Your Preferred Name

Student ID #

- Do not open this quiz until you are told to begin. You will have 30 minutes for the quiz.
- Check that you have a complete quiz. There are 3 questions for a total of 28 points.
- You are allowed to have one index card of handwritten notes (both sides). Only basic non-graphing scientific calculators are allowed, though you should not need one.
- Cheating will result in a zero and be reported to the Dean's Academic Conduct Committee.
- **Show all your work**. Unless explicitly stated otherwise in a particular question, if there is no work supporting your answer, you will not receive credit for the problem. If you need more space to answer a question, continue on the back of the page, and indicate that you have done so.

Question	Points	Score
1	12	
2	7	
3	9	
Total:	28	

1. Consider the vectors

$$\vec{v} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \qquad \vec{u}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \qquad \vec{u}_2 = \begin{pmatrix}1\\-1\\2\end{pmatrix}, \qquad \vec{u}_3 = \begin{pmatrix}3\\4\\5\end{pmatrix}.$$

(a) (6 points) Determine if $\vec{v}$ is in the span of $\vec{u}_1, \vec{u}_2, \vec{u}_3$.
If so, write $\vec{v}$ as an explicit linear combination of the other vectors.

Solution: We need to solve the linear system with augmented matrix

$$\begin{pmatrix}1 & 1 & 3 & 1\\1 & -1 & 4 & 2\\1 & 2 & 5 & 3\end{pmatrix}$$

which row reduces to

$$\begin{pmatrix}1 & 0 & 0 & -2\\0 & 1 & 0 & 0\\0 & 0 & 1 & 1\end{pmatrix}$$

so the system has a unique solution given by $c_1 = -2, c_2 = 0, c_3 = 1$. Hence $\vec{v}$ is in the span, and moreover

$$\vec{v} = -2\vec{u}_1 + \vec{u}_3.$$

(b) (3 points) Are $\vec{u}_1, \vec{u}_2, \vec{u}_3$ linearly dependent? Explain.

Solution: We could repeat the computation in (a) but replacing the last column with 0's. The result would be the system with unique solution $c_1 = c_2 = c_3 = 0$, so only the trivial linear combination of $\vec{u}_1, \vec{u}_2, \vec{u}_3$ results in $\vec{0}$, so they are linearly independent.

(c) (3 points) Are $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{v}$ linearly dependent? Explain.

Solution: There are more vectors than dimensions, so from class they must be linearly dependent. Alternatively, we found an explicit linear dependence relation in (a), namely

$$\vec{v} + 2\vec{u}_1 - \vec{u}_3 = \vec{0}.$$

2. Consider the function $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \int_0^\pi (x_1\theta + x_2)\cos\theta\, d\theta \\ \int_0^\pi (x_1\theta + x_2)\sin\theta\, d\theta \end{bmatrix}$$

(*Hints*: $\int \theta\cos\theta\, d\theta = \cos\theta + \theta\sin\theta + C$ and $\int \theta\sin\theta\, d\theta = \sin\theta - \theta\cos\theta + C$.)

(a) (4 points) Is T a linear transformation? Justify your answer by verifying the relevant properties or by giving an explicit example where they fail.

Solution: T is linear essentially because integration is linear. More precisely, we can check both properties at once as follows:

$$\begin{aligned} T\left(c_1\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + c_2\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) &= T\left(\begin{bmatrix} c_1x_1 + c_2y_1 \\ c_1x_2 + c_2y_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} \int_0^\pi ((c_1x_1 + c_2y_1)\theta + (c_1x_2 + c_2y_2))\cos\theta\, d\theta \\ \int_0^\pi ((c_1x_1 + c_2y_1)\theta + (c_1x_2 + c_2y_2))\sin\theta\, d\theta \end{bmatrix} \\ &= \begin{bmatrix} c_1\int_0^\pi (x_1\theta + x_2)\cos\theta\, d\theta + c_2\int_0^\pi (y_1\theta + y_2)\cos\theta\, d\theta \\ c_1\int_0^\pi (x_1\theta + x_2)\sin\theta\, d\theta + c_2\int_0^\pi (y_1\theta + y_2)\sin\theta\, d\theta \end{bmatrix} \\ &= c_1\begin{bmatrix} \int_0^\pi (x_1\theta + x_2)\cos\theta\, d\theta \\ \int_0^\pi (x_1\theta + x_2)\sin\theta\, d\theta \end{bmatrix} + c_2\begin{bmatrix} \int_0^\pi (y_1\theta + y_2)\cos\theta\, d\theta \\ \int_0^\pi (y_1\theta + y_2)\sin\theta\, d\theta \end{bmatrix} \\ &= c_1T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + c_2T\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right). \end{aligned}$$

(b) (3 points) If T is linear, find the matrix of T. If T is not linear, compute $T(\vec{e}_1)$ and $T(\vec{e}_2)$.

Solution: T is linear. From the hints we quickly find that

$$\int_0^\pi (x_1\theta + x_2)\cos\theta\, d\theta = -2x_1$$
$$\int_0^\pi (x_1\theta + x_2)\sin\theta\, d\theta = \pi x_1 + 2x_2$$

so that

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -2x_1 \\ \pi x_1 + 2x_2 \end{bmatrix}$$

and the matrix of T can be read off as

$$[T] = \begin{bmatrix} -2 & 0 \\ \pi & 2 \end{bmatrix}.$$

3. Give examples matching the following specifications. You do <u>not</u> need to justify your answers.

   (a) (3 points) Three vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^4$ which are linearly independent and where no coordinate is 0.

   > **Solution:** There are many options. One is
   >
   > $$\vec{v}_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \vec{v}_1 = \begin{bmatrix} 1 & -1 & 1 & 1 \end{bmatrix}, \qquad \vec{v}_1 = \begin{bmatrix} 1 & 1 & -1 & 1 \end{bmatrix}.$$

   (b) (3 points) A linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ where $[T] \neq I$ yet $[T]^2 = I$.

   > **Solution:** Geometrically, rotation about the origin by π has the property that doing it twice does nothing, i.e. $[T]^2 = I$, but clearly $[T] \neq I$. Algebraically, this is
   >
   > $$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ -y \end{bmatrix}.$$

   (c) (3 points) Matrices A and B such that

   $$AB = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \text{and} \qquad BA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

   > **Solution:** One approach is to compute
   >
   > $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}.$$
   >
   > Setting $f = h = 0$ forces the last column to be zero. By symmetry we'll also then want $b = d = 0$, so the products in question are
   >
   > $$\begin{bmatrix} ae & 0 \\ ce & 0 \end{bmatrix}, \qquad \begin{bmatrix} ea & 0 \\ ga & 0 \end{bmatrix}.$$
   >
   > Set $e = 1 = a$ and $c = 1$, $g = 0$ to satisfy the remaining constraints. That is,
   >
   > $$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$