

# 308 INVERSES NOTES

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ABSTRACT. This document summarizes our basic material on inverses. It is largely covered in §3.3 of Holt with a few additions.

## 1. INVERSES, THEORY

**Definition 1.** A function  $f: X \rightarrow Y$  is **invertible** if there is some function  $g: Y \rightarrow X$  where both

- (a)  $(g \circ f)(x) = x$  for all  $x \in X$ , and
- (b)  $(f \circ g)(y) = y$  for all  $y \in Y$ .

**Remark 2.**

- (a) alone is equivalent to  $f$  being one-to-one. In this case,  $g$  is called a **left inverse of  $f$** .
- (b) alone is equivalent to  $f$  being onto. In this case,  $g$  is called a **right inverse of  $f$** .
- If both a left and right inverse exist, they are equal. It follows that in this case all three types of inverses agree and are unique. Hence we can talk about “the inverse” of  $f$  if it has a two-sided inverse, this is unique, and we can write  $f^{-1}$  for it.
- Hence  $f$  is invertible if and only if  $f$  is one-to-one and onto. In this case  $g$  is called a **two-sided inverse of  $f$**  or just an **inverse of  $f$** .

**Example 3.** Let  $R(\theta)$  be a  $2 \times 2$  rotation matrix, so  $T(\mathbf{u}) := R(\theta)\mathbf{u}$  rotates  $\mathbf{u}$  counterclockwise by  $\theta$ . This function is invertible: simply rotate clockwise by  $\theta$ . Hence  $U(\mathbf{u}) := R(-\theta)\mathbf{u}$  is  $T^{-1}$ .

**Theorem 4.** *If  $T: \mathbb{R}^m \rightarrow \mathbb{R}^n$  is linear, then the following are equivalent.*

- (a)  $T$  is invertible.
- (b)  $m = n$  and any of the equivalent conditions in the big theorem hold.

*Proof.* (b)  $\Rightarrow$  (a): If the big theorem holds, then  $T$  is one-to-one and onto, so  $T$  is invertible.

(a)  $\Rightarrow$  (b): If  $T$  is invertible, then  $T$  is one-to-one and onto, so the  $m$  columns of the matrix of  $T$  are linearly independent and span  $\mathbb{R}^n$ . Linear independence says that  $m \leq n$ , and that they span  $\mathbb{R}^n$  says that  $m \geq n$ , so  $m = n$ .  $\square$

**Remark 5.**

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- To most people, the “big theorem” is thought of as a bunch of equivalent conditions for a linear transformation to be invertible.
- Recall one condition in the big theorem is that  $A\mathbf{x} = \mathbf{b}$  has a unique solution  $\mathbf{x}$  for all  $\mathbf{b}$  in  $\mathbb{R}^n$ . In this case,  $\mathbf{x} = A^{-1}\mathbf{b}$  since  $AA^{-1}\mathbf{b} = I\mathbf{b} = \mathbf{b}$ .

**Corollary 6.** *If  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is linear, then it is invertible if and only if it has a left or right inverse. In this case, all three types of inverses are equivalent.*

**Exercise 7.** If  $T$  is linear and  $T^{-1}$  exists, then  $T^{-1}$  is also linear.

**Definition 8.** Let  $A$  be a matrix.  $A$  is **invertible** whenever its linear transformation  $T(\mathbf{x}) := A\mathbf{x}$  is. In this case, the matrix of  $T^{-1}$  is denoted  $A^{-1}$ . Equivalently,  $A$  is invertible if there is a matrix  $B$  such that  $AB = I = BA$ .

- A square matrix which is not invertible is **singular**.
- A square matrix which is invertible is called **nonsingular**.

**Remark 9.**

- Holt’s definition is different but equivalent. He says that  $A$  is invertible if  $A$  is square and there is a matrix  $B$  such that  $AB = I$ . Hence Holt is really saying  $A$  has a right inverse and is leveraging the very special fact that for linear maps given by square matrices, all three types of inverses coincide.
- Down-to-earth version: if  $B = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_n]$ , then  $I = [\mathbf{e}_1 \ \cdots \ \mathbf{e}_n]$  and  $AB = I$  says precisely that  $[A\mathbf{b}_1 \ \cdots \ A\mathbf{b}_n] = [\mathbf{e}_1 \ \cdots \ \mathbf{e}_n]$ , i.e.  $A\mathbf{b}_i = \mathbf{e}_i$ .

