s}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}.$$

Compute a basis for $S^{\perp}$.

5. (a) (2 points) What is the (smaller) angle between the vectors

$$\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad \text{and} \qquad \vec{v} = \begin{bmatrix} -3 \\ 2 \\ -1 \\ 0 \end{bmatrix}?$$

(b) (6 points) Show that $\operatorname{null}(A) \subset \operatorname{null}(A^T A)$. Conclude that $\operatorname{rank}(A) \geq \operatorname{rank}(A^T A)$.

(c) (4 points) Find all least squares solutions to the system $A\vec{x} = \vec{y}$ where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

6. (a) (4 points) Consider the two linear systems $A\vec{x} = \vec{y}$ and $A\vec{u} = \vec{0}$. Suppose that $\vec{x}_p$ is a solution to the first system. For any solution $\vec{u}$ of the second system, show that $\vec{x} = \vec{x}_p + \vec{u}$ is a solution of the first system.

   (b) (5 points) Solve the two linear systems $A\vec{x} = \vec{u}$ and $A\vec{x} = \vec{v}$ where

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}, \quad \vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

   (Hint: inverse.)

(c) (5 points) Let

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\}, \quad \mathcal{B}_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Given that $\mathcal{B}_1, \mathcal{B}_2$ are bases for the same subspace of $\mathbb{R}^3$, compute the change of basis matrix C from $\mathcal{B}_1$ to $\mathcal{B}_2$.

7. The matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

has characteristic polynomial

$$\det(A - \lambda I) = (1-\lambda)^3(1+\lambda)^2.$$

(a) (8 points) Find bases for the eigenspaces of A.

(b) (2 points) What are the geometric multiplicities of the eigenvalues of A?

(c) (2 points) Is A diagonalizable?

(d) (3 points) If some matrix B is diagonalizable, show that B^2 is diagonalizable.

8. (a) (4 points) Determine the *eigenvalues* and *algebraic multiplicities* of the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

(b) (3 points) Suppose some matrix B has eigenvalue λ. Show that B^2 has eigenvalue λ^2.

(c) (5 points) Let

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

For which value(s) of $0 \leq \theta < 2\pi$ does R_θ have (real) eigenvalues? What are those eigenvalues?