1. (20 points) Indicate whether the statement is true (T) or false (F). Circle your response. You are not required to show any work.

(a) Applying row operations does not change the eigenvalues of a matrix.

ANSWER: (circle one) T F

(b) Any vector $\mathbf{x}$ with the property that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ is an eigenvector of A.

$\bar{X}$ cannot $= \bar{0}$!

ANSWER: (circle one) T F

(c) If $A = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, then $\mathbf{v}$ is an eigenvector of A.

ANSWER: (circle one) T F

$A\bar{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\bar{v}$

(d) If $\mathbf{u}_1$ and $\mathbf{u}_2$ are eigenvectors of a matrix A and $\{\mathbf{u}_1, \mathbf{u}_2\}$ is linearly independent, then $\mathbf{u}_1$ and $\mathbf{u}_2$ must correspond to different eigenvalues.

ANSWER: (circle one) T F

(e) Every basis of $\mathbb{R}^2$ is an orthogonal basis.

ANSWER: (circle one) T F

(f) If S is a subspace of $\mathbb{R}^3$ with dimension 1, then so is $S^\perp$.

ANSWER: (circle one) T F

(g) If $\mathbf{u}$ is orthogonal to $\mathbf{v}$, then $||\mathbf{u} + \mathbf{v}|| = ||\mathbf{u} - \mathbf{v}||$.

$||u+v|| = ||u|| + ||v|| = ||u-v||$

because they are orthogonal

ANSWER: (circle one) T F

(h) The zero vector in $\mathbb{R}^n$ is orthogonal to every vector in $\mathbb{R}^n$.

$\bar{0} \cdot \bar{v} = 0$ for any $\bar{v}$

ANSWER: (circle one) T F

(i) If $\mathbf{u}$ is in the subspace S, then $\text{proj}_S\mathbf{u} = \mathbf{u}$.

ANSWER: (circle one) T ~~F~~

(j) If A is any 2×2 diagonalizable matrix with eigenvalues $\lambda = 1$ and $\lambda = -1$, then $A^k = A$ for every positive integer k.

ANSWER: (circle one) T F

2. (12 points) Give an example of each of the following. You are not required to show any work.

   (a) a 4×4 matrix with determinant 12

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

   (b) a basis $\mathcal{B}$ of $\mathbb{R}^2$ such that

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

$\mathcal{B} = \{\bar{u}_1, \bar{u}_2\}$

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix}_{\mathcal{B}} = 4\bar{u}_1 + 3\bar{u}_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\bar{u}_1 = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}, \quad \bar{u}_2 = \begin{bmatrix} 0 \\ 5/3 \end{bmatrix}$$

   (c) an orthogonal set of three vectors in $\mathbb{R}^3$, none of which is a scalar multiple of any of the standard basis vectors

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

   (d) the characteristic equation for a matrix that has exactly two real eigenvalues, one with multiplicity 2 and the other with multiplicity 4

$$p(\lambda) = (\lambda - 2)^2 (\lambda - 4)^4$$

3. (12 points) Find the characteristic polynomial, the eigenvalues, and a basis for each eigenspace of the matrix A. Mark your responses clearly.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 & 0 \\ -1 & -1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)\left((-1-\lambda)(1-\lambda) - 1\right) - 2(-1)(1-\lambda)$$

$$\boxed{\text{characteristic polynomial} = \lambda^2(1-\lambda)}$$

$$\boxed{\text{eigenvalues} = 0,\ 1}$$

For $\lambda = 0$:

$A - 0I = A$ reduces to

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so eigenspace is

$$\left\{ s \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$\boxed{\text{basis for eigenspace of } \lambda = 0: \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}}$$

For $\lambda = 1$

$A - 1 \cdot I$ reduces to

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so eigenspace is

$$\left\{ s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\boxed{\text{basis for eigenspace of } \lambda = 1: \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

4. (6 points) Let $S = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

Find $\text{proj}_S \mathbf{u}$ if $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$.

$$\text{proj}_S \bar{u} = \frac{\bar{v}_1 \cdot \bar{u}}{\|\bar{v}_1\|^2} \bar{v}_1 + \frac{\bar{v}_2 \cdot \bar{u}}{\|\bar{v}_2\|^2} \bar{v}_2$$

$$= \frac{1}{9} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + \frac{5}{9} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\text{proj}_S \bar{u} = \begin{bmatrix} 7/9 \\ 8/9 \\ 11/9 \end{bmatrix}$$

5. (15 points) Find the value of k so that the matrix A is diagonalizable and then find the matrices P and D such that $A = PDP^{-1}$.

$$A = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & -2 & 1 \\ 0 & 1-\lambda & k \\ 0 & 0 & 1-\lambda \end{bmatrix} = (2-\lambda)(1-\lambda)(1-\lambda)$$

eigenvalues : 2 , 1

mult. 1   mult. 2

Need to find k such that eigenspace of $\lambda = 1$ has dimension = 2 :

$$A - \lambda I = A - 1 \cdot I = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & k \\ 0 & 0 & 0 \end{bmatrix}$$ the nullspace is 2 dimensional if and only if $k=0$

$\Longrightarrow$ A diagonalizable if and only if $\boxed{k=0}$

basis for eigenspace of $\lambda = 2$: ($k=0$) :

$$\begin{bmatrix} 0 & -2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

eigenspace = $\left\{ s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

basis: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

basis for eigenspace of $\lambda = 1$: ($k=0$)

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

eigenspace = $\left\{ s_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

basis: $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$\boxed{P = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

6. (15 points) Consider $\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$, a basis for $\mathbb{R}^3$.

(a) Compute $\mathbf{x}_{\mathcal{B}_1}$ if $\mathbf{x} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$.

$$\left(\bar{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \bar{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \bar{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right)$$

$$U = [\bar{u}_1 \ \bar{u}_2 \ \bar{u}_3]$$

$$\bar{x}_{\mathcal{B}_1} = U^{-1}\bar{x}$$

$$U = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Find U^{-1}: $U^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}$

then

$$\bar{x}_{\mathcal{B}_1} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}_{\mathcal{B}_1}$$

$$\boxed{\bar{x}_{\mathcal{B}_1} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}_{\mathcal{B}_1}}$$

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(b) Let $\mathcal{B}_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$. (Note that $\mathcal{B}_2$ is also a basis for $\mathbb{R}^3$.)

Find the change of basis matrix from $\mathcal{B}_2$ to $\mathcal{B}_1$.

$$\bar{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \bar{v}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \quad \bar{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$V = [\bar{v}_1 \; \bar{v}_2 \; \bar{v}_3]$$

$$\bar{x}_{\mathcal{B}_1} = \underbrace{U^{-1} V}_{\text{C.O.B. matrix from } \mathcal{B}_2 \text{ to } \mathcal{B}_1} \bar{x}_{\mathcal{B}_2}$$

$$U^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$V = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\boxed{U^{-1} V = \begin{bmatrix} 1 & 5 & -2 \\ 0 & -1 & 1 \\ -1 & -4 & 2 \end{bmatrix}}$$