**Proposition 10.** *Let  $A$  and  $B$  be invertible  $n \times n$  matrices and let  $C$  and  $D$  be  $n \times m$  matrices. Then:*

- $A^{-1}$  is invertible with inverse  $(A^{-1})^{-1} = A$ .
- $AB$  is invertible with inverse  $(AB)^{-1} = B^{-1}A^{-1}$ .
- If  $AC = AD$ , then  $C = D$ .
- If  $AC = 0$ , then  $C = 0$ .

*Proof.*

- Write  $X$  for  $A^{-1}$  for clarity. Since  $A$  is invertible,  $XA = I$ , so  $A = X^{-1} = (A^{-1})^{-1}$ .
- Class exercise.
- Left multiply by  $A^{-1}$  to get  $A^{-1}AC = A^{-1}AD$ , so  $IC = ID$ , so  $C = D$ .
- Use  $D = 0$  in (c).

□

## 2. INVERSES, COMPUTATION

**Example 11.** Diagonal matrices are a simple, explicit, and important case. Recall that  $D := \text{diag}(d_1, \dots, d_n)$  denotes the  $n \times n$  matrix with  $d_1, \dots, d_n$  along the main diagonal and zeros elsewhere.

Class exercise: when is  $D$  invertible, and if it is invertible what is its inverse? Geometric intuition:  $D$  scales the  $i$ th axis by a factor of  $d_i$ . To undo this, scale again by a factor of  $1/d_i$ . This breaks if we scaled by 0, but it couldn't be undone then anyway.

More formally, if some  $d_i$  is zero, then  $D\mathbf{e}_i = d_i\mathbf{e}_i = \mathbf{0}$ , so  $D$  is not invertible. If no  $d_i$  is zero, then we can check that  $\text{diag}(d_1, \dots, d_n)^{-1} = \text{diag}(d_1^{-1}, \dots, d_n^{-1})$ . (How? Explicit matrix multiplication; compute the composite's action on standard basis vectors; the geometric statement above; general composition rule for diagonal matrices; . . .)

**Remark 12.** Here is a general technique for computing matrix inverses. For simplicity of notation, suppose  $A$  is  $3 \times 3$ . We wish to find  $B$  such that  $AB = I$ . If  $B = [\mathbf{b}_1 \ \mathbf{b}_2 \ \mathbf{b}_3]$ , we need  $A\mathbf{b}_i = \mathbf{e}_i$ , so we could solve three systems of equations  $A\mathbf{x} = \mathbf{e}_1$ ,  $A\mathbf{x} = \mathbf{e}_2$ ,  $A\mathbf{x} = \mathbf{e}_3$ . For each, we could form the augmented matrix  $[A \mid \mathbf{e}_i]$  and row reduce it. If  $A$  is invertible, this system will have a unique solution, so the augmented matrix must have 3 pivots without a pivot in the last column, so it will be of the form

$$\begin{bmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & \mathbf{b}_i \\ 0 & 0 & 1 & \end{bmatrix}.$$

Note that the particular elementary row operations in Gauss–Jordan elimination here do *not* depend on  $i$ . Hence we can do everything at once by row reducing the “very augmented” matrix  $[A \mid I]$  and reading off the columns of  $B$  from the right half.

To summarize: to compute a matrix inverse, row reduce  $[A \mid I]$ . If the result is of the form  $[I \mid B]$ , then  $B = A^{-1}$ . Otherwise,  $A$  is not invertible.

**Example 13.** For two-by-two matrices, there is a relatively common, general formula for the matrix inverse. By Proof Homework 1, #2,  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is invertible if and only if  $ad \neq bc$ , so if and only if  $ad - bc \neq 0$ . In that case, one may show

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Holt calls this the “quick formula”.