Your Preferred Name

Student ID #

- Do not open this quiz until you are told to begin. You will have 30 minutes for the quiz.
- Check that you have a complete quiz. There are 3 questions for a total of 23 points.
- You are allowed to have one index card of handwritten notes (both sides). Only basic non-graphing scientific calculators are allowed, though you should not need one.
- Cheating will result in a zero and be reported to the Dean's Academic Conduct Committee.
- **Show all your work**. Unless explicitly stated otherwise in a particular question, if there is no work supporting your answer, you will not receive credit for the problem. If you need more space to answer a question, continue on the back of the page, and indicate that you have done so.

Question	Points	Score
1	6	
2	10	
3	7	
Total:	23	

1. (6 points) Consider the polynomial

$$p(x) = 3x^3 + 4x^2 + x + 1.$$

Express $p(x)$ in the form

$$p(x) = a(x+1)^3 + b(x+1)^2 + c(x+1) + d$$

by solving an appropriate linear system. (*Hint*: $(u+v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$.)

Solution: We compute

$$\begin{aligned} p(x) &= a(x+1)^3 + b(x+1)^2 + c(x+1) + d \\ &= (ax^3 + 3ax^2 + 3ax + a) + (bx^2 + 2bx + b) + (cx + c) + d \\ &= ax^3 + (3a+b)x^2 + (3a+2b+c)x + (a+b+c+d). \end{aligned}$$

Equating coefficients gives the linear system

$$\begin{aligned} a &= 3 \\ 3a + b &= 4 \\ 3a + 2b + c &= 1 \\ a + b + c + d &= 1. \end{aligned}$$

Reversing the order of the variables and equations would make the system triangular. Solving for a, b, c, d in that order quickly gives

$$a = 3, b = -5, c = 2, 1.$$

Hence

$$p(x) = 3x^3 + 4x^2 + x + 1 = 3(x+1)^3 - 5(x+1)^2 + 2(x+1) + 1.$$

2. Consider the following linear system:

$$\begin{aligned} 4x_1 + 2x_2 - x_3 + 2x_4 &= 6 \\ 2x_1 + x_2 + x_3 + 4x_4 &= 3 \\ -x_1 + 0x_2 - x_3 + x_4 &= 0 \end{aligned}$$

(a) (4 points) Give an echelon form matrix equivalent to the augmented matrix of the linear system.

Solution: Beginning with

$$\begin{pmatrix} 4 & 2 & -1 & 2 & 6 \\ 2 & 1 & 1 & 4 & 3 \\ -1 & 0 & -1 & 1 & 0 \end{pmatrix}$$

apply ERO's $2R_2 - R_1 \Rightarrow R_2$, $4R_3 + R_1 \Rightarrow R_3$, $R_2/3 \Rightarrow R_2$, and $R_2 \Leftrightarrow R_3$ to get

$$\begin{pmatrix} 4 & 2 & -1 & 2 & 6 \\ 0 & 2 & -5 & 6 & 6 \\ 0 & 0 & 1 & 2 & 0 \end{pmatrix}$$

which is in echelon form.

(b) (4 points) Give a reduced echelon form matrix equivalent to the augmented matrix of the linear system.

Solution: Continuing from the solution to (a), apply $R_2 + 5R_3 \Rightarrow R_2$, $R_1 + R_3 \Rightarrow R_1$, $R_2/2 \Rightarrow R_2$, $R_1/2 \Rightarrow R_1$, $R_1 - R_2 \Rightarrow R_1$, $R_1/2 \Rightarrow R_1$ to get

$$\begin{pmatrix} 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 8 & 3 \\ 0 & 0 & 1 & 2 & 0 \end{pmatrix}$$

which is in reduced echelon form.

(c) (2 points) What is the general solution of the linear system?

Solution: From (b), x_4 is the only free variable. Reading off the general solution gives

$$\begin{aligned} x_1 &= 3s \\ x_2 &= -8s + 3 \\ x_3 &= -2s \\ x_4 &= s. \end{aligned}$$

3. Give examples satisfying the following conditions. You are not required to justify your answers.

(a) (2 points) A linear system *not* in echelon form but whose augmented matrix *is* in echelon form.

Solution: One example is

$$\begin{aligned} x + y &= 0 \\ 0x + 0y &= 1 \end{aligned}$$

The second row has no leading variable, so the system is not in echelon form. The augmented matrix is

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which is in echelon form.

(b) (3 points) Two different linear combinations which result in the same vector. Give a sketch of the situation.

Solution: One example is

$$\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

(Sketch omitted; this point is the midpoint of two line segments.)

(c) (2 points) A point $\vec{u}$ in $\mathbb{R}^4$ which is not a linear combination of four vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ in $\mathbb{R}^4$.

Solution: One example is $(0, 0, 0, 1)$ not being a linear combination of

$$(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (1, 1, 1, 0)$$

since any linear combination of these four vectors has 0 in the fourth coordinate.